Zeros of Systems of $p$-adic Quadratic Forms

D.R. Heath-Brown
Mathematical Institute, Oxford

Abstract

We show that a system of $r$ quadratic forms over a $p$-adic field, in at least $4r + 1$ variables, will have a non-trivial zero as soon as the cardinality of the residue field is large enough. In contrast, the Ax-Kochen theorem [2] requires the characteristic to be large in terms of the degree of the field over $\mathbb{Q}_p$. The proofs use a $p$-adic minimization technique, together with counting arguments over the residue class field, based on considerations from algebraic geometry.

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1 Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$ with associated prime ideal $p$, and let $q_i(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be quadratic forms, for $1 \leq i \leq r$. It would follow from the conjecture of Artin [1, Preface] that these forms have a simultaneous non-trivial zero in $K^n$ providing only that $n > 4r$. Although Artin’s conjecture is known to be false in general (see Terjanian [14], for example), this particular consequence of the conjecture is still open. The cases $r = 1$ and $r = 2$ have been successfully handled, the former being due to Hasse [9] and the latter to Demyanov [6]. For $r = 3$ it has been shown by Schuur [13] that $n \geq 13$ suffices when the residue field has odd characteristic and cardinality at least 11. No analogous result for $r \geq 4$ has been established until now. However it follows from the work of Ax and Kochen [2] that if the degree $[K : \mathbb{Q}_p] = D$ is given, then $n \geq 4r + 1$ variables suffice as
soon as $p \geq p(r, D)$, for some prime $p(r, D)$. The proof uses methods from mathematical logic, and does not yield a practical value for $p(r, D)$.

If one is willing to allow more variables, then further results are available. Thus Leep [7] has shown that it suffices to have $n \geq 2r^2 + 2r - 3$ as soon as $r \geq 2$, for any $p$-adic field $K$, and Martin [12] has improved this further to allow $n \geq 2r^2 + 3$ if $r$ is odd, and $n \geq 2r^2 + 1$ if $r$ is even. One can do a little better for large $r$ but the bound on $n$ is asymptotically $2r^2$ in all such results.

The purpose of the present paper is to develop an analytic method which will establish the following result.

**Theorem** Let $K$ have residue field $F$ and suppose that $\#F = q$. Then the quadratic forms $q^{(1)}, \ldots, q^{(r)}$ have a non-trivial common zero over $K$ as soon as $n \geq 4r + 1$, providing that $q \geq (2r)^r$. More specifically it suffices that $q > n \geq 4r + 1$ and $\sigma_1 + \sigma_2 < 1$, where

$$\sigma_1 = q^{-n} + \sum_{[n/2r]-1 \leq t \leq n/2} q^{-t}(\frac{q}{2t+1})^{[4rt/n]}(2t+1)^r$$

and

$$\sigma_2 = \frac{1}{q-1} \sum_{\rho=2([n/2r]-1)}^{\rho=n-1} \sum_{0 \leq t \leq (n-\rho)/2} C_{\rho,t}q^{-\rho-t+[2\rho/n]+[2r(\rho+2t)/n]}$$

with

$$C_{\rho,t} = (\rho + 1)^{r-[2\rho/n]}(2t+1)^r-[2r(\rho+2t)/n].$$

Here we use the notation

$$[\theta] = \min\{n \in \mathbb{Z} : n \geq \theta\}.$$

Some small improvements in the values of $\sigma_1$ and $\sigma_2$ are possible, but these have little effect on the range of $q$ which one may handle.

It should be emphasized that the Ax-Kochen theorem gives no information about fields with a fixed characteristic $p$. Thus it leaves open the possibility that Artin’s conjecture is never true for dyadic fields, for example. In contrast our result shows that it is sufficient to have $\#F$ large enough.

We have the following corollary. The case $r = 8$ will be of relevance later.

**Corollary 1** It suffices to have $n \geq 4r + 1$ in the following cases.

(i) $r = 3$ and $q \geq 37$;

(ii) $r = 4$ and $q \geq 191$;
(iii) $r = 8$ and $q \geq 271919$.

As an indication of what can be achieved for larger values of $n$ we investigate the condition $n > r^2$, which may be compared with Martin's result [12] mentioned above in which one requires $n \geq 2r^2 + 3$ if $r$ is odd, and that $n \geq 2r^2 + 1$ if $r$ is even, for any $q$.

**Corollary 2** It suffices to have $n \geq r^2 + 1$ providing that $r \geq 5$ and $q \geq (4 \times 10^8)r^2$.

The coefficient in front of $r^2$ can certainly be improved, but the importance of the result is that we require a lower bound for $q$ which is only a power of $r$. However we have been unable to eliminate entirely the need for a lower bound on $q$, even for $n$ as large as $2r^2$.

The case $r = 8$ is of relevance to the problem of $p$-adic zeros of quartic forms. The author [10] has shown that if $p \neq 2, 5$, any quartic form over $\mathbb{Q}_p$ in $n$ variables has a non-trivial $p$-adic zero, providing that any system of 16 linear forms and 8 quadratic forms also has a non-trivial zero. Our results therefore have the following corollary.

**Corollary 3** A quartic form over $\mathbb{Q}_p$ in at least 49 variables has a non-trivial $p$-adic zero providing that $p \geq 271919$.

Our proofs use a $p$-adic minimization technique, for which see Birch and Lewis [3, Lemma 12]. Let $F$ be the residue field. Then, as in [3, §§3 & 4], it suffices to prove our theorem for “minimized” systems of forms $Q^{(i)}$. Such forms will have $p$-adic integer coefficients, and we write $Q^{(i)}(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ for their reductions in $F$. In view of Hensel’s Lemma it will suffice to find a non-singular zero in $F^n$ for the system $Q^{(i)} = 0$. The minimization process ensures that the forms $Q^{(i)}$ will satisfy a key condition, given by (2) of [3, Lemma 12]. We proceed to explain this condition.

Suppose $S^{(1)}, \ldots, S^{(s)}$ are linearly independent forms taken from the $F$-pencil generated by the $Q^{(i)}$. Suppose further that, after a linear change of variables, the forms

$$S^{(i)}(0, \ldots, 0, x_{w+1}, \ldots, x_n) \quad (1 \leq i \leq s)$$

all vanish identically. Then if the original system $q^{(i)}$ was minimized, part (2) of [3, Lemma 12] tells us that

$$w \geq \frac{sn}{2r}.$$  

(1)

In particular, if $n > 4r$ we must have $w > 2s$. As an example of the minimization condition (1), take $n > 4r$ and $s = 1$, whence we deduce that $w \geq 3$. 

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Thus no form $S$ in the pencil can be annihilated by setting 2 variables equal to zero. In particular, if there were any form $S$ in the $F$-pencil which had rank at most 2 we could express it as a function of $x_1$ and $x_2$ only, allowing $w = 2$, and thereby giving a contradiction. Indeed if there were a form of rank 3 it could be written as $S(x_1, x_2, x_3)$, and by Chevalley’s Theorem we could take $S(0, 0, 1) = 0$, which again permits $w = 2$. We therefore conclude that if $n > 4r$ the condition (1) implies that every non-zero form in the $F$-pencil has rank at least 4.

We can now focus on systems $Q^{(i)}$ over the finite field $F$. As noted above, it suffices to find a non-singular zero, given the key minimization condition (1). This will be done by a counting argument, in which we first give a lower bound estimate for the total number of solutions to the system $Q^{(i)} = 0$, and then give an upper bound on the number of singular solutions. Here a major rôle will be played by singular forms in the $F$-pencil generated by the $Q^{(i)}$. We will therefore be forced to consider how many forms of a given rank the pencil can contain, and this problem is the key point in the proof. Our treatment will use some algebraic geometry ultimately motivated by the work of Davenport [5, §2], and it is at this point that the minimization condition (1) is applied.

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2 Geometric Considerations

In discussing the geometry of our system of quadratic forms we shall work over the algebraic closure $\overline{F}$. Thus when we speak of a point on a variety $V$, we shall mean an $\overline{F}$-point, unless we explicitly write $V(F)$. We shall take special care to include the case in which $F$ is dyadic. We write $\chi(F)$ for the characteristic of $F$. Although $F$ will be a finite field in our application, for the generalities discussed below it suffices that $F$ is a perfect field. However the situation can be different when $F$ is not perfect. To begin with we will
not assume that condition (1) holds.

We start by attaching a symmetric $n \times n$ matrix $M^{(i)}$ to each form $Q^{(i)}$. In general, if

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j,$$

then the associated matrix will have entries

$$M_{ij} = \begin{cases} a_{ij}, & i < j, \\ 2a_{ii}, & i = j, \\ a_{ji}, & i > j. \end{cases}$$

(2)

When $\chi(F) \neq 2$ this corresponds to the usual definition. For $\chi(F) = 2$ the matrix $M$ is skew-symmetric, and always has even rank.

By the rank of a quadratic form $Q$ we mean the minimal $r$ such that there is a form $Q'$ over $F$, in $r$ variables, and linear forms $L_1(x_1, \ldots, x_n), \ldots, L_r(x_1, \ldots, x_n)$ over $F$ for which $Q(x_1, \ldots, x_n) = Q'(L_1, \ldots, L_r)$. It is not hard to show that the rank of a form is independent of the field over which one works. When $\chi(F) \neq 2$ one has $\text{Rank}(Q) = \text{Rank}(M)$, but this is not true in general if $\chi(F) = 2$. However we always have

$$\text{Rank}(M) = 2\lceil \text{Rank}(Q)/2 \rceil$$

for dyadic fields.

When $\chi(F) \neq 2$ the condition $\text{Rank}(Q) \leq R$ is equivalent to the vanishing of all the $(R + 1) \times (R + 1)$ minors of $M$. When $\chi(F) = 2$ and $R$ is odd we have $\text{Rank}(Q) \leq R$ if and only if $\text{Rank}(M) \leq R - 1$. Hence in this case a necessary and sufficient condition is that the $R \times R$ minors of $M$ all vanish. When $\chi(F) = 2$ and $R$ is even the picture is slightly more complicated. A necessary and sufficient condition for the rank of $Q$ to be at most $R$ is that $\text{Rank}(M) \leq R$ and that if $\text{Rank}(M) = R$ then $Q$ should vanish on a set of generators for the null space of $M$. However, if $\text{Rank}(M) = R$ then the null space is generated by vectors $v_1, \ldots, v_{n-R}$, whose components are $R \times R$ minors of $M$, while if $\text{Rank}(M) < R$ these vectors will vanish. It follows that if $\chi(F) = 2$ and $R$ is even then $\text{Rank}(Q) \leq R$ if and only if $\text{Rank}(M) \leq R$ and $Q(v_i) = 0$ for $i \leq n - R$. Thus in each case there is a set of polynomial conditions on the coefficients of $Q$, which determines whether or not $\text{Rank}(Q) \leq R$. If we now define

$$V_R = \{ [u] \in \mathbb{P}^{r-1} : \text{Rank}\{\sum_{i=1}^r u_i Q^{(i)}\} \leq R\}$$

(3)
it follows that $V_R$ is an algebraic set. We have shown that these polynomial conditions defining $V_R$ are of degree at most $R + 1$ in $u$ unless $\chi(F) = 2$ and $R$ is even, in which case they have degree $2R + 1$. In the final section of this paper we will establish the following improvement.

**Lemma 1** When $F$ is a perfect field with $\chi(F) = 2$ and $R$ is even there is a set of forms of degree $R + 1$ in the coefficients of the quadratic form $Q$ which vanish if and only if $\text{Rank}(Q) \leq R$.

Suppose that we have a point $[u_0]$ which lies in $V_R(F)$ but not in $V_{R-1}$, where we conventionally write $V_{-1} = \emptyset$. Then $[u_0]$ will belong to some component $W$, say, of $V_R$. We proceed to bound the dimension of $W$.

Let $k = n - R$ and let $G$ be the Grassmannian of $(k - 1)$-dimensional linear spaces $L \subseteq \mathbb{P}^{n-1}$. Then $\text{Rank}(Q) \leq R$ if and only if there is an $L \in G$ such that $Mx = 0$ and $Q(x) = 0$ for all $[x] \in L$. We use the notation $ML = 0$ and $Q(L) = 0$ for these latter conditions. If $\text{Rank}(Q) = R$ the space $L$ will be unique, and will be defined over $F$. If $N$ is the vector space corresponding to $L$, so that $\dim(N) = k$, we say that $N$ is the null space for $Q$.

Let

$$J = \{([u], L) \in W \times G : \left( \sum_{i=1}^{r} u_i M^{(i)} \right)L = 0, \left( \sum_{i=1}^{r} u_i Q^{(i)} \right)(L) = 0 \}.$$  

When we project from $J$ to $W$ the fibre above any point is non-empty, whence $\dim(J) \geq \dim(W)$.

It is now convenient to change the basis for the $F$-pencil generated by the forms $Q^{(i)}$ so that $u_0$ becomes $(1, 0, \ldots, 0)$. We then put $Q = Q^{(1)}$, so that $Q$ has rank exactly $R$. Let $N$ be the null space for $Q$, and make a linear change of variables so that $N$ is generated by the first $k$ unit vectors $e_1, \ldots, e_k$. We would like to examine the tangent space of $J$ at $([u_0], L_0)$, where $L_0$ is the projective linear space corresponding to $N$. This tangent space is most readily identified by switching to the affine setting. We therefore define

$$V = \{v = (v_2, \ldots, v_r) \in \mathbb{A}^{r-1} : [(1, v)] \in W \}$$

and

$$Y = \{y \in \mathbb{A}^{n} : y_1 = \ldots = y_k = 0 \}.$$  

Notice that $0 \in V$ and that $\dim(V) = \dim(W)$.

We now consider the algebraic set $Z \subseteq V \times Y^k$ specified by the condition that $v \in V$, along with the equations

$$\{M + \sum_{i=2}^{r} v_i M^{(i)} \}(e_j + y_j) = 0, \quad (1 \leq j \leq k)$$

when...
and
\[ \{ Q + \sum_{i=2}^{r} v_i Q^{(i)} \}(e_j + y_j) = 0, \quad (1 \leq j \leq k). \]

Thus we have \( nk + k \) equations, in addition to the condition \( v \in V \). Note that our equations imply that \( \{ Q + \sum_i v_i Q^{(i)} \}(w) = 0 \) for any \( w \) in the span of the vectors \( e_j + y_j \). Thus \( Z \) is an affine version of \( J \), with the linear space \( L \) corresponding to the vector space generated by \( e_j + y_j \) for \( 1 \leq j \leq k \). In particular it follows that \( \dim(Z) = \dim(J) \geq \dim(W) \).

One can now calculate the tangent space \( T = T(Z, (0, \ldots, 0)) \). One finds that \( T \) is the set of \((v, y_1, \ldots, y_k) \in T(V, 0) \times Y^k \) which satisfy the equations

\[ \{ \sum_{i=2}^{r} v_i M^{(i)} \} e_j + M y_j = 0, \quad (1 \leq j \leq k) \quad (4) \]

and
\[ \{ \sum_{i=2}^{r} v_i Q^{(i)} \}(e_j) + y_j^T \nabla Q(e_j) = 0, \quad (1 \leq j \leq k). \]

However we have \( \nabla Q(e_j) = M e_j = 0 \), so that the second set of conditions reduce to
\[ \{ \sum_{i=2}^{r} v_i Q^{(i)} \}(e_j) = 0, \quad (1 \leq j \leq k). \quad (5) \]

If \((v, y_1, \ldots, y_k) \in T \) we may pre-multiply the relation (4) by \( e_h^T \) for any \( h \leq k \) and use the fact that \( e_h^T M = 0^T \) to deduce that
\[ e_h^T \{ \sum_{i=2}^{r} v_i M^{(i)} \} e_j = 0, \quad (1 \leq j, h \leq k). \quad (6) \]

The two conditions (5) and (6) now imply that
\[ \{ \sum_{i=2}^{r} v_i Q^{(i)} \}(x) = 0 \text{ for all } x \in N. \quad (7) \]

Let \( \pi : \mathcal{T} \to \mathcal{T}(V, 0) \) be the natural projection. Then the relation (7) holds for any \( v \in \pi(\mathcal{T}) \). However \( \pi \) is a linear map between vector spaces, and
\[ \text{Ker}(\pi) = \{ (0, y_1, \ldots, y_k) \in \{ 0 \} \times Y^k : M y_j = 0, \quad (1 \leq j \leq k) \}. \]

When \( \chi(F) \neq 2 \) the matrix \( M \) has null space \( N \), so that we must have \( y_j = 0 \) for all \( j \), whence \( \text{Ker}(\pi) \) is trivial. When \( \chi(F) = 2 \) the matrix \( M \) will have
null space $N$ only when $R$ is even. Thus, in the dyadic case we now require $R$ to be even. Under this assumption we will have $\dim(\pi(T)) = \dim(T)$, whence

$$\dim(\pi(T)) = \dim(T) \geq \dim(Z) = \dim(J) \geq \dim(W),$$

since the tangent space of $Z$ at any point has dimension at least as large as $Z$ itself.

Since $Q^{(1)}(x) = Q(x) = 0$ for all $x \in N$ we now deduce that there is a linear space of quadratic forms in the $F$-pencil, with dimension at least $1 + \dim(W)$, all vanishing on the space $N$. However $N$ is defined over $F$ itself, whence

$$\{ u \in A^r : \{ \sum_{i=1}^r u_i Q^{(i)} \}(x) = 0 \text{ for all } x \in N \}$$

is also defined over $F$. We therefore draw the following conclusion.

**Lemma 2** Let $V_R$ be the variety (3). Suppose either that $\chi(F) \neq 2$, or that $\chi(F) = 2$ and that $R$ is even. Suppose further that we have a point $u \in F^r$ for which the form

$$Q = \sum_{i=1}^r u_i Q^{(i)}, \quad (8)$$

has rank $R$ and null-space $N$, and such that $[u]$ belongs to an irreducible component $W$ of $V_R$. Then there are at least $1 + \dim(W)$ linearly independent quadratic forms $S^{(i)}$ in the $F$-pencil (8), all of which vanish on the $F$-vector space $N$ of codimension $R$ in $F^n$.

To handle the case in which $\chi(F) = 2$ and $R$ is odd we need to make a small modification of the previous argument. We keep the same notation as before, but in addition to the null space $N$ of $Q$ we must now consider the null space $N_0$ of $M$. In the previous situation these coincided but now $N$ is strictly contained in $N_0$. If we now write $G_0$ for the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{F}^{n-1}$ then $N$ and $N_0$ will correspond to some pair of linear spaces $L \in G$ and $L_0 \in G_0$, with $L \subset L_0$. We now define

$$J_0 = \{ ([u], L, L_0) \in W \times G \times G_0 : L \subset L_0, \quad \sum_{i=1}^r u_i M^{(i)} L_0 = 0, \quad \sum_{i=1}^r u_i Q^{(i)}(L) = 0 \}. $$

As before, when we project from $J_0$ to $W$, the fibre above any point is non-empty, whence $\dim(J_0) \geq \dim(W)$. 

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Following the previous analysis we switch to affine coordinates. We change variables as before, so that $Q = Q^{(1)}$, and so that $N$ and $N_0$ are generated by $e_1, \ldots, e_k$ and $e_1, \ldots, e_{k+1}$ respectively. We use the same set $V$ as before, but take

$$Y = \{ y \in \mathbb{A}^n : y_1 = \ldots = y_{k+1} = 0 \}.$$ 

This time we define a set $Z_0 \subseteq V \times Y^{k+1}$ specified by the condition that $v \in V$, along with the equations

$$\{ M + \sum_{i=2}^r v_i M^{(i)} \} (e_j + y_j) = 0, \quad (1 \leq j \leq k+1)$$

and

$$\{ Q + \sum_{i=2}^r v_i Q^{(i)} \} (e_j + y_j) = 0, \quad (1 \leq j \leq k).$$

Again we note that $Z_0$ is an affine version of $J_0$, whence $\dim(Z_0) = \dim(J_0) \geq \dim(W)$.

The tangent space $T_0 = T(Z_0, (0, \ldots, 0))$ is the set of $(v, y_1, \ldots, y_{k+1}) \in T(V, 0) \times Y^{k+1}$ which satisfy the equations

$$\{ \sum_{i=2}^r v_i M^{(i)} \} e_j + M y_j = 0, \quad (1 \leq j \leq k+1)$$

and

$$\{ \sum_{i=2}^r v_i Q^{(i)} \} (e_j) = 0, \quad (1 \leq j \leq k).$$

As before, these imply that

$$\{ \sum_{i=2}^r v_i Q^{(i)} \} (x) = 0 \text{ for all } x \in N.$$ 

If $\pi_0 : T_0 \to T(V, 0)$ is the natural projection then the above relation holds for any $v \in \pi(T_0)$. However

$$\ker(\pi_0) = \{ (0, y_1, \ldots, y_{k+1}) \in \{ 0 \} \times Y^{k+1} : M y_j = 0, \quad (1 \leq j \leq k+1) \}.$$ 

Since $M$ has null space $N_0$ we must have $y_j = 0$ for all $j$, whence $\ker(\pi_0)$ is trivial. We may now complete the argument as before, leading to the following conclusion.
Lemma 3 Let $V_R$ be the variety (3). Suppose that $\chi(F) = 2$ and that $R$ is odd. Suppose further that we have a point $u \in F^r$ for which the form

$$Q = \sum_{i=1}^{r} u_i Q^{(i)}, \quad (9)$$

has rank $R$ and null-space $N$, and such that $[u]$ belongs to an irreducible component $W$ of $V_R$. Then there are at least $1 + \dim(W)$ linearly independent quadratic forms $S^{(i)}$ in the $F$-pencil (9), all of which vanish on the $F$-vector space $N$ of codimension $R$ in $F^n$.

If we now assume the fundamental minimization condition (1) then we may take $n - w = \dim(N)$, so that

$$R = n - \dim(N) = w \geq \frac{n}{2r}(1 + \dim(W)),$$

and therefore $1 + \dim(W) \leq 2rR/n$.

Lemma 4 Suppose that (1) holds. Let $V_R$ be the variety (3). Then any point $[u] \in \mathbb{P}^{r-1}(F)$ for which the form (8) has rank $R$ will belong to an irreducible component $W$ of $V_R$ having $1 + \dim(W) \leq 2rR/n$.

This lemma is the most novel part of our argument. Notice that it tells us nothing about those components $W$ of $V_R$ which do not contain a point defined over $F$, or for which the only such points are in the subvariety $V_{R-1}$.

We next estimate how many points can lie in each component $W$.

Lemma 5 Suppose that $V \subseteq \mathbb{A}^r$ is an algebraic set of pure dimension $w$ and degree $d$. Then

$$\#V(F) \leq dq^w,$$

where $q = \#F$.

This is a relatively standard result, proved along the lines given by Browning and the author [4, page 91]. We use induction on $w$, the case $w = 0$ being trivial. Clearly we can assume that $V$ is absolutely irreducible, by additivity of the degree. When $w \geq 1$ there is always at least one index $i$ such that $V$ intersects the hyperplane $u_i = \alpha$ properly for every $\alpha \in F$. (If this were not the case, then $V$ must be contained in a hyperplane $u_i = \alpha_i$ for each index
i, so that V could contain at most the single point \((\alpha_1, \ldots, \alpha_r)\). Fixing a suitable index \(i\) we conclude that

\[
\#V(F) \leq \sum_{\alpha \in F} \#(V \cap \{u_i = \alpha\}).
\]

Since \(V \cap \{u_i = \alpha\}\) has dimension at most \(w - 1\) and degree at most \(d\) we may use the induction hypothesis to conclude that

\[
\#(V \cap \{u_i = \alpha\}) \leq dq^{w-1},
\]

whence the required induction bound follows.

In order to estimate the contribution from all the relevant components \(W\) of \(V_R\) we will need information on their degrees as well as their dimensions, and for this we use the following result.

**Lemma 6** Let \(V \subseteq \mathbb{A}^r\) be an algebraic set defined by the vanishing of polynomials \(f_1, \ldots, f_N\) each having total degree at most \(d\). Suppose that \(V\) decomposes into irreducible components as \(V = \bigcup_{i=1}^I V_i\). Then

\[
\sum_{i=1}^I \deg(V_i)d^{\dim(V_i)} \leq d^r.
\]

This is proved by induction on \(N\), the case \(N = 1\) being trivial. We proceed to assume that the result holds for the case \(N\), and prove it for the case \(N + 1\). Let us write \(H = \{f_{N+1} = 0\}\) for convenience, and suppose that \(V_i \cap H\) decomposes into irreducible components as \(\bigcup_{j=1}^{J(i)} V_{ij}\). We claim that

\[
\sum_{j=1}^{J(i)} \deg(V_{ij})d^{\dim(V_{ij})} \leq \deg(V_i)d^{\dim(V_i)}.
\]  

(10)

Once this is established we will have

\[
\sum_{i=1}^I \sum_{j=1}^{J(i)} \deg(V_{ij})d^{\dim(V_{ij})} \leq \sum_{i=1}^I \deg(V_i)d^{\dim(V_i)} \leq d^r,
\]

by the induction hypothesis. We will therefore have completed the induction step.

To prove the statement (10) we factor \(f_{N+1}\) into absolutely irreducible polynomials \(f_{N+1} = g_1 \cdots g_M\), say, and write \(H_k = \{g_k = 0\}\). If there is any index \(k\) such that \(V_i \subseteq H_k\) then \(V_i \subseteq H\), whence \(V_i \cap H = V_i\) is already
irreducible and (10) is trivial. On the other hand, if $V_i$ and $H_k$ intersect properly for every $k$ then $V_i \cap H_k$ is a union of components $V_{ij}$ for $j$ in some set $S(k) \subseteq \{1, \ldots, J(i)\}$, with $\dim(V_{ij}) = \dim(V_i) - 1$ and

$$\sum_{j \in S(k)} \deg(V_{ij}) \leq \deg(V_i) \deg(g_k)$$

by Bézout’s Theorem. Summing over $k$ then yields

$$\sum_{j=1}^{J(i)} \deg(V_{ij}) \leq \deg(V_i)d,$$

and (10) follows in this case too. This completes the proof of Lemma 6.

We now combine Lemmas 4, 5 and 6 to produce the following result.

**Lemma 7** Suppose that the quadratic forms $q^{(i)}$ form a minimized system. Then the number $N(R)$ of quadratic forms (8) of rank $R$, with $u \in F^r$, satisfies

$$N(R) \leq \left(\frac{q}{R+1}\right)^{2rR/n}(R+1)^r$$

whenever $q \geq R + 1$. Moreover any non-zero form in the $F$-pencil has rank at least $2(\lceil n/2r \rceil - 1)$.

Suppose that $V_R$ is a union

$$V_R = \bigcup_{i=1}^{I} W_i$$

of irreducible components, and that the points $[u] \in V_R(F)$ lie in components $W_1, \ldots, W_L$. Then, applying Lemma 5 to the affine cone over each $W_i$, we find that

$$N(R) \leq \sum_{i=1}^{L} \deg(W_i)q^{1+\dim(W_i)}.$$

However according to our remarks at the beginning of §2, and Lemma 1 in particular, the set $V_R$ is defined by equations of degree at most $R + 1 = d$, say, whence Lemma 6 yields

$$\sum_{i=1}^{L} \deg(W_i)(R + 1)^{1+\dim(W_i)} \leq \sum_{i=1}^{I} \deg(W_i)(R + 1)^{1+\dim(W_i)} \leq (R + 1)^r.$$
However Lemma 4 shows that $1 + \dim(W_i) \leq [2rR/n]$ for $i \leq L$, so that if $q \geq R + 1$ we will have

$$N(R) \leq \sum_{i=1}^{L} \deg(W_i)(R+1)^{1+\dim(W_i)} \left(\frac{q}{R+1}\right)^{1+\dim(W_i)}$$

$$\leq \left(\frac{q}{R+1}\right)^{[2rR/n]} \sum_{i=1}^{L} \deg(W_i)(R+1)^{1+\dim(W_i)}$$

$$\leq \left(\frac{q}{R+1}\right)^{[2rR/n]}(R+1)^r$$

as required.

For the final observation we extend the remark made in §1, in connection with the condition (1). Any form of rank $R$ over $F$ will vanish on a vector space of codimension $(R+1)/2$, if $R$ is odd, or of codimension $(R+2)/2$ if $R$ is even. We may therefore take $w = 1+[R/2]$ and deduce that $1+[R/2] \geq n/2r$, which gives the required lower bound on $R$. Note that this argument uses only the minimization condition, and does not require either Lemmas 2, 3 or 4.

3 Counting Zeros

We begin by considering zeros of a system of quadratic forms

$$S^{(i)}(x_1, \ldots, x_k) \in F[x_1, \ldots, x_k], \quad (1 \leq i \leq I).$$

Consider the set

$$A = \{(u, x) \in F^I \times F^k : \sum_{i=1}^{I} u_i S^{(i)}(x_1, \ldots, x_k) = 0\}.$$ 

We shall count elements of $A$ in two ways. Firstly we consider how many choices of $u$ correspond to each $x$. If $S^{(i)}(x) = 0$ for each index $i$ then there are $q^I$ possible vectors $u$, and otherwise $q^{I-1}$ choices. Hence if the system $S^{(i)}(x) = 0$ has $N$ zeros in total we will have

$$\#A = q^I N + q^{I-1}(q^k - N).$$

Alternatively we can count elements of $A$ according to the value $u$. Here we write

$$N(u) = \#\{x \in F^k : \sum_{i=1}^{I} u_i S^{(i)}(x_1, \ldots, x_k) = 0\}.$$
whence
\[ \#A = \sum_u N(u). \]

We therefore deduce that
\[
N = \frac{1}{q^I(q-1)} \left\{ -q^{I+k-1} + \sum_u N(u) \right\}
\]
\[
= \frac{1}{q^I(q-1)} \left\{ \sum_u (N(u) - q^{k-1}) \right\}
\]
\[
= q^{k-I} + \frac{1}{q^I(q-1)} \left\{ \sum_{u \neq 0} (N(u) - q^{k-1}) \right\},
\]
since \( N(0) = q^k \).

We proceed to consider the number \( N(S) \) of zeros of a single quadratic form \( S(x_1, \ldots, x_k) \). If \( \text{Rank}(S) = 0 \), then there are trivially \( q^k \) zeros, and if \( S \) has rank one there are \( q^{k-1} \) zeros. For rank 2 there will be \((2q - 1)q^{k-2}\) zeros if \( S \) factors over \( F \) and \( q^{k-2} \) zeros otherwise. For larger ranks there will be at least one non-singular zero, by Chevalley’s Theorem, and a linear change of variable will allow us to write \( S \) in the shape
\[
S(x_1, \ldots, x_k) = x_1x_2 + S'(x_3, \ldots, x_k).
\]

One then finds that there are \( 2q - 1 \) possibilities for \((x_1, x_2)\) if \( S' = 0 \) and \((q - 1)\) choices otherwise, so that \( N(S) = qN(S') + (q - 1)q^{k-2} \). An easy induction on \( k \) now shows that \( N(S) = q^{k-1} \) whenever \( S \) has odd rank, and that
\[
|N(S) - q^{k-1}| = (1 - q^{-1})q^{k-R/2}
\]
whenever \( S \) has even rank \( R \).

We may therefore conclude as follows.

**Lemma 8** Suppose we have a system of quadratic forms
\[
S^{(i)}(x_1, \ldots, x_k) \in F[x_1, \ldots, x_k], \quad (1 \leq i \leq I)
\]
with \( N \) zeros over \( F \). Write \( N_R \) for the number of vectors \( u \in F^I \) for which
\[
\sum_{i=1}^{I} u_i S^{(i)}(x_1, \ldots, x_k)
\]
has rank \( R \), and assume that such a linear combination vanishes only for \( u = 0 \). Then
\[
|N - q^{k-I}| \leq \sum_{1 \leq t \leq k/2} q^{k-I-t}N_{2t}.
\]
We may now apply Lemma 8 to count non-singular zeros of the system
\[ Q^{(1)}(x_1, \ldots, x_n), \ldots, Q^{(r)}(x_1, \ldots, x_n) \tag{12} \]
arising from a minimized system \( q^{(1)}, \ldots, q^{(r)} \). In view of Lemmas 5 and 7 the total number \( N \) of common zeros satisfies
\[ N \geq q^n - r \sum_{\lfloor n/2 \rfloor - 1 \leq t \leq n/2} q^{-t} \left( \frac{q}{2t + 1} \right)^{\lfloor 4rt/n \rfloor} (2t + 1)^r \] \( \tag{13} \)
providing that \( q > n \geq 4r + 1 \). This latter condition is enough to ensure that \( q \geq 2t + 1 \) whenever \( t \leq n/2 \). Note that if a non-trivial linear combination (11) were to vanish we would be able to take \( s = 1, w = 0 \) in (1), which is impossible. We remark that the sum in (13) is \( O_{r,n}(q^{-1}) \) as soon as \( n > 4r \), and indeed we will have \( N \sim q^{n-r} \) as \( q \to \infty \), for such \( n \). This is the behaviour we would have if the variety defined by \( q^{(1)} = \ldots = q^{(r)} = 0 \) were absolutely irreducible. However it is not clear whether the minimization condition ensures such irreducibility.

We have now to consider singular zeros for the system (12). Any such zero \( x \) is a singular zero of at least one non-zero form (11) in the pencil, \( S \) say. Unless \( x = 0 \) we may deduce that \( S \) is singular. We proceed to estimate how many zeros the system (12) has, which are singular zeros of a given form \( S \) of the shape (11). By changing the basis for the pencil we may indeed assume that \( S = Q^{(r)} \). Suppose that \( S \) has rank \( \rho < n \). Then the singular zeros of \( S \) form a vector space of dimension \( n - \rho = k \), say, which we may take to be
\[ \{ (x_1, \ldots, x_k, 0, \ldots, 0) \}, \]
after a suitable change of variable. It follows then that our problem is to count zeros of the new system
\[ S^{(1)}(x_1, \ldots, x_k), \ldots, S^{(r-1)}(x_1, \ldots, x_k), \]
where
\[ S^{(i)}(x_1, \ldots, x_k) = Q^{(i)}(x_1, \ldots, x_k, 0, \ldots, 0). \]
According to Lemma 8 there are at most
\[ q^{k-(r-1)} \sum_{0 \leq t \leq k/2} q^{-t} N_{2t} \] \( \tag{14} \)
such zeros, where \( N_R \) is the number of linear combinations
\[ \sum_{i=1}^{r-1} u_i S^{(i)}(x_1, \ldots, x_k) \] \( \tag{15} \).
which have rank $R$.

To estimate $N_R$ we will use Lemmas 2 and 3 in combination with Lemmas 5 and 6. If $R = 2t$ and $W \subseteq \mathbb{P}^{r-2}$ is an irreducible component of the variety of vectors counted by $N_R$, then Lemmas 2 and 3 show that we have at least $1 + \dim(W)$ linearly independent forms from the pencil (15) which vanish simultaneously on a vector space $X \subseteq F^k$ of codimension $R$. By extending these to forms on $F^n$ we obtain $1 + \dim(W)$ linearly independent forms from the pencil

$$
\sum_{i=1}^{r-1} u_i Q^{(i)}(x_1, \ldots, x_n)
$$

which vanish simultaneously on

$$\tilde{X} = \{(x_1, \ldots, x_k, 0, \ldots, 0) \in F^n : (x_1, \ldots, x_k) \in X\}.$$ 

However $Q^{(r)}$ also vanishes on $\tilde{X}$, whence the minimization condition (1) yields

$$n - \dim(\tilde{X}) \geq \frac{(2 + \dim(W))n}{2r}.$$ 

Since $\dim(\tilde{X}) = \dim(X) = k - R$ we deduce that

$$\dim(W) \leq \frac{2r(n - k + R)}{n} - 2.$$ (16)

This allows us to use Lemmas 5 and 6 to conclude that

$$N_R \leq \left(\frac{q}{R + 1}\right)^{2r(n - k + R)/n - 1}(R + 1)^{r-1},$$

for $q \geq R + 1$, as in the proof of Lemma 7.

Since $k = n - \rho$ we now find from (14) that the number of zeros of (12) which are singular for a particular $S$ of rank $\rho$ is at most

$$q^{n-\rho-r+1}\left\{ \sum_{0 \leq t \leq (n-\rho)/2} q^{-t}\left(\frac{q}{2t + 1}\right)^{2r(\rho+2t)/n} - 1(2t + 1)^{r-1} \right\}$$

$$= q^{n-\rho-r}\left\{ \sum_{0 \leq t \leq (n-\rho)/2} q^{-t}\left(\frac{q}{2t + 1}\right)^{2r(\rho+2t)/n}(2t + 1)^r \right\}.$$ 

To estimate the total number of singular zeros of (12) we must sum this over all singular forms $S$, and allow for the trivial singular zero $x = 0$. Although Lemma 7 estimates the number of singular forms of given rank, for our present purposes scalar multiples of a given form $S$ produce the same singular zeros. Hence it suffices to count only one form $S$ from each set of
scalar multiples. Thus Lemma 7 shows that the total number of non-trivial singular zeros for the system (12) is at most
\[ \frac{q^{n-r}}{q-1} \sum_{\rho=2}^{n-1} (\frac{q}{\rho+1})^{[2r\rho/n]} \frac{\rho+1)^r}{q^\rho} \sum_{0 \leq t \leq (n-\rho)/2} (\frac{q}{2t+1})^{[2r(\rho+2t)/n]} \frac{(2t+1)^r}{q^t} \]
for \( q > n \). Note that this latter condition will ensure that \( q \geq 2t + 1 \) and that \( q \geq \rho + 1 \). After allowing for \( x = 0 \) it now follows that the total number of non-singular zeros for the system (12) is at least
\[ q^n - r(1 - \sigma_1 - \sigma_2) \]
with \( \sigma_1 \) and \( \sigma_2 \) as in the theorem, and the sufficiency of the condition \( \sigma_1 + \sigma_2 < 1 \) follows.

4 Completion of the Proofs

We begin by examining the special case \( n = 4r + 1 \). With this value of \( n \) we have \([4rt/n] = t - 1\) for \( 2 \leq t \leq n/2 \), whence
\[ \sigma_1 = q^{-3r-1} + q^{-1} \sum_{2 \leq t \leq 2r} (2t + 1)^{r-t+1}. \]
To evaluate \( \sigma_2 \) we observe that for \( n = 4r + 1 \) the ranges for \( \rho \) and \( t \) are given by \( 4 \leq \rho \leq 4r \) and \( 0 \leq t \leq (n-\rho)/2 \). Moreover we have \([2r\rho/n] = (\rho-1)/2 \) and \([2r(\rho+2t)/n] = t + (\rho - 1)/2 \) if \( \rho \) is odd, while \([2r\rho/n] = \rho/2 - 1 \) and \([2r(\rho+2t)/n] = t + \rho/2 - 1 \) if \( \rho \) is even. Thus
\[ \sigma_2 = \frac{1}{q-1} \left\{ q^{-1} \sum_{\nu=2}^{2r-1} \sum_{0 \leq t \leq 2r-\nu} (2\nu + 2)^{r-\nu} (2t + 1)^{r-t-\nu} \right. \\
+ q^{-2} \sum_{\nu=2}^{2r} \sum_{0 \leq t \leq 2r-\nu} (2\nu + 1)^{r-\nu+1} (2t + 1)^{r-t-\nu+1} \right\}. \]
In the case \( r = 3 \) we calculate that
\[ \sigma_1 = q^{-10} + (32.11\ldots)q^{-1} \]
and
\[ \sigma_2 = (14.72\ldots)q^{-1}(q-1)^{-1} + (145.68\ldots)q^{-2}(q-1)^{-1}, \]
whence \( q \geq 37 \) is admissible. The other values for \( r = 4 \) and 8 are calculated similarly.
To prove the general bound it now suffices to assume that \( r \geq 5 \). We observe that \((2t + 1)^{r-t+1} \leq (2r)^{r-1}\) for \(2 \leq t \leq r - 1\), while \((2t + 1)^{r-t+1} \leq 4r + 1\) for \( r \leq t \leq 2r\). It follows that

\[
\sum_{2 \leq t \leq 2r} (2t + 1)^{r-t+1} \leq (r - 2)(2r)^{r-1} + (r + 1)(4r + 1) \leq (r - 1)(2r)^{r-1}.
\]

For \( \sigma_2 \) we recall that \( \nu \) and \( t \) are restricted by the conditions \( 2 \leq \nu \leq 2r \) and \( 0 \leq t \leq 2r - \nu \). We then note that \((2\nu + 2)^{r-\nu} \leq (2r)^{r-2}\) in each of the cases \( 2 \leq \nu \leq r - 1 \) and \( r \leq \nu \leq 2r - 1 \), and similarly that \((2t + 1)^{r-t-\nu} \leq (2r)^{r-2}\) in all cases. Thus

\[
\sum_{\nu=2}^{2r-1} \sum_{0 \leq t \leq 2r - \nu} (2\nu + 2)^{r-\nu}(2t + 1)^{r-t-\nu} \leq (2r)^{2r-2}.
\]

In the same way we have \((2\nu + 1)^{r-\nu+1} \leq (2r + 1)^{r-1}\) and \((2t + 1)^{r-t-\nu+1} \leq (2r - 1)^{r-1}\) in all cases, whence

\[
\sum_{\nu=2}^{2r} \sum_{0 \leq t \leq 2r - \nu} (2\nu + 1)^{r-\nu+1}(2t + 1)^{r-t-\nu+1} \leq (2r)^{2r}.
\]

The condition \( \sigma_1 + \sigma_2 < 1 \) is therefore satisfied if

\[
q^{-r} + (r - 1)(2r)^{r-1}q^{-1} + (2r)^{2r-2}q^{-1}(q - 1)^{-1} + (2r)^{2r}q^{-2}(q - 1)^{-1} < 1.
\]

One now readily verifies that the above inequality holds if \( r \geq 5 \) and \( q \geq (2r)^{r} \), as required for the theorem.

We turn now to Corollary 2. Since \( n \geq r^2 + 1 \) we have \( \lceil n/2r \rceil - 1 \geq (r - 1)/2 \). Thus if \( \phi = 1 - 4/r \) we have

\[
\sigma_1 \leq q^{-r} + \sum_{t \geq (r-1)/2} q^{-\phi t}(2t + 1)^{r}.
\]

In the infinite sum the ratio of the terms for \( t + 1 \) and \( t \) is

\[
q^{-\phi}(1 + \frac{2}{2t + 1})^r \leq q^{-\phi}(1 + \frac{2}{r})^r \leq q^{-\phi}e^2.
\]

Moreover, for a real variable \( t \) the function \( q^{-\phi t}(2t + 1)^r \) is decreasing for \( t \geq (r - 1)/2 \), providing only that \( q^\phi > e^2 \). It follows that the first term in the sum is at most \( q^{-\phi(r-1)/2r}r \), whence

\[
\sum_{t \geq (r-1)/2} q^{-\phi t}(2t + 1)^r \leq \frac{q^{-\phi(r-1)/2r}r}{1 - q^{-\phi}e^2} \quad \text{(17)}
\].

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and

\[ \sigma_1 \leq q^{-r} + \frac{q^{-\phi(r-1)/2}r^r}{1 - q^{-\phi}e^2} \]

if \( r \geq 5 \) and \( q^\phi > e^2 \).

Similarly we find that

\[ \sigma_2 \leq \frac{1}{q-1} \left\{ \sum_{\rho=r-1}^{\infty} \sum_{t=0}^{\infty} q^{-\rho^r} \rho + 1)^r (2t + 1)^r \right\}. \]

The double sum factors, and the summation over \( \rho \) is

\[ \sum_{\rho=r-1}^{\infty} q^{-\rho^r} (\rho + 1)^r \leq \frac{q^{-\phi(r-1)r^r}}{1 - q^{-\phi}e} \]

by an argument closely analogous to that above. For the \( t \)-summation we note that the real variable function \( f(\tau) = \tau^r q^{-\phi \tau^r/2} \) is maximal at \( \tau = 2r/(\phi \log q) \), with maximum value \( \{2r/(e\phi \log q)\}^r \leq (r/e)^r \) if \( q^\phi > e^2 \). Thus

\[ \sum_{0 \leq t \leq (r-2)/2} q^{-\phi t} (2t + 1)^r \leq \frac{r}{2} q^{\phi/2}(r/e)^r. \]

On combining this with (17) we deduce that

\[ \sigma_2 \leq \frac{1}{q-1} \left\{ \frac{q^{-\phi(r-1)r^r}}{1 - q^{-\phi}e} \right\} \left\{ \frac{r}{2} q^{\phi/2}(r/e)^r + \frac{q^{-\phi(r-1)/2}r^r}{1 - q^{-\phi}e^2} \right\}. \]

Assuming that \( q^\phi \geq 2e^2 \) we conclude that

\[ \sigma_2 \leq q^{-\phi(r-3/2)r^2r} \frac{1}{q-1} \left\{ \frac{1}{1 - 1/2e} \right\} \left\{ \frac{r}{2} e^{-r} + 2q^{-\phi r} \right\} \]

\[ \leq q^{-\phi(r-3/2)r^2r} \frac{2}{q} \left\{ \frac{r}{2} e^{-r} + 2e^{-2r} \right\} \]

\[ \leq q^{-\phi(r-1/2)r^2r} C_r \]

where

\[ C_r = \left\{ \frac{r}{2} e^{-r} + 2e^{-2r} \right\} \leq 1 \]

for \( r \geq 5 \).

One may now calculate that \( \phi_1 + \phi_2 < 1 \) providing that \( q^\phi \geq 4r^2(\geq 2e^2) \). However, the function \( (2r)^{1/(r-4)} \) is decreasing for \( r \geq 5 \), so that

\[ (4r^2)^{1/\phi} = (4r^2)^{(2r)^{1/(r-4)}} \leq 10^8(4r^2), \]

and Corollary 2 follows.
5 Ranks of Quadratic Forms in Characteristic 2

In this final section we will prove Lemma 1. Recall that $F$ is any perfect field of characteristic 2. Let $t_{ij}$ be indeterminates for $1 \leq i \leq j \leq n$, and write $t = (t_{11}, t_{12}, \ldots, t_{nn})$. Let

$$Q_t(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} t_{ij} x_i x_j$$

be the corresponding quadratic form, considered as a polynomial in $\mathbb{Z}[t_{11}, t_{12}, \ldots, t_{nn}, x_1, \ldots, x_n]$.

We associate a matrix $U(t)$ to $Q_t$, with entries

$$U_{ij} = \begin{cases} t_{ij}, & i < j, \\ 2t_{ii}, & i = j, \\ t_{ji}, & i > j. \end{cases}$$

If $I, J \subseteq \{1, \ldots, n\}$ with $\#I = \#J = R + 1$ we define $m_{I,J}^*(t)$ to be the $I, J$ minor of $U$. This has order $(R+1) \times (R+1)$, and is a form of degree $R + 1$ in the variables $t_{ij}$. If $R$ is even, as we are supposing, then $m_{I,I}^*(t)$ vanishes modulo 2, since it becomes the determinant of a skew-symmetric matrix of odd order when we reduce to $\mathbb{Z}_2$. Thus if we define

$$m_{I,J}(t) = \begin{cases} m_{I,J}^*(t), & I \neq J, \\ \frac{1}{2} m_{I,I}^*(t), & I = J, \end{cases}$$

then $m_{I,J}$ will be an integral form in the $t_{ij}$.

When $I = J$ this is the “half-determinant”, introduced by Kneser in the 1970’s, see [11]. A detailed discussion is given by Leep and Schueller [8, pp 395–397], but what we establish here will be sufficient for our purposes. We are grateful to the referee for pointing out these references.

We now map the various $m_{I,J}(t)$ to forms $m_{I,J}(t; F)$ in $F[t_{11}, \ldots, t_{nn}]$, using the obvious homomorphism from $\mathbb{Z}[t_{11}, \ldots, t_{nn}]$ to $F[t_{11}, \ldots, t_{nn}]$. Let

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} q_{ij} x_i x_j$$

be a quadratic form over a finite field $F$ of characteristic 2. Then we claim that a necessary and sufficient condition for $Q$ to have rank at most $R$, is that the forms $m_{I,J}(t; F)$ all vanish at $t_{ij} = q_{ij}$. This will clearly suffice
for Lemma 1. It will be convenient to call this condition on $Q$ the “Rank Condition”.

We now use the fact that any quadratic form over $F$, of rank at least 3, has a non-singular zero. This is an easy exercise. It follows that any quadratic form over $F$ can be reduced, via a sequence of elementary transformations, into a form of the shape

$$x_1 x_2 + \ldots + x_{2m-2} x_{2m} + q(x_{2m+1}, \ldots, x_n),$$

in which $q(x_{2m+1}, \ldots, x_n) = 0$, or $x_{2m+1}^2$, or $x_{2m+1}^2 + x_{2m+1} x_{2m+2} + \mu x_{2m+2}^2$.

In the third case $\mu \in F$ is such that $q$ is irreducible over $F$. The rank of the form will be $2m$ or $2m + 1$ or $2m + 2$ respectively. One can easily verify by explicit calculation that our claim holds if $Q$ is in one of these three canonical shapes.

We proceed to show that if forms $Q$ and $Q'$, with coefficients $q_{ij}$ and $q'_{ij}$ respectively, are related by an elementary transformation, then $Q$ satisfies the Rank Condition if and only if $Q'$ does. This will be sufficient to complete the proof. Indeed, since elementary transformations are invertible, it will be enough to assume that $Q$ satisfies the Rank Condition and to deduce that $Q'$ does.

Elementary transformations are of three types. The first kind interchanges two of the variables $x_i$ and $x_j$, and in this case our result is trivial, since the forms $m_{I,J}(t; F)$ will merely be permuted. The second type of transformation is $S(\lambda)$, say, which multiplies $x_1$ by a non-zero scalar $\lambda$. If we apply $S(v)$, with an indeterminate $v$, to the quadratic form (18), then the forms $m^*_{I,J}(t)$ will be multiplied by appropriate powers of $v$. It follows that $S(\lambda)$ will multiply each $m_{I,J}(q; F)$ by a power of $\lambda$. Hence again we see that if $Q$ satisfies the Rank Condition then so does $Q'$.

The third type of elementary transformation, which we denote by $T(\lambda)$, replaces $x_1$ by $x_1 + \lambda x_2$. The argument here is similar to that used for $S(\lambda)$. When $T(v)$ is applied to $Q$, the forms $m^*_{I,J}(t)$ get replaced by linear combinations of various $m^*_{K,L}(t)$, with coefficients $1, v$ or $v^2$. Hence when $T(\lambda)$ is applied to $Q$ the forms $m_{I,J}(q; F)$ get replaced by linear combinations of various $m_{K,L}(q; F)$, with coefficients $1, \lambda$ or $\lambda^2$. Again it is clear that if $Q$ satisfies the Rank Condition then so does $Q'$. This completes the proof of the lemma.
References


Mathematical Institute,
24–29, St. Giles’,
Oxford
OX1 3LB
UK

rhb@maths.ox.ac.uk