Optimal L2-error estimates for the semidiscrete Galerkin approximation to a second order linear parabolic initial and boundary value problem with nonsmooth initial data

by

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Abstract

In this article, we have discussed a priori error estimate for the semidiscrete Galerkin approximation of a general second order parabolic initial and boundary value problem with non-smooth initial data. Our analysis is based on an elementary energy argument without resorting to parabolic duality technique. The proposed technique is also extended to a semidiscrete mixed method for parabolic problems. Optimal $L^2$-error estimate is derived for both cases, when the initial data is in $L^2$.

Key words. error estimates, finite element, mixed finite element, parabolic equation, nonsmooth data, energy argument.

1 Introduction

In this paper, we discuss a priori $L^2$ error estimate for the semi-discrete Galerkin approximation to the following second order linear parabolic initial and boundary value problem

\[
\begin{align*}
    u_t - A(t)u &= 0 \quad \text{in } \Omega \times (0,T], \\
    u|_{\partial \Omega} &= 0 \quad u|_{t=0} = u_0,
\end{align*}
\]

when $u_0 \in L^2(\Omega)$. Here, $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded convex polygonal or polyhedral domain with boundary $\partial \Omega$, $0 < T < \infty$, $u_t = \frac{\partial u}{\partial t}$ and $A(t)$ is a second order elliptic operator of the form

\[
A(t) = \nabla \cdot (A \nabla + \bar{b}) - c,
\]

where $A(x,t) = (a_{ij}(x,t))_{1 \leq i,j \leq d}$ is a symmetric and uniformly positive definite matrix and $\bar{b}(x,t) = b_i(x,t)$ is a vector in $\mathbb{R}^d$, $d \geq 1$. All the coefficients $a_{ij}, b_i$ and $c$ are smooth functions of $x$ and $t$, with uniformly bounded derivatives on $\Omega \times (0,T)$ and $c \geq 0$.

Error estimates for semidiscrete and completely discrete approximations to parabolic equations for smooth and non-smooth initial data have been extensively studied in the literature (see [6], for references). When the initial data $u_0$ is in $L^2(\Omega)$, there are, in general, two major techniques applied to semi-discrete Galerkin approximation of a second order linear parabolic equation. While the technique introduced by Thomée et al.[1] is based on the use of the inverse of the associated elliptic operator and spectral theory, the energy argument in conjunction with parabolic duality is analyzed by Luskin and Rannacher[3] for a more general time dependent parabolic equation. In [4], Huang and Thomée have extended the analysis of [1] to discuss optimal $L^2$-estimates for time dependent parabolic
equation with nonself-adjoint elliptic part by using the inverse of the associated uniformly positive definite elliptic operator and a two parameter family of solution operator. In this paper, we use energy argument without resorting to the use of parabolic duality. In fact, the present tool can be thought of as a proper combination of the tools used in [4] and [3]. Essentially, we use integration in time, instead of using the inverse of the elliptic operator and the energy arguments without employing parabolic duality. The argument presented in this article is easily extended to semidiscrete mixed finite element method for parabolic boundary and initial value problem with nonsmooth initial data.

The remaining part of this paper is organized as follows. While in section 2, we discuss a priori estimates and the regularity results, in section 3, we present the $L^2$ error analysis for the semidiscrete Galerkin approximations to (1.1) when the initial data is in $L^2(\Omega)$. Finally, section 4 is devoted to the error analysis of semidiscrete mixed Galerkin approximations to (1.1), when the initial data is in $L^2(\Omega)$.

2 A priori estimates and regularity results

We use usual notations for the $L^2$, $H^1_0$ and $H^2$ spaces and their norms and semi-norms.

Let the bilinear form $A : H^1_0 \times H^1_0 \rightarrow \mathbb{R}$ be defined by

$$A(\phi, \psi) = (A\nabla \phi + \bar{b}\phi, \nabla \psi) + (c\phi, \psi), \quad \phi, \psi \in H^1_0.$$ 

Based on the assumptions on the coefficients, it is possible to show that the bilinear form $A$ satisfies Gårding’s inequality, i.e., there are positive constants $\rho_1$ and $\beta$, independent of $t$, such that

$$A(\phi, \phi) \geq \rho_1\|\phi\|_1^2 - \beta\|\phi\|^2, \quad \phi \in H^1_0.$$ 

Also, the domain being convex polygonal or polyhedral, there is a positive constant $\rho_2$, independent of $t$, such that

$$\|\phi\|_2 \leq \rho_2\|A(t)\phi\|, \quad \phi \in H^1_0 \cap H^2.$$ 

Finally, there is a positive constant $\rho_3$, independent of $t$, such that

$$|A(\phi, \psi)| \leq \rho_3\|\phi\|_1\|\psi\|_1, \quad \phi, \psi \in H^1_0.$$ 

We define the bilinear form $A_t(\cdot, \cdot) : H^1_0 \times H^1_0 \rightarrow \mathbb{R}$ by

$$A_t(\phi, \psi) = (A_t\nabla \phi + \bar{b}_t\phi, \nabla \psi) + (c_t\phi, \psi), \quad \phi, \psi \in H^1_0.$$ 

Since the coefficients and their derivatives are bounded uniformly in time, we conclude that, there exists a positive constant $\rho_4$, independent of $t$, such that

$$|A_t(\phi, \psi)| \leq \rho_4\|\phi\|_1\|\psi\|_1, \quad \phi, \psi \in H^1_0.$$ 

We now present a weak formulation for the problem (1.1). Find $u(t) \in H^1_0(\Omega), \; t \in (0, T]$ with $T > 0$ satisfying $u_0 \in L^2$ and

$$u_t + A(u, \phi) = 0, \quad \forall \phi \in H^1_0, \quad t \in (0, T],$$

$$u(0) = u_0.$$
We recall below \textit{a priori} estimates and regularity results for the solution of (2.5), when \( u_0 \in L^2 \). For a proof, we refer to [3].

**Lemma 2.1** Let \( u \) be the exact solution of the second order parabolic equation (1.1) and let \( 0 \leq i \leq 4 \) and \( 0 \leq j, k \leq 2 \).

(a) If \( 0 \leq k + 2j - i \leq 2 \), then for \( t \in J = (0, T] \)
\[
\| \partial_t^i \partial_t^j u(t) \|_k^2 \leq C \| u_0 \|_{k+2j-i}^2.
\]

(b) If \( 0 \leq k + 2j - i - 1 \leq 2 \), then \( t \in J = (0, T] \)
\[
\int_0^t s^i \| \partial_t^j u(t) \|_k^2 ds \leq C \| u_0 \|_{k+2j-i-1}^2.
\]

Throughout the rest of the paper, we use the following notation:
\[
\hat{\phi}(t) = \int_0^t \phi(s) ds.
\]

Integrate (1.1) with respect to time to obtain
\[
(u, \phi) + \mathcal{A}(\hat{u}, \phi) - \int_0^t \mathcal{A}_s(\hat{u}(s), \phi) ds = (u_0, \phi).
\]

Put \( \phi = \hat{u}(t) \) and integrate to arrive at
\[
\| \hat{u} \|^2 + \rho_1 \int_0^t \| \hat{u}(s) \|^2_1 ds \leq C \| u_0 \|^2 + C \rho_1 \int_0^t \{ \| \hat{u}(s) \|^2 + \rho_1 \int_0^s \| \hat{u}(\tau) \|^2_1 d\tau \} ds.
\]

Use Gronwall’s Lemma to obtain
\[
\| \hat{u} \|^2 + \int_0^t \| \hat{u}(s) \|^2_1 ds \leq C \| u_0 \|^2.
\]

Now, from (2.2), we find that
\[
\| \hat{u}(t) \|_2^2 \leq \rho_2 \| \mathcal{A}(\hat{u}) \|^2 \leq \rho_2 \{ \| u(t) \| + \| u_0 \| + \| \int_0^t \mathcal{A}_s(\hat{u}(s)) ds \| \}^2.
\]

Note that
\[
\| \int_0^t \mathcal{A}_s(\hat{u}(s)) ds \|^2 = \int_\Omega \int_0^t \mathcal{A}_s(\hat{u}(s))^2 ds dx,
\]
\[
\leq \int_\Omega \int_0^t |\mathcal{A}_s(\hat{u}(s))|^2 ds dx,
\]
\[
\leq C t \int_\Omega \int_0^t |\hat{u}(s)|_2^2 ds dx = C t \int_0^t \| \hat{u}(s) \|_2^2 ds.
\]

Therefore, we obtain
\[
\| \hat{u}(t) \|_2^2 \leq C \| u_0 \|^2 + C(T) \int_0^t \| \hat{u}(s) \|_2^2 ds.
\]

Using Gronwall’s Lemma, we now arrive at
\[
(2.7) \quad \| \hat{u}(t) \|_2 \leq C \| u_0 \|.
\]
3 Error Estimates for Galerkin Approximation

Let $h$ with $0 < h < 1$ be the discretizing parameter of a triangulation of $\Omega$. Let us denote, as $S_h$, the corresponding finite dimensional subspace of $H_0^1$ such that for all $v \in H_0^1 \cap H^2$, $k \in \{1, 2\}$,

$$\inf_{\phi_h \in S_h} \|v - \phi_h\|_j \leq \rho_5 h^{k-j} \|v\|_k, \quad j \in \{0, 1\},$$

where $\rho_5$ is independent of $h$ or $v$.

Below, we present a semi-discrete Galerkin formulation of our problem (1.1). Find $u_h(t) \in S_h$, $0 < t \leq T$, satisfying

$$\left( u_{ht}, \phi_h \right) + A(u_h, \phi_h) = 0, \quad \forall \phi_h \in S_h, \quad t \in (0, T]$$

with $u_h(0) = P_h u_0$, where $P_h u_0$ is the $L^2$-projection of $u_0$ onto $S_h$.

In this section, we discuss the $L^2$ estimate of the error $e = u - u_h$. More precisely, we shall prove the following theorem.

**Theorem 3.1** Let $u$ and $u_h$ be the solution of the equations (2.5) and (3.2), respectively, with $u(0) = u_0$ and $u_h(0) = P_h u_0$. Then

$$\|u - u_h\| \leq C h^2 t^{-1} \|u_0\|.$$

The proof of this theorem is achieved through a series of Lemmas presented below. First, we define the Ritz projection $R_h : H_0^1 \to S_h$ as

$$A(u - R_h u, \phi_h) = 0, \quad \forall \phi_h \in S_h, \quad u \in H_0^1.$$

With $\eta = u - R_h u$, we discuss below some estimates for $\eta$. For a proof, we again refer to [3].

**Lemma 3.1** For $\eta$ as defined above we have, for $k \in \{1, 2\}$, $j \in \{0, 1\}$ and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2(\Omega)$

$$\|\eta\|_j \leq C h^{k-j} \|u\|_k \leq C h^{k-j} t^{-k/2} \|u_0\|,$$

$$\|\eta_t\|_j \leq C h^{k-j} \{\|u\|_k + \|u_t\|_k\} \leq C h^{k-j} t^{-(1+k/2)} \|u_0\|.$$

Next, we present an estimate for $\dot{\eta}$.

**Lemma 3.2** For $\eta$ as defined above we have, for $k \in \{1, 2\}$, $j \in \{0, 1\}$ and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2(\Omega)$

$$\|\dot{\eta}\|_j \leq C h^{k-j} \|u_0\|.$$

**Proof.** We recall that for $\phi_h \in S_h$,

$$A(\eta, \phi_h) = 0.$$/
We integrate with respect to time to obtain

\[(3.4) \quad \mathcal{A}(\hat{\eta}, \phi_h) - \int_0^t \mathcal{A}_s(\hat{\eta}(s), \phi_h) ds = 0.\]

Using Gårding’s inequality (2.1) for \(\mathcal{A}(\cdot, \cdot)\), (2.3) and (3.4), it follows that

\[
\rho_1 \|\hat{\eta}\|_1^2 \leq \mathcal{A}(\hat{\eta}, \hat{\eta}) + \beta \|\hat{\eta}\|^2
\]

\[
= \mathcal{A}(\hat{\eta}, \hat{\eta} - \chi) + \int_0^t \mathcal{A}_s(\hat{\eta}(s), \chi - R_h \hat{\eta}) + \beta \|\hat{\eta}\|^2, \quad \chi \in S_h,
\]

\[
\leq \rho_3 \|\hat{\eta}\|_1 \|\hat{\eta} - \chi\|_1 + \int_0^t \mathcal{A}_s(\hat{\eta}(s), \chi - \hat{\eta} + \hat{\eta} - R_h \hat{\eta}) + \beta \|\hat{\eta}\|^2
\]

\[
\leq \rho_3 \|\hat{\eta}\|_1 \|\hat{\eta} - \chi\|_1 + C(\|\chi - \hat{\eta}\|_1 + \|\hat{\eta}\|_1) \int_0^t \|\hat{\eta}(s)\|_1 ds + \beta \|\hat{\eta}\|^2.
\]

Using Young’s inequality and, (3.1) and (2.7), we obtain

\[
\|\hat{\eta}\|_1^2 \leq C \|\hat{\eta} - \chi\|_1^2 + C \int_0^t \|\hat{\eta}(s)\|_1^2 ds + C \|\hat{\eta}\|^2
\]

\[
\leq Ch^2 \|u_0\|^2 + \beta \|\hat{\eta}\|^2 + C \int_0^t \|\hat{\eta}(s)\|_1^2 ds.
\]

Now, use Gronwall’s Lemma to arrive at

\[
\|\hat{\eta}\|_1^2 \leq Ch^2 \|u_0\|^2 + C \|\hat{\eta}\|^2 + C \int_0^t \|\hat{\eta}(s)\|_1^2 ds.
\]

And therefore,

\[(3.5) \quad \|\hat{\eta}\|_1 \leq C \left( h \|u_0\| + \|\hat{\eta}\| \right) + C \int_0^t \|\hat{\eta}(s)\| ds.
\]

For the \(L^2\) estimate, we use the Aubin-Nitche duality argument. Now consider the following auxiliary problem: Find \(\phi \in H_0^1(\Omega)\) such that

\[-\mathcal{A}^* \phi = \hat{\eta}, \quad \text{on } \Omega,
\]

\[
\phi|_{\partial\Omega} = 0,
\]

where \(\mathcal{A}^*\) is the formal adjoint of \(\mathcal{A}\). Note that following regularity result holds for the above problem:

\[(3.6) \quad \|\phi\|_2 \leq C \|\hat{\eta}\|.
\]

Now,

\[
\|\hat{\eta}\|^2 = -(\mathcal{A}^* \phi, \hat{\eta}),
\]

\[
= \mathcal{A}(\hat{\eta}, \phi - \chi) + \mathcal{A}(\hat{\eta}, \chi) \quad \text{for some } \chi \in S_h,
\]

\[
= \mathcal{A}(\hat{\eta}, \phi - \chi) + \int_0^t \mathcal{A}_s(\hat{\eta}(s), \chi) ds \quad \text{from (3.4),}
\]

\[
= \mathcal{A}(\hat{\eta}, \phi - \chi) + \int_0^t \mathcal{A}_s(\hat{\eta}(s), \chi - \phi) ds + \int_0^t \mathcal{A}_s(\hat{\eta}(s), \phi) ds.
\]
Since \( \mathbf{A}_s(\hat{\eta}, \phi) = -(\mathbf{A}_s^* \phi, \hat{\eta}) \leq C \| \hat{\eta} \| \| \phi \|_2, \) we, therefore, find that

\[
\| \hat{\eta} \|^2 \leq \rho_3 \| \hat{\eta} \|_1 \| \phi - \chi \|_1 + C \int_0^t (\| \hat{\eta}(s) \|_1 \| \phi - \chi \|_1 + \| \hat{\eta}(s) \| \| \phi \|_2) ds.
\]

Use (3.1), (3.6) and (3.5) to obtain

\[
\| \hat{\eta} \| \leq Ch(\| \hat{\eta} \|_1 + \int_0^t \| \hat{\eta}(s) \|_1 ds) + C \int_0^t \| \hat{\eta}(s) \| ds.
\]

We choose \( h \) sufficiently small such that \( 1 - Ch > 0 \) and then apply Gronwall’s Lemma to conclude

\[
\| \hat{\eta} \| \leq Ch^2 \| u_0 \|.
\]

Now, use (3.5) to complete the proof. \( \square \)

Next, we observe that the error \( e = u - u_h \) satisfies the following equation

(3.7) \( (e_t, \phi_h) + \mathbf{A}(e, \phi_h) = 0, \quad \forall \phi_h \in S_h, \quad t \in (0, T]. \)

Now integrate (3.7) once and twice and use \( u_h = P_h u_0 \) to obtain

(3.8) \( (e, \phi_h) + \mathbf{A}(\hat{e}, \phi_h) - \int_0^t \mathbf{A}_s(\hat{e}(s), \phi_h) = 0, \quad \phi_h \in S_h, \)

and

(3.9) \( (e, \phi_h) + \mathbf{A}(\hat{e}, \phi_h) - 2 \int_0^t \mathbf{A}_s(\hat{e}(s), \phi_h) + \int_0^t \int_0^s \mathbf{A}_{\tau\tau}(\hat{e}(\tau), \phi) d\tau ds = 0, \quad \phi_h \in S_h, \)

Note that we have used the fact that, for \( \phi \in H^1_0 \)

\[
\int_0^t \mathbf{A}(v(s), \phi) ds = \mathbf{A}(\hat{v}, \phi) - \int_0^t \mathbf{A}_s(\hat{v}(s), \phi) ds,
\]

and

\[
\int_0^t \int_0^s \mathbf{A}(v(\tau), \phi) d\tau ds = \mathbf{A}(\hat{v}, \phi) - 2 \int_0^t \mathbf{A}_s(\hat{v}(s), \phi) ds + \int_0^t \int_0^s \mathbf{A}_{\tau\tau}(\hat{v}(\tau), \phi) d\tau ds.
\]

Using Ritz projection \( R_h \), rewrite

(3.10) \( e = u - u_h = (u - R_h u) - (u_h - R_h u) =: \eta - \xi. \)

Now use (3.10) and (3.4) in (3.8) to obtain

(3.11) \( (\xi, \phi_h) + \mathbf{A}(\hat{\xi}, \phi_h) - \int_0^t \mathbf{A}_s(\hat{\xi}(s), \phi_h) ds = (\eta, \phi_h), \quad \forall \phi_h \in S_h. \)
We next integrate (3.4) to arrive at

\begin{equation}
\mathcal{A}(\hat{\eta}, \phi_h) - 2 \int_0^t \mathcal{A}_s(\hat{\eta}(s), \phi_h) \, ds + \int_0^t \int_0^s \mathcal{A}_{\tau \tau}(\hat{\eta}(\tau), \phi_h) \, d\tau \, ds = 0.
\end{equation}

Now, use (3.10) and (3.12) in (3.9) to obtain

\begin{equation}
(\hat{\xi}, \phi_h) + \mathcal{A}(\hat{\xi}, \phi_h) - 2 \int_0^t \mathcal{A}_s(\hat{\xi}(s), \phi_h) \, ds + \int_0^t \int_0^s \mathcal{A}_{\tau \tau}(\hat{\xi}(\tau), \phi_h) \, d\tau \, ds = (\hat{\eta}, \phi_h), \quad \forall \phi_h \in S_h.
\end{equation}

**Lemma 3.3** With \( \hat{\xi} \) satisfying (3.13), there exists a positive constant \( C \), independent of \( h \), such that

\begin{equation}
\|\hat{\xi}\|^2 + \int_0^t \|\hat{\xi}(s)\|^2 \, ds \leq Cth^4\|u_0\|^2,
\end{equation}

and

\begin{equation}
\|\hat{\xi}\|^2 + \int_0^t \|\hat{\xi}(s)\|^2 \, ds \leq Cth^4\|u_0\|^2.
\end{equation}

**Proof.** Choose \( \phi_h = \hat{\xi}(t) \) in (3.13) and use (2.1) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\hat{\xi}\|^2 + \rho_1 \|\hat{\xi}\|^2 \leq C \left\{ \int_0^t (\|\hat{\xi}(s)\|_1^1 ds + \int_0^t \int_0^s \|\hat{\xi}(\tau)\|_1^1 d\tau ds \right\} \|\hat{\xi}\|_1 + \|\hat{\eta}\|_1 \|\hat{\xi}\| + \beta \|\hat{\xi}\|^2.
\]

Observe that to estimate \( \mathcal{A}_{\tau \tau}(\cdot, \cdot) \), we have used similar arguments as in (2.4) to obtain (2.5). Now integrate and use Lemma 3.2 to arrive at

\[
\|\hat{\xi}\|^2 + \rho_1 \int_0^t \|\hat{\xi}(s)\|^2 \, ds \leq C \int_0^t (\|\hat{\xi}(s)\|^2 + \int_0^s \|\hat{\xi}(\tau)\|^2 \, d\tau) \, ds + C \int_0^t \|\hat{\eta}(s)\|^2 \, ds
\]

\[
\leq Cth^4\|u_0\|^2 + C \int_0^t (\|\hat{\xi}(s)\|^2 + \|\hat{\xi}(s)\|_1^1 d\tau ds.
\]

An appeal to Gronwall’s Lemma concludes (3.14).

Next, set \( \phi_h = \hat{\xi}(t) \) in (3.13) and rewrite the resulting equation as

\begin{equation}
\|\hat{\xi}\|^2 + (A\nabla \hat{\xi}, \nabla \hat{\xi}) + (\hat{b}\cdot \hat{\xi}, \nabla \hat{\xi}) + (\hat{c}\cdot \hat{\xi}, \hat{\xi}) = 2 \int_0^t \mathcal{A}_s(\hat{\xi}(s), \hat{\xi}) \, ds
\end{equation}

\[- \int_0^t \int_0^s \mathcal{A}_{\tau \tau}(\hat{\xi}(\tau), \hat{\xi}) \, d\tau \, ds + (\hat{\eta}, \hat{\xi}).
\]

Since

\[
\frac{d}{dt} \mathcal{A}(\hat{\xi}, \hat{\xi}) = \mathcal{A}_t(\hat{\xi}, \hat{\xi}) + 2(A\nabla \hat{\xi}, \nabla \hat{\xi}) + (\hat{b}\cdot \hat{\xi}, \nabla \hat{\xi}) + (\hat{b}\cdot \nabla \hat{\xi}, \hat{\xi}) + 2(\hat{c}\cdot \hat{\xi}, \hat{\xi}),
\]

we obtain from (3.16)

\[
\|\hat{\xi}\|^2 + \frac{1}{2} \frac{d}{dt} \mathcal{A}(\hat{\xi}, \hat{\xi}) - \frac{1}{2} \mathcal{A}_t(\hat{\xi}, \hat{\xi}) + \frac{1}{2} \left[ (\hat{b}\cdot \nabla \hat{\xi}) - (\hat{b}\cdot \nabla \hat{\xi}) \right] = (\hat{\eta}, \hat{\xi}) + 2 \int_0^t \mathcal{A}_s(\hat{\xi}(s), \hat{\xi}) \, ds
\]

\[- \int_0^t \int_0^s \mathcal{A}_{\tau \tau}(\hat{\xi}(\tau), \hat{\xi}) \, d\tau \, ds.
\]
Integrate the above equation with respect to time to arrive at
\[
\int_0^t \|\dot{\xi}(s)\|^2\,ds + \frac{1}{2} A(\dot{\xi}, \dot{\xi}) = \frac{1}{2} \int_0^t A_s(\dot{\xi}(s), \dot{\xi}(s))\,ds
\]
\[+ \frac{1}{2} \int_0^t \left[ (b \dot{\xi}(s), \nabla \dot{\xi}(s)) - (\dot{b}(s), \nabla \dot{\xi}(s)) \right] \,ds \]
\[+ \int_0^t (\dot{\eta}(s), \dot{\xi}(s)) + 2 \int_0^t \int_0^s A_r(\dot{\xi}(\tau), \dot{\xi}(s)) \,d\tau \,ds \]
\[- \int_0^t \int_0^s A_{rr}(\dot{\xi}(\tau), \dot{\xi}(s)) \,d\tau' \,d\tau \]
(3.17)
\[= \frac{1}{2} \int_0^t A_s(\dot{\xi}(s), \dot{\xi}(s)) + I_1 + \int_0^t (\dot{\eta}(s), \dot{\xi}(s)) + I_2.
\]
We now rewrite \(I_1\) and \(I_2\), as
\[I_1 = \frac{1}{2} (b \dot{\xi}, \nabla \dot{\xi}) - \int_0^t (b \dot{\xi}(s), \nabla \dot{\xi}(s)) \,ds - \frac{1}{2} \int_0^t (b_s \dot{\xi}(s), \nabla \dot{\xi}(s)) \,ds,
\]
and
\[I_2 = 2 \int_0^t A_s(\dot{\xi}(s), \dot{\xi}(s)) \,ds - 2 \int_0^t A_s(\dot{\xi}(s), \dot{\xi}(s)) \,ds - \int_0^t \int_0^s A_{rr}(\dot{\xi}(\tau), \dot{\xi}(s)) \,d\tau \,ds
\]
\[- \int_0^t \int_0^s A_{rr}(\dot{\xi}(\tau), \dot{\xi}(s)) \,d\tau' \,d\tau \,ds.
\]
Using (2.1) and (2.4), we obtain
\[
\frac{\rho_1}{2} \|\dot{\xi}\|^2_1 + \int_0^t \|\dot{\xi}(s)\|^2 \,ds \leq C \left( \|\dot{\xi}\| + \int_0^t \|\dot{\xi}(s)\|_1 \,ds + \int_0^t \int_0^s \|\dot{\xi}(\tau)\|_1 \,d\tau \,ds \right) \|\dot{\xi}\|_1
\]
\[+ C \int_0^t \left( \|\dot{\xi}(s)\|_1 + \|\dot{\eta}(s)\| \right) \|\dot{\xi}(s)\|_1 \,ds
\]
\[+ C \int_0^t \left( \|\dot{\xi}(s)\|_1 + \|\dot{\xi}(s)\| \right) \|\dot{\xi}(s)\|_1 \,ds
\]
\[+ C \int_0^t \left( \|\dot{\xi}(s)\|_1 \int_0^s \|\dot{\xi}(\tau)\|_1 \,d\tau \right) \,ds + \frac{\beta}{2} \|\dot{\xi}\|^2.
\]
Using Young’s inequality, we arrive at
\[
\rho_1 \|\dot{\xi}\|^2_1 + \int_0^t \|\dot{\xi}(s)\|^2 \,ds \leq C \left( \|\dot{\xi}\|^2 + \int_0^t \left( \|\dot{\xi}(s)\|^2_1 + \|\dot{\xi}(s)\|^2 \right) \,ds \right) + C \int_0^t \|\dot{\eta}\|^2 \,ds.
\]
Now use (3.14) and Lemma 3.2 to arrive at (3.15). This completes the rest of the proof. □

**Lemma 3.4** With \(\dot{\xi}\) satisfying (3.11), there exists a positive constant \(C\), independent of \(h\), such that
\[
\int_0^t \|\dot{\xi}\|^2 \,ds + 
\int_0^t \|\dot{\xi}(s)\|^2_1 \,ds \leq C t h^4 u_0^2.
\]
and
\[
\frac{1}{2} \int_0^t s^2 \|\dot{\xi}\|^2 \,ds \leq C t h^4 u_0^2.
\]

8
Proof. Choose $\phi_n = t\hat{\xi}(t)$ in (3.11) and rewrite the integral term to find that
\[
\frac{1}{2} \frac{d}{dt} \{t||\hat{\xi}|^2\} + \rho_1 t||\hat{\xi}|^2 \leq \frac{1}{2}||\hat{\xi}|^2 + t||\eta||\hat{\xi}|| + tA_t(\hat{\xi}, \hat{\xi}) - t \int_0^t A_{ss}(\hat{\xi}(s), \hat{\xi})ds + \beta t||\hat{\xi}|^2.
\]
Integrate with respect to time and use Cauchy-Schwarz inequality to obtain
\[
t||\hat{\xi}|^2 + \rho_1 \int_0^t s||\hat{\xi}(s)||^2 ds \leq \int_0^t (||\xi(s)||^2 + ||\hat{\xi}(s)||^2) ds + C \int_0^t (||\xi(s)||^2 + ||\hat{\xi}(s)||^2) ds.
\]
Use Lemma 3.1 and (3.15) to arrive at (3.18).
Set $\phi_n = t^2\xi(t)$ in (3.11) to obtain
\[
t^2||\xi||^2 + t^2A(\xi, \xi) - t^2 \int_0^t A_s(\xi(s), \xi)ds = t^2(\eta, \xi).
\]
Note that
\[
\frac{1}{2} \frac{d}{dt} \{t^2A(\xi, \xi)\} = tA(\xi, \xi) + \frac{t^2}{2}A_t(\xi, \xi) + t^2A(\xi, \xi) + \frac{t^2}{2} \left\{ \langle \hat{b}\xi, \nabla \hat{\xi} \rangle - \langle \hat{b}\xi, \nabla \xi \rangle \right\},
\]
and, hence,
\[
t^2||\xi||^2 + \frac{1}{2} \frac{d}{dt} \{t^2A(\xi, \xi)\} = t^2(\eta, \xi) + tA(\xi, \xi) + \frac{t^2}{2}A_t(\xi, \xi) + t^2 \left\{ \langle \hat{b}\xi, \nabla \hat{\xi} \rangle - \langle \hat{b}\xi, \nabla \xi \rangle \right\} + t^2 \int_0^t A_s(\xi(s), \xi)ds.
\]
Integrate with respect to time and use (2.1) to obtain
\[
\int_0^t s^2||\xi(s)||^2 ds + \frac{\rho_1}{2}t^2||\hat{\xi}||^2 \leq \int_0^t \left( s^2(\eta(s), \xi(s)) + sA(\xi(s), \xi(s)) + \frac{s^2}{2}A_s(\xi(s), \xi(s)) \right) ds
\]
\[
\quad + \int_0^t \frac{s^2}{2} \left\{ \langle \hat{b}\xi(s), \nabla \hat{\xi}(s) \rangle - \langle \hat{b}\xi(s), \nabla \xi(s) \rangle \right\} ds
\]
\[
\quad + \int_0^t s^2 \int_0^s A_t(\hat{\xi}(\tau), \xi(s))d\tau ds + \frac{\beta}{2}t^2||\hat{\xi}||^2
\]
\[
\quad = \int_0^t \left( s^2(\eta(s), \xi(s)) + sA(\xi(s), \xi(s)) + \frac{s^2}{2}A_s(\xi(s), \xi(s)) \right) ds
\]
\[
\quad + I_3 + I_4 + \frac{\beta}{2}t^2||\hat{\xi}||^2 \quad \text{(say)}.
\]
Use integration by parts to rewrite $I_3$ and $I_4$, respectively, as follows
\[
I_3 = -\frac{t^2}{2} \langle \hat{b}\xi, \nabla \hat{\xi} \rangle + \int_0^t s^2 \langle \hat{b}\xi(s), \nabla \xi(s) \rangle ds + \frac{1}{2} \int_0^t ((2s\hat{b} + s^2\hat{b}s)\xi(s), \nabla \xi(s)) ds,
\]
and
\[
I_4 = \int_0^t s^2 \int_0^s A_t(\hat{\xi}(\tau), \xi(s))d\tau ds
\]
\[
= t^2 \int_0^t A_s(\xi(s), \xi)ds - t^2 \int_0^t 2s \int_0^s A_t(\hat{\xi}(\tau), \xi(s))d\tau ds - \int_0^t s^2 A_s(\xi(s), \xi(s))ds
\]
\[
= t^2A_t(\hat{\xi}, \hat{\xi}) - t^2 \int_0^t A_{ss}(\hat{\xi}(s), \hat{\xi})ds - \int_0^t 2sA_s(\hat{\xi}(s), \hat{\xi}(s))ds
\]
\[
\quad + \int_0^t 2s \int_0^s A_{ss}(\hat{\xi}(\tau), \xi(s))d\tau ds - \int_0^t s^2 A_s(\xi(s), \xi(s))ds.
\]
Repeated use of (2.1) and (2.4) help us to arrive at
\[
\int_0^t s^2 \|\xi(s)\|^2 ds + \frac{\rho_1 t^2}{2} \|\dot{\xi}\|^2_1 \leq C t^2 \left( \|\dot{\xi}\| + \|\ddot{\xi}\|_1 + \int_0^t \|\dot{\xi}(s)\|_1 ds \right) \|\dot{\xi}\|_1 \\
+ C \int_0^t s^2 \left( \|\dot{\xi}(s)\|_1 + \|\eta(s)\|_1 \right) \|\xi(s)\| ds \\
+ C \int_0^t s \left( (1 + s) \|\dot{\xi}(s)\|^2_1 + s \|\dot{\xi}(s)\|^2 \right) ds \\
+ C \int_0^t s \left( (2 + s) \|\dot{\xi}(s)\| + \|\ddot{\xi}(s)\|_1 \right) \\
+ \int_0^s \|\ddot{\xi}(\tau)\|_1 d\tau \right) \|\dot{\xi}(s)\|_1 ds + \frac{\beta}{2} t^2 \|\dot{\xi}\|^2.
\]

Using Young’s inequality, we obtain
\[
\int_0^t s^2 \|\xi(s)\|^2 ds + \rho_1 t^2 \|\dot{\xi}\|^2_1 \leq C t^2 \left( \|\dot{\xi}\|^2 + \|\ddot{\xi}\|^2_1 + \int_0^t \|\dot{\xi}(s)\|^2_1 ds \right) \\
+ C \int_0^t s^2 \left( \|\dot{\xi}(s)\|^2 + \|\eta(s)\|^2 \right) ds \\
+ \int_0^t (1 + s^2) \|\dot{\xi}(s)\|^2 ds
\]

Now, use (3.14), (3.15), (3.18) and Lemma 3.1 to conclude (3.19). This completes the rest of the proof. \(\square\)

**Proof of the Theorem 3.1.**
We recall the error equation (3.7)
\[
(e_t, \phi_h) + A(e, \phi_h) = 0, \quad \forall \phi_h \in S_h, \quad t \in (0, T).
\]
Observing the fact that \(A(\eta, \phi_h) = 0\) and using (3.10), we obtain from (3.7)
\[(\xi_t, \phi_h) + A(\xi, \phi_h) = (\eta_t, \phi_h), \quad \forall \phi_h \in S_h,\]
Choose \(\phi_h = t^3 \xi(t)\) in (3.20) to find that
\[
\frac{1}{2} \frac{d}{dt}(t^3 \|\xi\|^2) + \rho_1 t^3 \|\xi\|^2_1 \leq \frac{3\rho^2}{2} \|\xi\|^2 + t\|\eta\|_1 \|\xi\| + \beta t^3 \|\xi\|^2.
\]
Integrate with respect to time to arrive at
\[
\frac{t^3}{2} \|\xi\|^2 + \rho_1 \int_0^t s^3 \|\xi(s)\|^2 ds \leq \frac{1}{2} \int_0^t s^4 \|\eta_h(s)\|^2 ds + C \int_0^t s^2 \|\xi(s)\|^2 ds.
\]
Since, from Lemma 3.1, we have
\[
\|\eta\| \leq Ch^2 t^{-2} \|u_0\|
\]
we, therefore, obtain
\[
\int_0^t s^4 \|\eta_h(s)\|^2 ds \leq Ch^4 t \|u_0\|^2.
\]
Using (3.19) and (3.22) in (3.21), we obtain
\[ t^3 \| \xi \|^2 + \int_0^t s^3 \| \xi(s) \|^2 ds \leq C t h^4 \| u_0 \|^2, \]
and hence,
\[ (3.23) \quad \| \xi \| \leq C h^2 t^{-1} \| u_0 \|. \]
Now, from Lemma 3.1 with \( k = 2 \) and \( j = 0 \) and (3.23), we conclude that
\[ \| e \| \leq C h^2 t^{-1} \| u_0 \|. \]
This completes the rest of the proof. \( \square \)

Remarks: (i) We can even obtain superconvergence in our case. Set \( \phi_h = t^4 \xi_t(t) \) in (3.20) to obtain
\[ t^4 \| \xi_t \|^2 + t^4 A(\xi, \xi_t) = t^4 (\eta_t, \xi_t). \]
Observe that
\[ \frac{1}{2} \frac{d}{dt} [t^4 A(\xi, \xi)] = 2 t^3 A(\xi, \xi) + \frac{t^4}{2} A_t(\xi, \xi) + t^4 A(\xi, \xi_t) + \frac{t^4}{2} ([\bar{b} \xi_t, \nabla \xi] - (\bar{b} \xi, \nabla \xi_t)]. \]
Therefore,
\[ t^4 \| \xi_t \|^2 + \frac{1}{2} \frac{d}{dt} [t^4 A(\xi, \xi)] = t^4 (\eta_t, \xi_t) + 2 t^3 A(\xi, \xi) + \frac{t^4}{2} A_t(\xi, \xi) + \frac{t^4}{2} ([\bar{b} \xi_t, \nabla \xi] - (\bar{b} \xi, \nabla \xi_t)]. \]
Now, we integrate the above equation and use integration by parts to rewrite the last term on the right hand side to arrive at the following
\[ \frac{a_0}{2} t^4 \| \xi_t \|^2 + \int_0^t s^4 \| \xi(s) \|^2 ds \leq \int_0^t s^4 \eta_s(s) \| \xi_s(s) \| ds + C \int_0^t s^3 (1 + s) \| \xi(s) \|^2 ds \]
\[ - \frac{t^4}{2} (\bar{b} \xi, \nabla \xi) + \int_0^t s^4 (\bar{b} \xi(s), \nabla \xi(s)) ds + \beta t^4 \| \xi \|^2. \]
Use Young’s inequality to obtain
\[ t^4 \| \xi \|^2 + \int_0^t s^4 \| \xi_s(s) \|^2 ds \leq C \int_0^t (s^4 \| \eta_s(s) \|^2 + s^3 \| \xi(s) \|^2) ds + C t^4 \| \xi \|^2. \]
We conclude, using above obtained estimates, that
\[ t^4 \| \xi \|^2 + \int_0^t s^4 \| \xi_s(s) \|^2 ds \leq C h^4 t \| u_0 \|^2, \]
and therefore,
\[ (3.24) \quad \| \xi \|_1 \leq C h^2 t^{-3/2} \| u_0 \|. \]

(ii) Assuming that the triangulation is quasi-uniform and \( d = 2 \), we have, from the subspace-Sobolev inequality for elements in \( S_h \),
\[ \| \chi \|_\infty \leq C |\log h|^{1/2} \| \chi \|_1, \quad \forall \chi \in S_h, \]
where, $\| \cdot \|_\infty$ denotes the $L^\infty$ norm. Now, from (3.24), we obtain
\[ \| \xi \|_\infty \leq C |\log h|^{1/2} \| \xi \|_1 \leq Ch^2 t^{-3/2} |\log h|^{1/2} \| u_0 \|. \]
Also we know, from [6], that, for $u \in W^2_\infty(\Omega) \cap H^1_0(\Omega)$ and $u_0 \in L_\infty(\Omega)$
\[ \| u - R_h u \|_\infty \leq Ch^2 t^{-1} |\log h|^{1/2} \| u_0 \|. \]
With above two estimates, we conclude that
\[ \| e \|_\infty \leq \| \eta \|_\infty + \| \xi \|_\infty \leq C h^2 \left( t^{-3/2} |\log h|^{1/2} \| u_0 \| + t^{-1} |\log h|^{1/2} \| u_0 \|_{L_\infty} \right). \]

(iii) Note that by simply modifying the arguments in Remark (i), we easily obtain
\[ \| u - u_h \|_{L^\infty(0, T; H^1_0(\Omega))} \leq C h^{-1} \| u_0 \|. \]

4 Error Estimates for Mixed Method

In this section, we consider the mixed formulation for the parabolic problem (1.1) and discuss optimal $L^2$-error estimates for the semidiscrete mixed Galerkin approximations to (1.1), when the initial data $u_0$ is in $L^2(\Omega)$. Let
\[ V = H(div, \Omega) = \{ \bar{\phi} \in (L^2(\Omega))^d : \nabla \cdot \bar{\phi} \in L^2(\Omega) \}, \]
be a Hilbert space equipped with norm $\| \bar{\phi} \|_V = (\| \bar{\phi} \|^2 + \| \nabla \cdot \bar{\phi} \|^2)^{1/2}$.

For the mixed formulation, we introduce
\[ \bar{\sigma}(t) = -(A \nabla u + \bar{b} u), \]
and set
\[ \alpha = A^{-1}, \quad \bar{\beta} = \alpha \bar{b}. \]

Then, we rewrite the equation (1.1) as
\[ (4.1) \quad u_t + \nabla \cdot \bar{\sigma} + cu = 0, \]
\[ (4.2) \quad \alpha \bar{\sigma}(t) + \nabla u + \bar{b} u = 0. \]

For our subsequent analysis, we assume, without loss of generality, that $c \geq c_1 > 0$, $\forall x \in \Omega$. When $c \geq 0$, the analysis can be easily modified to derive the desired results.

We now present the weak mixed formulation for (1.1). Find $(u, \bar{\sigma}) : (0, T] \to L^2(\Omega) \times H(div, \Omega)$ such that
\[ (4.4) \quad (u_t, w) + (\nabla \cdot \bar{\sigma}, w) + (c u, w) = 0 \quad \forall w \in L^2(\Omega), \]
\[ (4.5) \quad (\alpha \bar{\sigma}, \bar{v}) - (\nabla \cdot \bar{v}, u) + (\bar{\beta} u, \bar{v}) = 0 \quad \forall \bar{v} \in H(div, \Omega), \]
with $u(0) = u_0 \in L^2(\Omega)$. Since $A$ is uniformly positive definite, there exist two positive constants $a_0$ and $a_1$ such that
\[ (4.6) \quad a_0 \| \bar{\sigma} \| \leq \| \bar{\sigma} \|_{A^{-1}} \leq a_1 \| \bar{\sigma} \|, \quad \text{where } \| \bar{\sigma} \|^2_{A^{-1}} := (\alpha \bar{\sigma}, \bar{\sigma}). \]
For the mixed formulation, we consider $V_h$ and $W_h$ as finite element subspaces of $H(div, \Omega)$ and $L^2(\Omega)$, respectively (for a quick reference see [2]), satisfying

(i) $\nabla \cdot V_h \subset W_h$, and
(ii) there exists a linear operators $\Pi_h : V \rightarrow V_h$ such that $\nabla \cdot \Pi_h = P_h(\nabla \cdot )$,

where $P_h : W \rightarrow W_h$ is the $L^2$-projection defined by

$$(\phi - P_h \phi, w_h) = 0, \quad \forall \ w_h \in W_h, \quad \phi \in W.$$ 

Further, we assume that the finite element spaces satisfy the following approximation properties:

$$\| \sigma - \Pi_h \sigma \| \leq C h \| \nabla \cdot \sigma \|, \quad \| u - P_h u \| \leq C h^r \| u \|, \quad r = 1, 2.$$ 

From (4.1), we observe that

$$\| \nabla \cdot \sigma \| \leq C \| u \|_2,$$

and so

$$\| \sigma - \Pi_h \sigma \| \leq C h \| u \|_2, \quad \| u - P_h u \| \leq C h^r \| u \|_r, \quad r = 1, 2.$$ 

Also, we have

$$\langle \nabla \cdot (\sigma - \Pi_h \sigma), w_h \rangle = 0, \quad w_h \in W_h; \quad \langle u - P_h u, \nabla \cdot \bar{v}_h \rangle = 0, \quad \bar{v}_h \in V_h.$$ 

Now, we can define the corresponding semidiscrete mixed finite element approximation as a pair $(u_h, \sigma_h) : (0, T) \rightarrow W_h \times V_h$ such that

$$\begin{align*}
(u_{ht}, w_h) + (\nabla \cdot \sigma_h, w_h) + (\alpha u, w_h) = 0 & \quad \forall \ w_h \in W_h, \\
(\alpha \sigma_h, \bar{v}_h) - (\nabla \cdot \bar{v}_h, u_h) + (\beta u_h, \bar{v}_h) = 0 & \quad \forall \ \bar{v}_h \in V_h,
\end{align*}$$

with $u_h(0) = P_h u_0$.

Our main aim, in this section, is to accomplish the following theorem.

**Theorem 4.1** Let $(u, \sigma)$ and $(u_h, \sigma_h)$ satisfy (4.4)-(4.5) and (4.9)-(4.10), respectively, with $u(0) = u_0$ and $u_h(0) = P_h u_0$. Then

$$\| u - u_h \| \leq C h^2 t^{-1} \| u_0 \|.$$ 

We achieve the proof of the theorem through a series of Lemmas.

Define $e_u = u - u_h$ and $e_\sigma = \sigma - \sigma_h$. Then, from (4.4)-(4.5), (4.9)-(4.10), $e_u$ and $e_\sigma$ satisfy the following equations

$$\begin{align*}
(e_{ut}, w_h) + (\nabla \cdot e_\sigma, w_h) + (\alpha e_u, w_h) = 0 & \quad \forall \ w_h \in W_h, \\
(\alpha e_\sigma, \bar{v}_h) - (\nabla \cdot \bar{v}_h, e_u) + (\beta e_u, \bar{v}_h) = 0 & \quad \forall \ \bar{v}_h \in V_h.
\end{align*}$$

We introduce intermediate projections $\tilde{u}_h \in W_h$ and $\tilde{\sigma}_h \in V_h$ of $u$ and $\sigma$, respectively. For given $u$ and $\sigma$, let $\tilde{u}_h$ and $\tilde{\sigma}_h$ satisfy

$$\begin{align*}
(\nabla \cdot (\sigma - \tilde{\sigma}_h), w_h) + (c(u - \tilde{u}_h), w_h) = 0 & \quad w_h \in W_h, \\
(\alpha (\sigma - \tilde{\sigma}_h), \bar{v}_h) - (\nabla \cdot \bar{v}_h, u - \tilde{u}_h) = 0 & \quad \bar{v}_h \in V_h.
\end{align*}$$
For error estimates of $\bar{\sigma} - \tilde{\sigma}_h$ and $u - \bar{u}_h$, set
\[
\eta_u = u - \bar{u}_h = (u - P_h u) - (\bar{u}_h - P_h u) =: \theta_u - \zeta_u,
\]
\[
\eta_{\bar{\sigma}} = \bar{\sigma} - \tilde{\sigma}_h = (\bar{\sigma} - \Pi_h \bar{\sigma}) - (\tilde{\sigma}_h - \Pi_h \tilde{\sigma}) =: \theta_{\bar{\sigma}} - \zeta_{\bar{\sigma}}.
\]
Using (4.8), we arrive at
\[
(\nabla \cdot \zeta, w_h) + (c \zeta, w_h) = (c \theta_u, w_h), \quad w_h \in W_h,
\]
\[
(\alpha \zeta_{\bar{\sigma}}, \tilde{v}_h) - (\nabla \cdot \tilde{v}_h, \zeta_u) = (\alpha \theta_{\bar{\sigma}}, \tilde{v}_h), \quad \tilde{v}_h \in V_h.
\]

**Lemma 4.1** Let $\bar{u}_h$ and $\tilde{\sigma}_h$ be defined as in (4.13)-(4.14). Then, there exist a positive constant $C$, independent of $h$, such that
\[
\| \eta_u \| \leq C h^2 t^{-1} \| u_0 \|, \quad \| \eta_{\bar{\sigma}} \| \leq C h t^{-1} \| u_0 \|.
\]

**Proof.** Choose $w_h = \zeta_u$ and $\tilde{v}_h = \zeta_{\bar{\sigma}}$ in (4.15) and (4.16), respectively and add the two resulting equations to obtain
\[
c_1 \| \zeta_u \|^2 + a_0 \| \zeta_{\bar{\sigma}} \|^2 \leq C (\| \theta_u \| \| \zeta_u \| + \| \theta_{\bar{\sigma}} \| \| \zeta_{\bar{\sigma}} \|),
\]
on the basis of the assumption that $c$ is bounded below by some constant $c_1$ uniformly.
Now, use Young’s inequality and (4.7) to conclude that
\[
\| \zeta_u \|^2 + \| \zeta_{\bar{\sigma}} \|^2 \leq C (\| \theta_u \|^2 + \| \theta_{\bar{\sigma}} \|^2) \\
\leq C h^2 t^{-2} \| u_0 \|^2.
\]
Using triangle inequality, we obtain
\[
\| \eta_u \| + \| \eta_{\bar{\sigma}} \| \leq C h t^{-1} \| u_0 \|.
\]

For optimal order estimate of $\| \eta_u \|$, we consider the following auxiliary problem:
\[-\nabla \cdot (A \nabla \phi) + c \phi = \eta_u \quad \text{in } \Omega,
\]
\[\phi = 0, \quad \text{on } \partial \Omega.
\]
Set $p = -A \nabla \phi$ and rewrite the above problem in weak form as
\[
(\alpha p, \bar{v}) - (\phi, \nabla \cdot \bar{v}) = 0, \quad \bar{v} \in V,
\]
\[
(\nabla \cdot p, w) + (c \phi, w) = (\eta_u, w), \quad w \in L^2(\Omega).
\]
From the regularity results, we note that
\[
\| \phi \|_{2}, \| \nabla \cdot p \| \leq C \| \eta_u \|.
\]
Now, using (4.13)-(4.14), we find that
\[
\| \eta_u \|^2 = \langle \nabla \cdot p, \eta_u \rangle + (c \phi, \eta_u)
\]
\[
= \langle \nabla \cdot (p - \Pi_h p), \eta_u \rangle + (\nabla \cdot \Pi_h p, \eta_u) + (c(\phi - P_h \phi), \eta_u) + (c P_h \phi, \eta_u)
\]
\[
= \langle \nabla \cdot (p - \Pi_h p), \eta_u \rangle + (\alpha \eta_{\bar{\sigma}}, \Pi_h p) + (c (\phi - P_h \phi), \eta_u) - (\nabla \cdot \eta_{\bar{\sigma}}, P_h \phi)
\]
\[
= \langle \nabla \cdot (p - \Pi_h p), \eta_u \rangle - (\alpha \eta_{\bar{\sigma}}, p - \Pi_h p) + (c(\phi - P_h \phi), \eta_u) + (\phi - P_h \phi, \nabla \cdot \eta_{\bar{\sigma}})
\]
\[
= \langle \nabla \cdot (p - \Pi_h p), \eta_u \rangle - (\alpha \eta_{\bar{\sigma}}, p - \Pi_h p) + (c(\phi - P_h \phi), \eta_u) - (\nabla (\phi - P_h \phi), \eta_{\bar{\sigma}}).
\]
Using (4.18)-(4.19) and Lemma 2.1, we estimate each of the terms as follows:

\[(\nabla \cdot (p - \Pi_h p), \theta_a) \leq C h^2 \|\nabla \cdot p\|_2 \leq C h^2 t^{-1} \|u_0\| \|\eta_a\|,\]

\[(a_{\eta a}, p - \Pi_h p) \leq C h \|\nabla \cdot p\| \|\eta_a\| \leq C h^2 t^{-1} \|u_0\| \|\eta_a\|,\]

\[(c(\phi - P_h \phi), \eta_a) \leq C h^2 \|\phi\|_2 \|\eta_a\| \leq C h^2 t^{-1} \|u_0\| \|\eta_a\|,\]

\[\left(\nabla (\phi - P_h \phi), \eta_a\right) \leq C h^2 t^{-1} \|u_0\| \|\eta_a\|.\]

Combine all the above estimates, we complete the rest of the proof. \(\square\)

**Lemma 4.2** Let \(\tilde{u}_h\) and \(\tilde{\sigma}_h\) be defined as in (4.13)-(4.14). Then, there exist a positive constant \(C\), independent of \(h\), such that

\[
\|\tilde{\eta}_a\| = \|\hat{u} - \tilde{u}_h\| \leq C h^2 \|u_0\|, \quad \|\tilde{\eta}_\sigma\| = \|\hat{\sigma} - \tilde{\sigma}_h\| \leq C h \|u_0\|, \quad \|\tilde{\eta}_h\| = \|\hat{\eta} - \tilde{\sigma}_h\| \leq C h t \|u_0\|, \quad \|\tilde{\eta}_\sigma\|_{(H(\text{div,} \Omega))^*} \leq C h^2 \|u_0\|,
\]

where \((H(\text{div,} \Omega))^*\) is the dual of \(H(\text{div,} \Omega)\).

**Proof.** Integrate (4.15)-(4.16) and put \(w_h = \hat{\zeta}_u\) and \(\bar{v}_h = \hat{\zeta}_\sigma\) and then, add the two resulting equations to arrive at

\[
(c\hat{\zeta}_u, \hat{\zeta}_u) + (\alpha \hat{\zeta}_\sigma, \hat{\zeta}_\sigma) = (c\hat{\theta}_u, \hat{\zeta}_u) + (\alpha \hat{\theta}_\sigma, \hat{\zeta}_\sigma) + \int_0^t \left((c_s(s)\hat{\zeta}_u(s), \hat{\zeta}_u) - (c_s\hat{u}_u(s), \hat{\zeta}_u) + (\alpha_s(s)\hat{\zeta}_\sigma(s), \hat{\zeta}_\sigma) - (\alpha_s\hat{\theta}_\sigma(s), \hat{\zeta}_\sigma)\right) ds.
\]

Use Cauchy-Schwarz inequality to obtain

\[
c_1\|\hat{\zeta}_u\|^2 + a_0\|\hat{\zeta}_\sigma\|^2 \leq C \int_0^t \left(\|\hat{\zeta}_u(s)\|^2 + \|\hat{\theta}_u(s)\|^2 + \|\hat{\zeta}_\sigma(s)\|^2 + \|\hat{\theta}_\sigma(s)\|^2\right) ds + C \left(\|\hat{\theta}_u\|^2 + \|\hat{\theta}_\sigma\|^2\right).
\]

Note that

\[
\|\hat{\theta}_u\| \leq C h \|\nabla \cdot \hat{\sigma}\| \leq C h \|\hat{\eta}\|_2 \leq C h \|u_0\|,
\]

and

\[
\|\hat{\theta}_\sigma\| \leq C h \|\nabla \cdot \hat{\sigma}\| \leq C h \|\hat{\eta}\|_2 \leq C h \|u_0\|.
\]

Hence,

\[
c_1\|\hat{\zeta}_u\|^2 + a_0\|\hat{\zeta}_\sigma\|^2 \leq C h^2 \|u_0\|^2 + C(c_1, a_0) \int_0^t (c_1\|\hat{\zeta}_u(s)\|^2 + a_0\|\hat{\zeta}_\sigma(s)\|^2) \, ds
\]

Use Gronwall's Lemma to conclude that

\[
\|\hat{\zeta}_u\| + \|\hat{\zeta}_\sigma\| \leq C h \|u_0\|,
\]

and thus, using Lemma 2.1

(4.20) \[\|\tilde{\eta}_h\| \leq C h (\|\nabla \cdot \tilde{\sigma}\| + \|u_0\|) \leq C h \|u_0\|.
\]

15
For optimal order estimate of $\|\hat{\eta}_u\|$, we again consider the following auxiliary problem

$$-\nabla \cdot (A\nabla \phi) + c\phi = \hat{\eta}_u \quad \text{in } \Omega,$$

$$\phi = 0, \quad \text{on } \partial \Omega.$$

Set $p = -A\nabla \phi$ and rewrite the above problem in weak form as

$$\text{(4.21)} \quad (\alpha p, \bar{v}) - (\phi, \nabla \cdot \bar{v}) = 0, \quad \bar{v} \in V,$$

$$\text{(4.22)} \quad (\nabla \cdot p, w) + (c\phi, w) = (\hat{\eta}_u, w), \quad w \in L^2(\Omega).$$

From the regularity results, we note that

$$\|\phi\|_2, \|\nabla \cdot p\| \leq C\|\hat{\eta}_u\|.$$

Also, after integrating (4.13)-(4.14), we obtain

$$\begin{align*}
(\nabla \cdot \hat{\eta}_u, w_h) + (c\hat{\eta}_u, w_h) - \int_0^t (c_s(s)\hat{\eta}_u(s), w_h) \, ds &= 0, \quad w_h \in W_h, \\
(\alpha \hat{\eta}_u, \bar{v}_h) - \int_0^t (\alpha_s(s)\hat{\eta}_u(s), \bar{v}_h) \, ds - (\nabla \cdot \bar{v}_h, \hat{\eta}_u) &= 0, \quad \bar{v}_h \in V_h.
\end{align*}$$

Setting $w = \hat{\eta}_u$ in (4.22), we find that

$$\begin{align*}
\|\hat{\eta}_u\|^2 &= (\nabla \cdot p, \hat{\eta}_u) + (c\phi, \hat{\eta}_u) \\
&= (\nabla \cdot (p - \Pi_h p), \hat{\eta}_u) + (\alpha \hat{\eta}_u, \Pi_h p) - \int_0^t (\alpha_s(s)\hat{\eta}_u(s), \Pi_h p) \, ds + (c(\phi - P_h \phi), \hat{\eta}_u) \\
&- (\nabla \cdot \hat{\eta}_u, P_h \phi) + \int_0^t (c_s(s)\hat{\eta}_u(s), P_h \phi) \, ds \\
&= (\nabla \cdot (p - \Pi_h p), \hat{\eta}_u) - (\alpha \hat{\eta}_u, p - \Pi_h p) + \int_0^t (\alpha_s(s)\hat{\eta}_u(s), p - \Pi_h p) \, ds \\
&- \int_0^t (c_s(s)\hat{\eta}_u(s), \phi - P_h \phi, \hat{\eta}_u)) + (\phi - P_h \phi, \nabla \cdot \hat{\eta}_u) \\
&- \int_0^t (c_s(s)\hat{\eta}_u(s), \phi - P_h \phi) \, ds + \int_0^t (c_s(s)\hat{\eta}_u(s), \phi) \, ds \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 \quad \text{(say)}.
\end{align*}$$

As in the previous Lemma, we easily obtain the following estimates

$$J_1 + J_2 + J_3 + J_5 + J_6 + J_7 \leq C h^2 \|u_0\| \|\hat{\eta}_u\|,$$

$$J_8 \leq C \|\hat{\eta}_u\| \int_0^t \|\hat{\eta}_u(s)\| \, ds,$$

and

$$J_4 \leq C \|\hat{\eta}_u\| \int_0^t \|\hat{\eta}_u\|_{(H(div, \Omega))^*} \, ds.$$

On combining the estimates of $J_1 - J_8$ in (4.24), we find that

$$\|\hat{\eta}_u\| \leq C h^2 \|u_0\| + C \int_0^t (\|\hat{\eta}_u(s)\|_{(H(div, \Omega))^*} + \|\hat{\eta}_u(s)\|) \, ds.$$
Using Gronwall’s Lemma, we obtain

\begin{equation}
\| \dot{\eta}_u \| \leq \| \eta_0 \| + C \int_0^t \| \dot{\eta}_\theta(s) \|_{(H(\text{div}, \Omega))^*} \, ds.
\end{equation}

Observe that for \( w \in V \)

\[
(\alpha \dot{\eta}_\theta, w) = (\alpha \dot{\eta}_\theta, w - \Pi_h w) + (\alpha \dot{\eta}_\theta, \Pi_h w)
\]

\[
= (\alpha \dot{\eta}_\theta, w - \Pi_h w) + (\nabla \cdot \Pi_h w, \dot{\eta}_u) + \int_0^t (\alpha_s(s) \dot{\eta}_\theta(s), \Pi_h w) \, ds.
\]

and hence,

\[
\| \dot{\eta}_\theta \|_{(H(\text{div}, \Omega))^*} \leq \| \dot{\eta}_\theta \| + \| \eta_0 \| + C \int_0^t \| \dot{\eta}_\theta(s) \|_{(H(\text{div}, \Omega))^*} \, ds.
\]

From (4.20) and (4.25), we now arrive at

\[
\| \dot{\eta}_\theta \|_{(H(\text{div}, \Omega))^*} \leq \| \eta_0 \| + C \int_0^t \| \dot{\eta}_\theta(s) \|_{(H(\text{div}, \Omega))^*} \, ds.
\]

Again a use Gronwall’s Lemma yields

\[
\| \dot{\eta}_\theta \|_{(H(\text{div}, \Omega))^*} \leq \| \eta_0 \|.
\]

Now from (4.25), it follows that

\[
\| \dot{\eta}_u \| \leq \| \eta_0 \| + C \int_0^t \| \dot{\eta}_u(s) \| \, ds.
\]

The remaining estimates can be obtained in a similar fashion and this completes the rest

Next, set

\[
\xi_u := u_h - \tilde{u}_h, \quad \xi_\theta := \tilde{\sigma}_h - \bar{\sigma}_h.
\]

From (4.11)-(4.14), we note that, \( \xi_u \) and \( \xi_\theta \) satisfy the following equations

\begin{align}
(\xi_{u,t}, w_h) + (\nabla \cdot \xi_\theta, w_h) + (c \xi_u, w_h) &= (\eta_{u,t}, w_h), \\
(\alpha \xi_\theta, \bar{v}_h) - (\nabla \cdot \bar{v}_h, \xi_u) + (\beta \xi_u, \bar{v}_h) &= (\beta \eta_u, \bar{v}_h).
\end{align}

Integrate (4.26)-(4.27) with respect to time, once and twice, to obtain

\begin{align}
(\xi_u, w_h) + (\nabla \cdot \xi_\theta, w_h) + (c \xi_u, w_h) - \int_0^t (c_s \xi_u(s), w_h) \, ds &= (\eta_u, w_h), \\
(\alpha \xi_\theta, \bar{v}_h) - \int_0^t (\alpha \xi_\theta(s), \bar{v}_h) \, ds - (\nabla \cdot \bar{v}_h, \xi_u) + (\beta \xi_u, \bar{v}_h) - \int_0^t (\beta_s \xi_u(s), \bar{v}_h) \, ds &= (\beta \eta_u, \bar{v}_h) - \int_0^t (\beta_s(s) \eta_u(s), \bar{v}_h) \, ds,
\end{align}

\[
(\alpha \xi_\theta, \bar{v}_h) - \int_0^t (\alpha \xi_\theta(s), \bar{v}_h) \, ds - (\nabla \cdot \bar{v}_h, \xi_u) + (\beta \xi_u, \bar{v}_h) - \int_0^t (\beta_s \xi_u(s), \bar{v}_h) \, ds
\]

\[
= (\beta \eta_u, \bar{v}_h) - \int_0^t (\beta_s(s) \eta_u(s), \bar{v}_h) \, ds,
\]

17
and

\begin{align*}
(\hat{\xi}_u, w_h) &+ (\nabla \cdot \hat{\xi}_\sigma, w_h) + (c\hat{\xi}_u, w_h) - 2 \int_0^t (c_s\hat{\xi}_u(s), w_h) ds \\
&+ \int_0^t \int_0^s (c_{\tau\tau}\hat{\xi}_u(\tau), w_h) d\tau \, ds = (\hat{\eta}_u, w_h), \\
(\alpha\hat{\xi}_\sigma, \bar{v}_h) &- 2 \int_0^t (\alpha_s\hat{\xi}_\sigma(s), \bar{v}_h) ds + \int_0^t \int_0^s (\alpha_{\tau\tau}\hat{\xi}_\sigma(\tau), \bar{v}_h) d\tau \, ds - (\nabla \cdot \bar{v}_h, \hat{\xi}_u) \\
&+ (\beta\hat{\xi}_u, \bar{v}_h) - 2 \int_0^t (\beta_s\hat{\xi}_u(s), \bar{v}_h) ds + \int_0^t \int_0^s (\beta_{\tau\tau}\hat{\xi}_u(\tau), \bar{v}_h) d\tau \, ds \\
(\hat{\xi}_u, w_h) &- 2 \int_0^t (\hat{\xi}_u(s)\hat{\eta}_u(s), \bar{v}_h) ds + \int_0^t \int_0^s (\beta_{\tau\tau}\hat{\eta}_u(s), \bar{v}_h) d\tau \, ds.
\end{align*}

(4.30)

We present below, a series of Lemmas that will establish the required estimate for $\xi_u$.

**Lemma 4.3** Let $\hat{\xi}_u$, $\hat{\xi}_u$, and $\hat{\xi}_\sigma$ satisfy (4.28), (4.29) and (4.31), respectively. Then, there exists positive constants $C$, independent of $h$, such that

\begin{align*}
\|\hat{\xi}_u\|^2 + \int_0^t \|\hat{\xi}_u(s)\|^2 ds &\leq Cth^4\|u_0\|^2, \\
\|\hat{\xi}_\sigma\|^2 + \int_0^t \|\hat{\xi}_\sigma(s)\|^2 ds &\leq Cth^4\|u_0\|^2.
\end{align*}

(4.32)

(4.33)

**Proof.** Choose $w_h = \hat{\xi}_u(t)$ and $\bar{v}_h = \hat{\xi}_\sigma(t)$ in the equations (4.30) and (4.31), respectively. Then, add the resulting equations to obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt}\|\hat{\xi}_u\|^2 + a_0\|\hat{\xi}_\sigma\|^2 &\leq (\hat{\eta}_u, \hat{\xi}_u) - (c\hat{\xi}_u, \hat{\xi}_u) + 2 \int_0^t (c_s\hat{\xi}_u(s), \hat{\xi}_u) ds \\
&- \int_0^t \int_0^s (c_{\tau\tau}\hat{\xi}_u(\tau), \hat{\xi}_u) d\tau \, ds + 2 \int_0^t (\alpha_s\hat{\xi}_\sigma(s), \hat{\xi}_\sigma) ds - \int_0^t \int_0^s (\alpha_{\tau\tau}\hat{\xi}_\sigma(\tau), \hat{\xi}_\sigma) d\tau \, ds \\
&- (\beta\hat{\xi}_u, \hat{\xi}_\sigma) + 2 \int_0^t (\beta_s\hat{\xi}_u(s), \hat{\xi}_\sigma) ds - \int_0^t \int_0^s (\beta_{\tau\tau}\hat{\xi}_u(\tau), \hat{\xi}_\sigma) d\tau \, ds + (\beta\hat{\eta}_u, \hat{\xi}_\sigma) \\
&- 2 \int_0^t (\hat{\xi}_u(s)\hat{\eta}_u(s), \hat{\xi}_\sigma) ds + \int_0^t \int_0^s (\beta_{\tau\tau}\hat{\eta}_u(s), \hat{\xi}_\sigma) d\tau \, ds.
\end{align*}

Use Young’s inequality and kickback arguments to arrive at

\begin{align*}
\frac{d}{dt}\|\hat{\xi}_u\|^2 + a_0\|\hat{\xi}_\sigma\|^2 &\leq C(\|\hat{\eta}_u\|^2 + \|\hat{\eta}_u\|^2 + \|\hat{\xi}_u\|^2) + C \int_0^t (\|\xi_u(s)\|^2 + \|\hat{\xi}_u(s)\|^2 + \|\hat{\xi}_\sigma(s)\|^2 + \|\hat{\eta}_u(s)\|^2) \, ds.
\end{align*}

Integrate with respect to time and then use Lemma 4.2 to obtain

\begin{align*}
\|\hat{\xi}_u\|^2 + a_0 \int_0^t \|\hat{\xi}_u(s)\|^2 ds &\leq Cth^4\|u_0\|^2 + C \int_0^t (\|\xi_u(s)\|^2 + \|\hat{\xi}_u(s)\|^2 + \|\hat{\xi}_\sigma(s)\|^2) \, ds.
\end{align*}
Use Gronwall’s Lemma to conclude (4.32).

Next, set \( w_h = \hat{\xi}_u(t) \) and \( \bar{v}_h = \hat{\xi}_\sigma(t) \) in (4.30) and (4.29), respectively, and add them to obtain

\[
\|\hat{\xi}_u\|^2 + \frac{1}{2} \frac{d}{dt}(\alpha \hat{\xi}_\sigma, \hat{\xi}_\sigma) = (\hat{\eta}_u, \hat{\xi}_u) - (c \hat{\xi}_u, \hat{\xi}_u) + 2 \int_0^t (c_s \hat{\xi}_u(s), \hat{\xi}_u) ds \\
- \int_0^t \int_0^s (c_{\tau\tau} \hat{\xi}_u(\tau), \hat{\xi}_u) d\tau ds + \frac{1}{2} (\alpha \hat{\xi}_\sigma, \hat{\xi}_\sigma) + \int_0^t (\alpha_s \hat{\xi}_\sigma(s), \hat{\xi}_\sigma) ds \\
- (\beta \hat{\xi}_u, \hat{\xi}_\sigma) + \int_0^t (\beta_s \hat{\xi}_u(s), \hat{\xi}_\sigma) ds + (\beta \hat{\eta}_u, \hat{\xi}_\sigma) - \int_0^t (\beta_s(s) \hat{\eta}_u(s), \hat{\xi}_\sigma) ds.
\]  

(4.34)

We use integration by parts to rewrite the sixth term in the right hand side of (4.34) to obtain

\[
\int_0^t (\alpha_s \hat{\xi}_\sigma(s), \hat{\xi}_\sigma) ds = (\alpha \hat{\xi}_\sigma, \hat{\xi}_\sigma) - \int_0^t (\alpha_s \hat{\xi}_\sigma(s), \hat{\xi}_\sigma) ds.
\]  

(4.35)

Substitute (4.35) in (4.34). Then use Cauchy-Schwarz and Young’s inequalities to arrive at

\[
\|\hat{\xi}_u\|^2 + \frac{d}{dt}(\alpha \hat{\xi}_\sigma, \hat{\xi}_\sigma) \leq C \left( \|\hat{\eta}_u\|^2 + \|\hat{\xi}_u\|^2 + \|\hat{\xi}_\sigma\|^2 + \int_0^t \|\hat{\xi}_u(s)\|^2 + \|\hat{\xi}_u(s)\|^2 + \|\hat{\eta}_u(s)\|^2 \right) ds.
\]

Integrate, use (4.32) and Lemma 4.2 to obtain

\[
a_0 \|\hat{\xi}_\sigma\|^2 + \int_0^t \|\hat{\xi}_u(s)\|^2 ds \leq C t h^4 \|u_0\|^2 + C(a_0) \int_0^t (a_0 \|\hat{\xi}_\sigma(s)\|^2 + \int_0^s \|\hat{\xi}_u(\tau)\|^2 d\tau) ds.
\]

We use Gronwall’s Lemma to conclude (4.33). This completes the rest of the proof. \(\Box\)

**Lemma 4.4** Let \( \xi_u, \hat{\xi}_u \) and \( \xi_\sigma, \hat{\xi}_\sigma \) satisfy (4.26)-(4.27) and (4.28)-(4.29). Then there exists positive constants \( C, \) independent of \( h, \) such that

\[
t \|\hat{\xi}_u\|^2 + \int_0^2 s \|\hat{\xi}_\sigma(s)\|^2 ds \leq C t h^4 \|u_0\|^2,
\]  

(4.36)

and

\[
t^2 \|\hat{\xi}_\sigma\|^2 + \int_0^2 s^2 \|\xi_u(s)\|^2 ds \leq C t h^4 \|u_0\|^2.
\]  

(4.37)

**Proof.** Choose \( w_h = t\hat{\xi}_u(t) \) and \( \bar{v}_h = t\hat{\xi}_\sigma(t) \) in (4.28) and (4.29) respectively, and add them to obtain

\[
\frac{1}{2} \frac{d}{dt}(t \|\hat{\xi}_u\|^2) + a_0 t \|\hat{\xi}_\sigma\|^2 \leq t(\eta_u, \hat{\xi}_u) - \frac{1}{2} \|\hat{\xi}_u\|^2 - t(c \hat{\xi}_u, \hat{\xi}_u) + t \int_0^t (c_s \hat{\xi}_u(s), \hat{\xi}_u) ds \\
+ t \int_0^t (\alpha_s \hat{\xi}_\sigma(s), \hat{\xi}_\sigma) ds - t(b \hat{\xi}_u, \hat{\xi}_\sigma) + t \int_0^t (b_s \hat{\xi}_u(s), \hat{\xi}_\sigma) ds \\
+ t(\beta \hat{\eta}_u, \hat{\xi}_\sigma) - t \int_0^t (\beta_s(s) \hat{\eta}_u(s), \hat{\xi}_\sigma) ds.
\]  

(4.38)
To estimate the fifth term on the right hand side of (4.38), we use integration by parts to rewrite it as

\begin{equation}
(4.39) \quad t \int_0^t \langle \alpha_s \xi_h(s), \xi_h \rangle \, ds = t\langle \alpha_t \xi_h, \xi_h \rangle - t \int_0^t \langle \alpha_s \xi_h(s), \xi_h \rangle \, ds.
\end{equation}

On substituting (4.39) in (4.38), we use Young’s inequality to arrive at

\begin{equation}
\frac{d}{dt} (t \| \xi_h \|^2) + a_t t \| \xi_h \|^2 \leq t^2 \| \eta_a \|^2 + \| \xi_h \|^2 + Ct \left( \| \dot{\xi}_h \|^2 + \| \xi_h \|^2 + \| \eta_a \|^2 \right)
+ \int_0^t \left( \| \dot{\xi}_h(s) \|^2 + \| \xi(s) \|^2 + \| \eta_a(s) \|^2 \right) \, ds.
\end{equation}

Integrate and use (4.33) and Lemmas 4.2 and 4.1 to conclude (4.36). Next, put \( w_h = t^2 \xi_h(t) \) and \( \dot{v}_h = t^2 \dot{\xi}_h(t) \) in (4.28) and (4.27), respectively, and add them to obtain

\begin{equation}
(4.39) \quad t^2 \| \xi_h \|^2 + \frac{1}{2} \frac{d}{dt} [t^2 (\alpha \xi_h, \xi_h)] = t \langle \alpha \xi_h, \xi_h \rangle + t^2 (\eta_a, \xi_h) - t^2 (c \xi_h, \xi_h) + t^2 \int_0^t (c \xi_h(s), \xi_h) \, ds
+ t^2 (\beta \eta_a, \dot{\xi}_h) + t^2 (\beta \xi_h, \dot{\xi}_h).
\end{equation}

Using Young’s inequality and kickback argument, we arrive at

\begin{equation}
(4.39) \quad t^2 \| \xi_h \|^2 + \frac{d}{dt} [t^2 (\alpha \xi_h, \xi_h)] \leq Ct^2 \left( \| \eta_a \|^2 + \| \xi_h \|^2 + \int_0^t \| \dot{\xi}_h(s) \|^2 \, ds \right) + Ct \| \dot{\xi}_h \|^2.
\end{equation}

Integrate and use (4.36) and Lemma 4.1 to obtain (4.37). This completes the rest of the proof.

\begin{proof}
Proof of the Theorem 4.1.
\end{proof}

We choose \( w_h = t^3 \xi_h(t) \) and \( \dot{v}_h = t^3 \dot{\xi}_h(t) \) in (4.26) and (4.27) respectively, and add them to obtain

\begin{equation}
\frac{d}{dt} (t^3 \| \xi_h \|^2) + a_t t^3 \| \xi_h \|^2 \leq t^4 \| \eta_a,t \|^2 + 4t^2 \| \xi_h \|^2 + Ct^3 \left( \| \eta_a \|^2 + \| \xi_h \|^2 \right).
\end{equation}

Integrate and use (4.37) to have

\begin{equation}
(4.40) \quad t^3 \| \xi_h \|^2 + \int_0^2 s^3 \| \xi_h(s) \|^2 \, ds \leq Cth^4 \| u_0 \|^2,
\end{equation}

and, therefore,

\begin{equation}
\| \xi_h \| \leq Ch^{2t^{-1}} \| u_0 \|.
\end{equation}

Now, from Lemma 4.1, we conclude that

\begin{equation}
\| e_u(t) \| \leq Ch^{2t^{-1}} \| u_0 \|.
\end{equation}

\begin{proof}
Remarks: (i) We can also obtain a superconvergence result for \( \| \xi_h \| \). For this, we differentiate the equation (4.27) to arrive at, for \( \dot{v}_h \in V_h \)

\begin{equation}
(\alpha_t \xi_h, \dot{v}_h) + (\alpha \xi_h,t, \dot{v}_h) + (\nabla \cdot \dot{v}_h, \xi_h,t) - (\beta \xi_h, \dot{v}_h) - (\beta \xi_h,t, \dot{v}_h)
= (\beta \eta_a, \dot{v}_h) + (\beta \eta_a,t, \dot{v}_h).
\end{equation}

\end{proof}
We choose \( w_h = t^4 \xi_{u,t} \) and \( \bar{v}_h = t^4 \xi_{\bar{\sigma}} \) in (4.26) and (4.41), respectively and add them to obtain

\[
\begin{align*}
t^4 \| \xi_{u,t} \|^2 + \frac{1}{2} \frac{d}{dt} [t^4 (\alpha \xi_{\sigma}, \xi_{\sigma})] &= t^4 (\eta_{u,t}, \xi_{u,t}) - t^4 (c \xi_{u}, \xi_{u,t}) + 2t^3 (\alpha \xi_{\bar{\sigma}}, \xi_{\bar{\sigma}}) \\
& - \frac{t^4}{2} (\alpha \xi_{\sigma}, \xi_{\sigma}) + t^4 (\bar{\beta} t \xi_{u}, \xi_{\sigma}) + t^4 (\bar{\beta} \xi_{u,t}, \xi_{\sigma}) \\
& - t^4 (\bar{\beta} t \eta_{u}, \xi_{\sigma}) - t^4 (\bar{\beta} \eta_{u,t}, \xi_{\sigma}).
\end{align*}
\]

Use Young’s inequality and simplify to arrive at the following

\[
\begin{align*}
t^4 \| \xi_{u,t} \|^2 + \frac{d}{dt} [t^4 (\alpha \xi_{\bar{\sigma}}, \xi_{\bar{\sigma}})] &\leq C t^4 (\eta_{u})^2 + \| \eta_{u,t} \|^2 + \| \xi_{u} \|^2 + \| \xi_{\bar{\sigma}} \|^2) + C t^3 \| \xi_{\bar{\sigma}} \|^2.
\end{align*}
\]

Integrate and use (4.40) to conclude

\[
\begin{align*}
t^4 \| \xi_{\bar{\sigma}} \|^2 + \int_0^t s^4 \| \xi_{u,s} (s) \|^2 &\leq C t h^4 \| u_0 \|^2,
\end{align*}
\]

and, therefore,

\[
(4.42) \quad \| \xi_{\bar{\sigma}} \| \leq C h^2 t^{-3/2} \| u_0 \|.
\]

\((ii)\) When \( d = 2 \), under the assumption of quasiuniform triangulation, we obtain from [5] (Remark 2 on page 358)

\[
\| \xi_{u} \|_{\infty} \leq C |\log h| \| \xi_{\bar{\sigma}} \| \leq C h^2 t^{-3/2} |\log h| \| u_0 \|.
\]

And, from [6], we now find that

\[
\| \eta_{u} \| \leq \| \theta_{u} \|_{\infty} + \| \xi_{u} \|_{\infty} \leq C h^2 t^{-1} |\log h| \| u_0 \|_{\infty} + C h^2 t^{-1} |\log h|^2 \| u_0 \|_{\infty} \leq C h^2 t^{-1} |\log h|^2 \| u_0 \|_{\infty}.
\]

Therefore, we conclude that

\[
\| \epsilon_{u} \| \leq C h^2 |\log h| \left\{ t^{-3/2} \| u_0 \| + t^{-1} |\log h| \| u_0 \|_{\infty} \right\}.
\]

\((iii)\) After making appropriate modification in Remark \((i)\), it is possible to obtain

\[
\| (\bar{\sigma} - \bar{\sigma}_h) (t) \| \leq C h^{-1} \| u_0 \|,
\]

which is optimal.

\((iv)\) Note that, we have proved the result in Theorem 4.1 when \( c \geq c_1 > 0 \). However, for \( c \geq 0 \), the result of Theorem 4.1 holds under appropriate modifications in Lemmas 4.1-4.2.

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<thead>
<tr>
<th>Date</th>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>26/09</td>
<td>Functional differential equations arising in cell-growth</td>
<td>Wake Begg</td>
</tr>
<tr>
<td>27/09</td>
<td>A Cell Growth Model Revisited</td>
<td>Derfel van Brunt Wake</td>
</tr>
<tr>
<td>28/09</td>
<td>Quasi-steady state reduction of molecular motor-based models of</td>
<td>Newby</td>
</tr>
<tr>
<td></td>
<td>directed intermittent search</td>
<td>Bressloff</td>
</tr>
<tr>
<td>29/09</td>
<td>All-at-once preconditioning in PDE-constrained optimization</td>
<td>Rees Stoll Wathen</td>
</tr>
<tr>
<td>30/09</td>
<td>An hp-Local Discontinuous Galerkin method for Parabolic Integro-Differential Equations</td>
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<td>Bressloff</td>
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