Optimal Selling Strategy With Piecewise Linear Drift Function

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Abstract

In this paper the optimal decision to sell a stock in a given time is investigated when the drift term in Black Scholes setting is a piecewise linear function of time. The goal is to minimize the expected relative error between the discounted selling price and the discounted maximum price over a given time horizen. With the drift changing to a piecewise linear function, we are interested in that if the trend of the stock price changes during the same time horizen, and what would be the impact on the selling strategy.

Keywords: piecewise linear drift function, optimal stopping time, value function, selling region, holding region

1 Introduction

In a standard Black Sholes setting, we have a stock with an appreciation rate a and a volatility rate $\sigma > 0$. The risk free interest rate is r. The stock price follows the process

$$dP_t = (a - r)P_t dt + \sigma P_t dB_t, P_0 = 1 \tag{1}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion with $B_0 = 0$ under P.We assume $\mathcal{F}_t = \sigma(B_s, s \leq t)$. From (1) we can get

$$P_t = e^{\lambda t + \sigma B_t}, \tag{2}$$

where $\lambda = a - r - (\sigma^2/2)$ is a constant. Shiryaev, Xu and Zhou(2008) have researched to minimize the expected relative error between the selling price and the maximum price over the horizon under this setting with a wonderful Bang-bang result. With the running maximum price process

$$M_t = max_{0 \le s \le t} P_s, t \ge 0 \tag{3}$$

they optimized the following problem to sell a stock in a given period $[0, T]$

$$\min E[\frac{M_T - P_\tau}{M_T}] \tag{4}$$

which means that the investor wants to minimize the expected relative error between the discounted selling price and the discounted maximum price over $\tau \in \mathcal{T}$, the set of all$\mathcal{F}_t$-stopping time $\tau \in [0, T]$. They define a goodness index of stock $\alpha = \frac{a-r}{\sigma^2}$;and show that one should hold on to the stock until the end if the stock goodness index is no less than 0.5, while one should sell immediately if the index is no greater than 0.

In this paper we want to change the assumption of λ. We consider the case that there could be sudden changes in the stock price trend. For example, under some big event announcement such as the bankruptcy of the Lehman Brothers, the stock price will have a slower uprising trend or will change to a negative trend in the rest time horizon. We investigate the optimal selling strategy under the above framework but with λ changing to a piecewise linear function $\mu(t)$

$$\mu(t) = \begin{cases} at, & t \le T_1 \\ aT_1 + b(t - T_1), & t > T_1 \end{cases} \tag{5}$$

Notice that $\mu(t)$ is a continuous function. We can also assume $\mu(t)$ to have a jump at some time point in the time horizon. To simplify the following numerical calculation, we here assume $\mu(t)$ is continuous.

The rest of the paper is organized as follows. In the next section, we formulate the problem (4) as an optimal stopping problem. In section 3, we construct complementarity formulation to get the value function. In section 4, we do numerical computation to get the selling region and holding region.

2 Optimal Stopping Problem

In this section we want to optimize problem (4) with our new assumption of $\mu(t)$.We have the simplicity assumption that $\sigma = 1$. In other words,we do not consider the stochastic volatility or volatility change during the time horizon. We have the following definitions based on (5)

$$B_t^\mu := \begin{cases} at + B_t, & t \le T_1 \\ aT_1 + b(t - T_1) + B_t, & t > T_1 \end{cases} \tag{6}$$

$$S_t^\mu := \max_{0 \le s \le t} B_s^\mu \tag{7}$$

Hence problem (4) is equivalent to the following stopping time problem

$$\sup_{\tau \in \mathcal{T}} E\left[\frac{e^{B_\tau^\mu}}{e^{S^\mu_T}}\right] \tag{8}$$

From our definitions (6) and (7) we can calculate

$$\begin{aligned} E\left[\frac{e^{B_\tau^\mu}}{e^{S_T^\mu}}\right] &= E\left[e^{B_\tau^\mu} / \max\left\{e^{S_\tau^\mu}, e^{\max_{\tau \le t \le T} B_t^\mu}\right\}\right] \\ &= E\left[\min\{e^{-(S_\tau^\mu - B_\tau^\mu)}, e^{-\max_{\tau \le t \le T}(B_t^\mu - B_\tau^\mu)}\}\right] \\ &= E[E\left[e^{S_\tau^\mu - B_\tau^\mu} \wedge e^{-\max_{\tau \le s \le T}((\mu(s)+B_s)-(\mu(\tau)+B_\tau))} | \mathcal{F}_\tau\right]] \\ &= E[\tilde{E}\left[\min(e^{-x}, e^{-\max_{0 \le s \le T-t}(\mu(s+t)-\mu(t)+\tilde{B}_s)})\right] | x = S_\tau^\mu - B_\tau^\mu] \\ &= E[G(\tau, x) | x = S_\tau^\mu - B_\tau^\mu] \end{aligned}$$

Here $\tilde{B}$ is a new Brownian motion with $\tilde{B}_s = B_{s+\tau} - B_\tau$ and $\tilde{B}_0 = 0$ under a new probability measure $\tilde{P}$. $\mu(t)$ is a function of time as in (5) rather than a constant.We define the G function as below

$$G(t, x) = \tilde{E}\left[\min(e^{-x}, e^{-\max_{0 \le s \le T-t}(\mu(s+t)-\mu(t)+\tilde{B}_s)})\right], (t, x) \in [0, T] \times [0, \infty) \tag{9}$$

Hence

$$G(t, x) = \int_x^\infty e^{-z} P(\max_{0 \le s \le T-t}(\mu(s+t) - \mu(t) + \tilde{B}_s) \le z) dz \tag{10}$$

$$\begin{aligned} P(\max_{0 \le s \le T-t}(\mu(s+t) - \mu(t) + \tilde{B}_s) \le z) = &\int_0^z \int_{-\infty}^m (\Phi\left[\frac{z - w - b(T - T_1)}{\sqrt{T - T_1}}\right] \\ &- e^{2b(z-w)} \Phi\left[\frac{-(z - w) - b(T - T_1)}{\sqrt{T - T_1}}\right]) \frac{2(2m - w)}{(T_1 - t)\sqrt{2\pi(T_1 - t)}} e^{-\frac{(2m-w)^2}{2(T_1-t)}} dwdm \end{aligned}$$

The calculation of $P(\max_{0\le s\le T-t}(\mu(s+t)-\mu(t)+\tilde{B}_s)\le z)$ is put in appendix A. Also from (11) we can see that $G_x(t,x) = -e^{-x}P(S^{\mu}_{T-t}\le x)$ hence

$$G_x(t, 0+) = 0 \tag{11}$$

The optimization problem (8) is now equivalent to

$$\sup_{\tau\in\mathcal{T}} E[G(\tau, X_\tau)] \tag{12}$$

This is similar to an American option with terminal payoff G and an underlying state process

$$X_t = S^{\mu}_t - B^{\mu}_t, X_0 = 0 \tag{13}$$

When the $\mu(t)$ a piecewise linear function, the theory of optimal stopping and the dynamic programming approach still apply. We introduce the same value function as when λ is a constant

$$\begin{aligned} V(t,x) &= \sup_{\tau\in\mathcal{T}_{T-t}} E_{t,x}[G(t+\tau, X_{t+\tau})] && (14)\\ &= \sup_{0\le\tau\le T-t} E[G(t+\tau, X^x_\tau)] && (15)\end{aligned}$$

where X^x_t under P is explicitly given as

$$X^x_t = x \vee S^{\mu}_t - B^{\mu}_t, t\ge 0 \tag{16}$$

We would like to find the two sets

$$C = \{(t,x)\in[0,T)\times[0,\infty) : V(t,x) > G(t,x)\} \tag{17}$$

and

$$D = \{(t,x)\in[0,T]\times[0,\infty) : V(t,x) = G(t,x)\} \tag{18}$$

where C is the area of continuation of observation or holding region and D is the stopping area or selling region.

3 Linear Complementarity Formulation

First with X_t^x is defined as (16) above, we still have

$$X_\cdot^x \overset{law}{=} |Y_\cdot| \tag{19}$$

where Y. is the unique strong solution to the SDE

$$\begin{cases} dY_t = -asign(Y_t)dt + d\tilde{B}_t, t \leq T_1 \\ dY_t = -bsign(Y_t)dt + d\tilde{B}_t, t > T_1 \\ Y_0 = x \end{cases} \tag{20}$$

where $\tilde{B}$ is a standard Brownian motion. The proof is in Appendix B. Hence according to *Itô* Tanaka formula

$$\begin{cases} dX_t = -a1_{(Y_t \neq 0)}dt + sign(Y_t)d\tilde{B}_t + dL_0^t, t \leq T_1 \\ dX_t = -b1_{(Y_t \neq 0)}dt + sign(Y_t)d\tilde{B}_t + dL_0^t, t > T_1 \end{cases} \tag{21}$$

Thus we can set up the following linear complementarity formulation to solve the value funtion.

$$\begin{cases} LV = V_t + \frac{1}{2}V_{xx} - aV_x, t \leq T_1 \\ LV = V_t + \frac{1}{2}V_{xx} - bV_x, t > T_1 \\ \min\left(-LV(t,x), V(t,x) - G(t,x)\right) = 0 \\ V(T,x) = G(T,x) \\ V_x(t,0+) = G_x(t,0+) = 0 \\ (t,x) \in [0,T] \times [0,\infty) \end{cases} \tag{22}$$

4 Numerical Computation

In this section we use the explicit Euler scheme of finite difference methods to calculate value function $V(t,x)$.

And use Monte Carlo method to calculate function $G(t,x)$.

First, according to the definition of $G(t,x)$ in (9) above, $G(t,x) \geq 0$ hence $V(t,x) \geq 0$. When x is large, $G(t,x)$ is close to zero. The maximum of x is set to be 7 since $e^{-7} = 0.00091$, making the error to be less than 0.0001.

Further we set $T = 1$,$T_1 = 0.5$, $\Delta t = \frac{1}{35}$ or time steps equal 35, and $\Delta x = \frac{7}{40}$ or x steps equal 40, thus satisfying the convergence condition of $\Delta t/(\Delta x)^2 < 1$.

We get for every t step and x step the value of $V(t,x)$ and $G(t,x)$. Finally we get $V(t,x) - G(t,x)$ for every t and x step; hence we can decide the selling region D and holding region C from (17) and (18) above.

The results are for different cases of a and b:

4.1 $a \leq b$

We selected two cases:(1) $a = 0.2, b = 0.6$ and (2) $a = -0.2, b = 0.6$.

In the graphs below, the x axis is x, which equals $S_t^\mu - B_t^\mu$. The y axis is time t. The blue area stands for the selling region or $V(t,x) = G(t,x)$, while the grey area stands for the holding region or $V(t,x) > G(t,x)$. The line between the two is the transition point when $V(t,x) = G(t,x)$.

We can see the optimal selling time is T for both cases. It is reasonable since $a \leq b$ means the stock has a larger uprising trend during the time horizon. So the investor should hold the stock until the end. In more general case, we can prove the following theorem: If $\mu(t)$ is increasing in t (may not be continuous or deterministic), then the optimal selling time is T.

Proof. It is enough to show that

$$E\left[e^{B_T^\mu - S_T^\mu}|\mathcal{F}_t\right] \geq E\left[e^{B_t^\mu - S_t^\mu}|\mathcal{F}_t\right], \forall t \in [0,T)$$

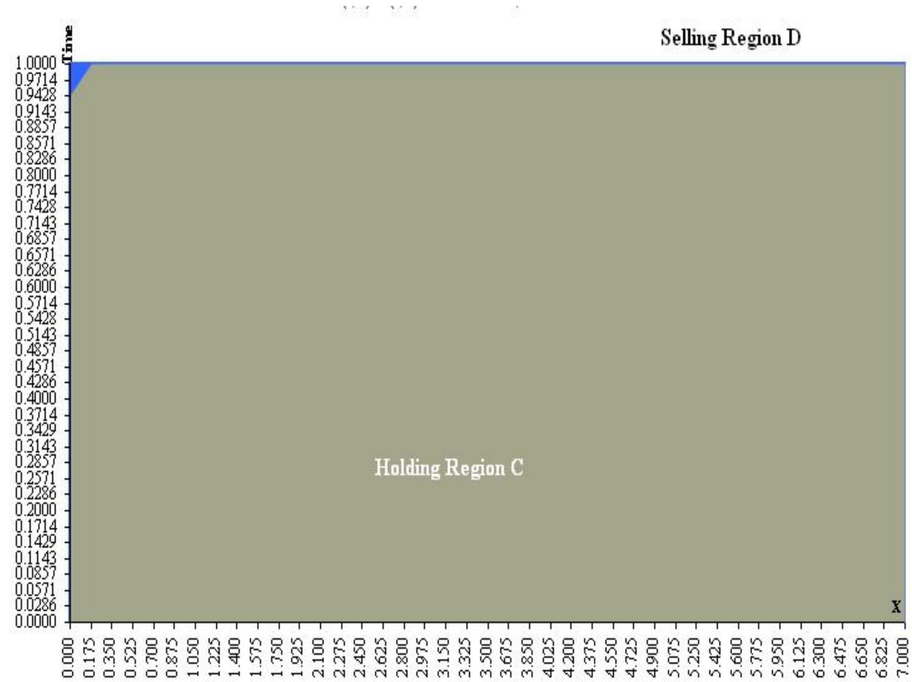

Figure 1: Holding and Selling region when $a = 0.2, b = 0.6$

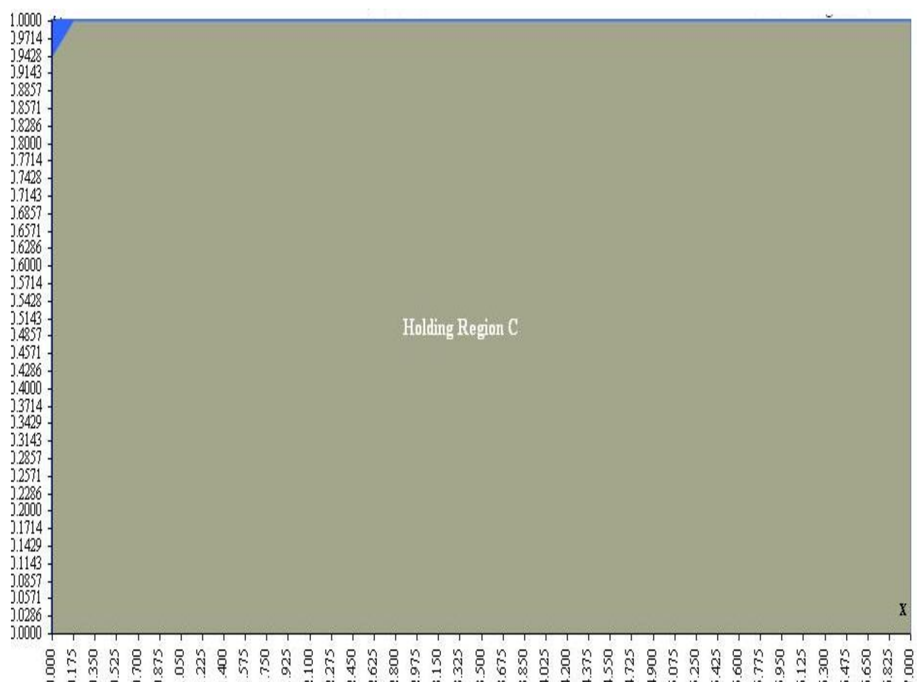

Figure 2: Holding and Selling region when $a = -0.2, b = 0.6$

In fact,

$$\begin{aligned}
E\left[e^{B_T^\mu - S_T^\mu}|\mathcal{F}_t\right] &= E\left[e^{B_T^\mu - B_t^\mu - (S_T^\mu - B_t^\mu)}|\mathcal{F}_t\right] \\
&= E\left[e^{B_T^\mu - B_t^\mu - (S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s^\mu - B_t^\mu)}|\mathcal{F}_t\right] \\
&= E\left[e^{B_T - B_t + \mu(T) - \mu(t) - (S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s + \mu(s) - B_t - \mu(t))}|\mathcal{F}_t\right] \\
&\geq E\left[e^{B_T - B_t + \mu(T) - \mu(t) - (S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s + \mu(T) - B_t - \mu(t))}|\mathcal{F}_t\right] \\
&= E\left[e^{B_T - B_t - (S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s - B_t)}|\mathcal{F}_t\right] \\
&= E\left[e^{B_{T-t} - x\vee sup_{0\leq s\leq T-t} B_s}\right]|_{x = S_t^\mu - B_t^\mu}
\end{aligned}$$

Note from equation (38) of Shiryaev Albert, Xu Zuoquan and Zhou Xun Yu(2008),

$$\begin{cases} E[G(T, X_T^x) =] > G(0,x) \forall x > 0 and E[G(T, X_T^x) =] > G(0,x) for x = 0 \end{cases} \tag{23}$$

we have

$$E\left[e^{B_{T-t} - x\vee sup_{0\leq s\leq T-t} B_s}\right] \geq E\left[e^{-x\vee sup_{0\leq s\leq T-t} B_s}\right]$$

Hence

$$\begin{aligned}
E\left[e^{B_T^\mu - S_T^\mu}|\mathcal{F}_t\right] &\geq E\left[e^{-x\vee sup_{0\leq s\leq T-t} B_s}\right]|_{x = S_t^\mu - B_t^\mu} \\
&= E\left[e^{-(S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s - B_t)}|\mathcal{F}_t\right] \\
&\geq E\left[e^{-(S_t^\mu - B_t^\mu)\vee sup_{t\leq s\leq T}(B_s^\mu - B_t^\mu)}|\mathcal{F}_t\right] \\
&= E\left[e^{B_t^\mu - S_t^\mu}|\mathcal{F}_t\right]
\end{aligned}$$

□

4.2 $a > b$

If $a > b$, the stock will turn worse at transition time T_1.The optimal selling time depends. We have selected four cases: (1) $a = 0.6, b = 0.2$; (2) $a = -0.2, b = -0.6$;(3) $a = 0.6, b = -0.2$ and (4) $a = 0.2, b = -0.6$.

From the graphs, we can see some interesting phenomena.

Under (1), from Figure 3 the optimal selling region is T when the drawdown process $X_t$is small;when X_t is very large close to the maximum, the investor could sell the stock at initial time 0.

Under (2), from Figure 4 the optimal selling time is 0.5, or T_1 when $X_t$is between 0.350 and 6.825; when X_t is very small or very large, the investor could sell the stock at initial time 0.

Under (3), from Figure 5 the stock performes very good before T_1,the optimal selling time is T when $X_t$is between 0.350 and 6.650; when X_t is very small or very large, the investor should sell the stock at initial time 0.

Under (4),from Figure 6 the stock deteriorates seriously, the investor should sell the stock when the stock deteriorates at time T_1. And when X_t is very small or very large, the investor should sell the stock at initial time 0.

we can conclude that when $\mu(t)$ is a decreasing piecewise linear function, or the trend of the stock price changes negatively, the optimal selling time is afftected strongly by how much of the trend change.

5 Conclusion

We have explored the finite horizon stock selling model with the drift as a piecewise linear function. We constructed the optimal stopping time and the linear complementarity formulation to get the optimal selling region and holding region. The numerical computation justify that when the drift function $\mu(t)$ is increasing, the optimal selling time is T. And from our graphs of the selling regions for decreasing $\mu(t)$ functions, the conclusion is that the optimal selling time has strong relationship with the transition time T_1 of the drift function, and how much change of the trend.

References

[1] Shiryaev Albert, Xu Zuoquan and Zhou Xun Yu, Thou shalt buy and hold, Quantitative Finance, 8:8(2008), pp 765-776.

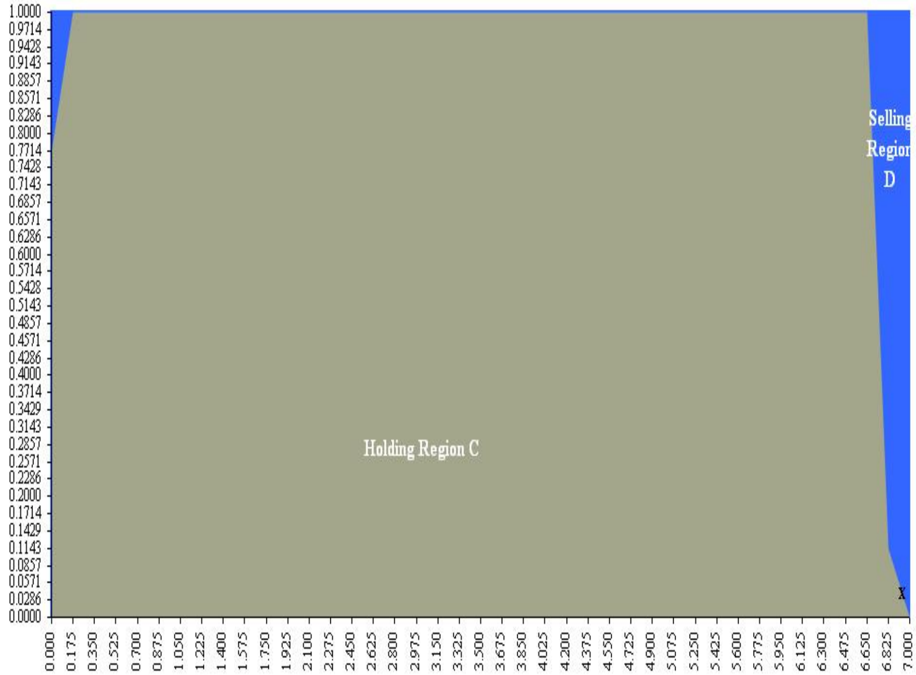

Figure 3: Holding and Selling region when $a = 0.6, b = 0.2$

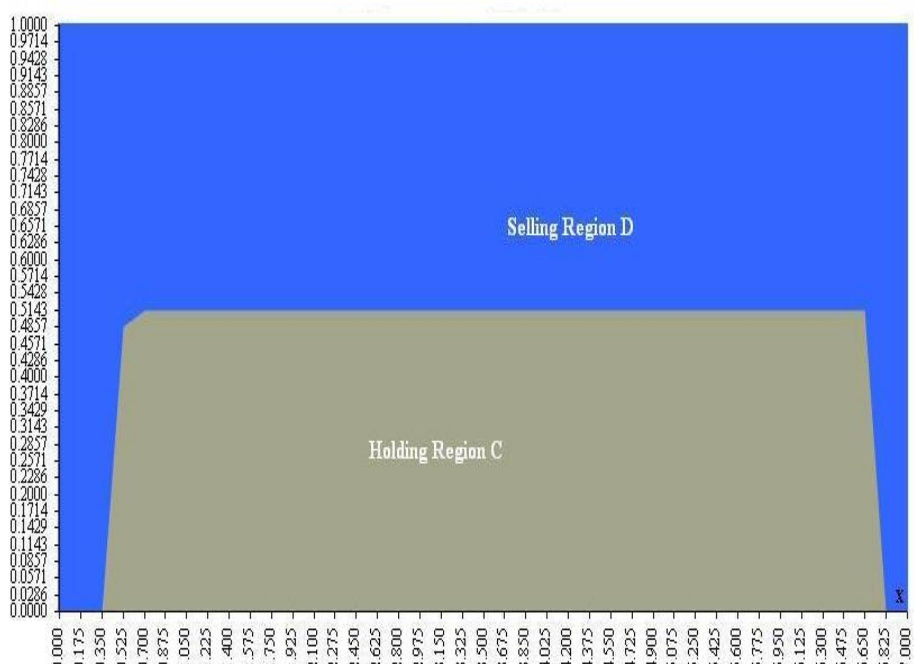

Figure 4: Holding and Selling region when $a = -0.2, b = -0.6$

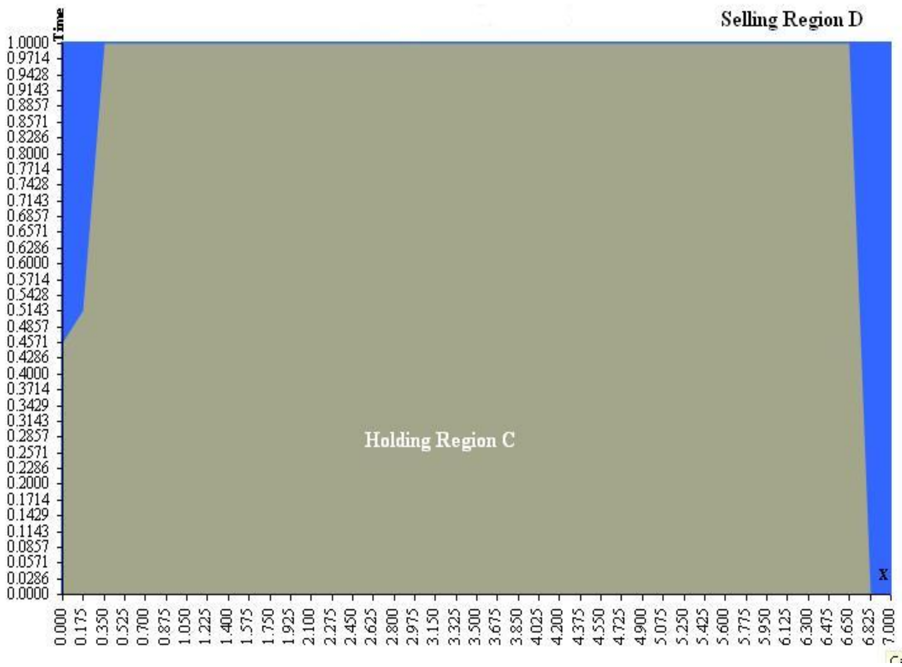

Figure 5: Holding and Selling region when $a = 0.6, b = -0.2$

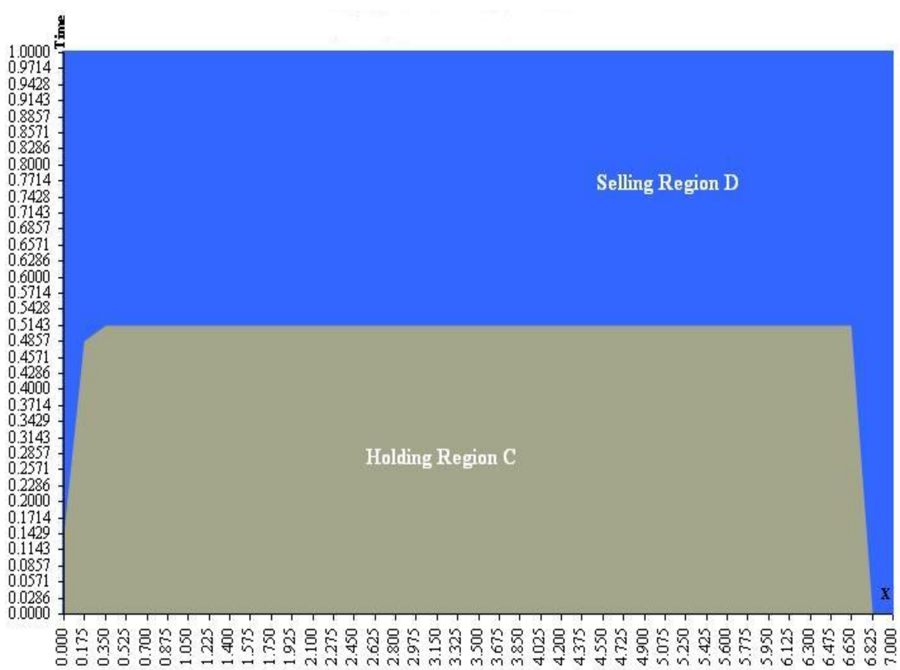

Figure 6: Holding and Selling region when $a = 0.2, b = -0.6$

[2] M. Dai and Y.F. Zhong, Optimal stock selling/buying strategy with reference to the ultimate average, Working Paper, July 2008.

[3] Shreve, Steven E., Stochastic Calculus for Finance II Continuous-Time Models, Springer Finance, 2004.

[4] Christoph Reisinger, Numerical Methods for Finance, Lecture Notes, University of Oxford, 2008.

[5] Jeff Dewynne and Sam Howison, American Options, Lecture Notes, University of Oxford, 2008.

Appendices

Appendix A: Calculation of $\tilde{P}(\max_{0\le s\le T-t}(\mu(s+t)-\mu(t)+\tilde{B}_s)\le z)$

$$
\begin{aligned}
&\tilde{P}(\max_{0\le s\le T-t}(\mu(s+t)-\mu(t)+\tilde{B}_s)\le z)\\
=&\begin{cases}
\tilde{P}(\max_{0\le s\le T-t}(bs+\tilde{B}_s)\le z), & t>T_1\\
\tilde{P}(\max_{0\le s\le T_1-t}(as+\tilde{B}_s)\vee\max_{T_1-t\le s\le T-t}(b(s+t)-at+T_1(a-b)+\tilde{B}_s)\le z), & t\le T_1
\end{cases}
\end{aligned}
$$

$t < T_1$,

$$
\begin{aligned}
& P(\max_{0\le s\le T_1-t}(as+\tilde{B}_s) \vee \max_{T_1-t\le s\le T-t}(b(s+t)+T_1(a-b)-at+\tilde{B}_s) \le z) \\
= \;& E[E[1_{(\max_{0\le s\le T_1-t}(as+\tilde{B}_s)\vee \max_{T_1-t\le s\le T-t}(b(s+t)+T_1(a-b)-at+\tilde{B}_s)\le z}|\mathcal{F}_{T_1-t})]] \\
= \;& E[E[1_{y\le z}1_{\max_{T_1-t\le s\le T-t}(b(s+t)+T_1(a-b)-at+\tilde{B}_s)} \le z|\mathcal{F}_{T_1-t}]|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le z}E[1_{\max_{T_1-t\le s\le T-t}(b-a)t+T_1(a-b)+bs+\tilde{B}_s-\tilde{B}_{T_1-t}\le z-\tilde{B}_{T_1-t}}|\mathcal{F}_{T_1-t}]|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le z}E[1_{\max_{0\le s\le T-T_1}((b-a)t+T_1(a-b)+b(s+T_1-t)+\overline{B}_s)\le\eta}|\eta = z-\tilde{B}_{T_1-t}]|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le z}[P(\max_{0\le s\le T-T_1}(aT_1-at+bs+\overline{B}_s)\le\eta)|\eta = z-\tilde{B}_{T_1-t}]|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le z}[P(\max_{0\le s\le T-T_1}(bs+\overline{B}_s)\le\eta+at-aT_1)|\eta = z-\tilde{B}_{T_1-t}]|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le z}(\Phi[\frac{(\eta+at-aT_1)-b(T-T_1)}{\sqrt{T-T_1}}] - e^{2b(\eta+at-aT_1)}\Phi[\frac{-(\eta+at-aT_1)-b(T-T_1)}{\sqrt{T-T_1}}])|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)] \\
= \;& E[1_{y\le x}(\Phi[\frac{z-\tilde{B}_{T_1-t}+at+bT_1-aT_1-bT)}{\sqrt{T-T_1}}] \\
& -e^{2b(z-\tilde{B}_{T_1-t}+at-aT_1)}\Phi[\frac{-(z-\tilde{B}_{T_1-t}+at-aT_1)-b(T-T_1)}{\sqrt{T-T_1}}])|y = \max_{0\le s\le T_1-t}(as+\tilde{B}_s)]
\end{aligned}
$$

Let

$$
\begin{aligned}
f(z-\tilde{B}_{T_1-t}) \quad = \quad & \Phi[\frac{x-\tilde{B}_{T_1-t}+at+bT_1-aT_1-bT)}{\sqrt{T-T_1}}] \\
& -e^{2b(x-\tilde{B}_{T_1-t}+at-aT_1)}\Phi[\frac{-(x-\tilde{B}_{T_1-t}+at-aT_1)-b(T-T_1)}{\sqrt{T-T_1}}])
\end{aligned}
$$

Hence

$$
\begin{aligned}
& t < T_1, \\
& P(\max_{0\le s\le T_1-t}(as+\tilde{B}_s) \vee \max_{T_1-t\le s\le T-t}(b(s+t)+T_1(a-b)-at+\tilde{B}_s) \le z) \\
= \;& E[1_{y\le x} f(z-\tilde{B}_{T_1-t})] \\
= \;& E[1_{\max_{0\le s\le T_1-t}(as+\tilde{B}_s)\le z} f(z-\tilde{B}_{T_1-t}] \\
= \;& E[1_{\max_{0\le s\le T_1-t}\overline{B}_s\le z} f(z-\overline{B}_{T_1-t}+a(T_1-t))e^{\frac{1}{2}a^2(T_1-t)-a\overline{B}_{T_1-t}}] \\
= \;& \int_0^{+\infty}\int_{-\infty}^{m} 1_{m\le z} f(z-w+a(T_1-t))e^{\frac{1}{2}a^2(T_1-t)-aw}\frac{2(2m-w)}{(T_1-t)\sqrt{2\pi(T_1-t)}}e^{aw-\frac{1}{2}a^2(T_1-t)-\frac{1}{2(T_1-t)}(2m-w)^2}dwdm \\
= \;& \int_0^{z}\int_{-\infty}^{m} f(z-w+a(T_1-t))e^{\frac{1}{2}a^2(T_1-t)-aw}\frac{2(2m-w)}{(T_1-t)\sqrt{2\pi(T_1-t)}}e^{aw-\frac{1}{2}a^2(T_1-t)-\frac{1}{2(T_1-t)}(2m-w)^2}dwdm \\
= \;& \int_0^{z}\int_{-\infty}^{m} f(z-w+a(T_1-t))\frac{2(2m-w)}{(T_1-t)\sqrt{2\pi(T_1-t)}}e^{-\frac{(2m-w)^2}{2(T_1-t)}}dwdm \\
= \;& \int_0^{z}\int_{-\infty}^{m} (\Phi[\frac{z-w+a(T_1-t)+at-aT_1-b(T-T_1)}{\sqrt{T-T_1}}]-e^{2b(z-w+a(T_1-t)+at-aT_1)} \\
& \Phi[\frac{-(z-w+a(T_1-t)+at-aT_1)-b(T-T_1)}{\sqrt{T-T_1}}])\frac{2(2m-w)}{(T_1-t)\sqrt{2\pi(T_1-t)}}e^{-\frac{(2m-w)^2}{2(T_1-t)}}dwdm \\
= \;& \int_0^{z}\int_{-\infty}^{m} (\Phi[\frac{z-w-b(T-T_1)}{\sqrt{T-T_1}}] \\
& -e^{2b(z-w)}\Phi[\frac{-(z-w)-b(T-T_1)}{\sqrt{T-T_1}}])\frac{2(2m-w)}{(T_1-t)\sqrt{2\pi(T_1-t)}}e^{-\frac{(2m-w)^2}{2(T_1-t)}}dwdm
\end{aligned}
$$

Appendix B: Proof of $X^x_. \overset{law}{=} |Y.|$ Since

$$X^x_t = x \vee S^\mu_t - B^\mu_t, t \ge 0$$

and

$$
\begin{cases}
dY_t = -asign(Y_t)dt + d\tilde{B}_t, t \le T_1 \\
dY_t = -bsign(Y_t)dt + d\tilde{B}_t, t > T_1 \\
Y_0 = x
\end{cases}
$$

hence for $\forall 0 \leq t_1 < t_2 < ... < t_k \leq T_1 < t_{k+1} < ... < t_n, A_i \in \mathcal{F}$,

$$
\begin{aligned}
P(x_{t_i} \in A_i, i = 1, 2, ..., n) &= E[1_{x_{t_i} \in A_i, i=1,2,...,n}] \\
&= E\left[E(1_{x_{t_i} \in A_i}, i = 1, 2, ..., n)|\mathcal{F}_{T_1})\right] \\
&= E\left[1_{x_{t_i} \in A_i, i \leq k} E(1_{x_{t_i} \in A_i}, i \geq k)|\mathcal{F}_{T_1})\right] \\
&= E\left[1_{x_{t_i} \in A_i, i \leq k} E(1_{x_{t_i} \in A_i}, i \geq k)|x_{T_1})\right]
\end{aligned}
$$

Let

$$E[1_{x_{t_i} \in A_i, i > k}|x_{T_1}] = f(x_{T_1})$$

then

$$P(x_{t_i} \in A_i, i = 1, 2, ..., n) = E\left[1_{x_{t_i} \in A_i, i \leq k} f(x_{T_1})\right]$$

Note $X^x_. \overset{law}{=} |Y_.|, t > T_1$ hence

$$E[1_{|y_{t_i}| \in A_i, i > k}||y_{T_1}|] = f(|y_{T_1}|)$$

and $X^x_. \overset{law}{=} |Y_.|, t \leq T_1$. So

$$
\begin{aligned}
P(x_{t_i} \in A_i, i = 1, 2, ..., n) &= E\left[1_{|y_{t_i}| \in A_i, i \leq k} f(|y_{T_1}|)\right] \\
&= E\left[1_{|y_{t_i}| \in A_i, i \leq k} E(1_{|y_{t_i}| \in A_i, i > k}||y_{T_1}|)\right] \\
&= E\left[1_{|y_{t_i}| \in A_i, i=1,2,...,n}\right] \\
&= P(|y_{t_i}| \in A_i, i = 1, 2, ..., n)
\end{aligned}
$$

So

$$X^x_. \overset{law}{=} |Y_.|, t \geq 0$$