1 Introduction

This is the first in a series of papers constructing explicit examples of special Lagrangian submanifolds in $\mathbb{C}^m$. In it we will study special Lagrangian $m$-folds with large symmetry groups. The next five papers in the series are $[9, 10, 11, 12, 13]$. They use other methods to construct special Lagrangian $m$-folds, namely evolution equations, ruled submanifolds, and integrable systems.

The author’s principal motivation for studying special Lagrangian $m$-folds in $\mathbb{C}^m$ is that they provide local models for singularities of special Lagrangian $m$-folds in Calabi–Yau $m$-folds. In 1996 Strominger, Yau and Zaslow $[17]$ proposed an explanation of mirror symmetry (the SYZ conjecture) between Calabi–Yau 3-folds $X, \hat{X}$ in terms of dual ‘fibrations’ of $X$ and $\hat{X}$ by special Lagrangian $T^3$’s, with some singular fibres.

To make progress towards proving the SYZ conjecture, or even stating it precisely, will require a good understanding of the possible singularities that can develop in families of special Lagrangian 3-folds in a Calabi–Yau 3-fold. This paper is part of a programme to develop such an understanding.

Some first steps in this direction were taken by the author in $[8]$, which tried to define an invariant of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres with weights; proving (or disproving) the conjectures made in $[8]$ will also require an understanding of the singularities of special Lagrangian 3-folds.

Perhaps the most obvious kinds of local model for singularities of special Lagrangian $m$-folds are special Lagrangian cones in $\mathbb{C}^m$. The main results of the paper, in $\S\S 7$ and $\S\S 8$, are a study of $U(1)^{m-2}$-invariant special Lagrangian cones in $\mathbb{C}^m$, and the proof of the existence in $\mathbb{C}^m$ of large families of special Lagrangian cones on $T^{m-1}$ (for $m \geq 3$), $S^2 \times T^{m-3}$ (for $m \geq 4$) and $S^3 \times T^{m-4}$ (for $m \geq 5$).

We begin in $\S\S 2$ by introducing special Lagrangian geometry in $\mathbb{C}^m$, and then in $\S\S 3$ we give several results relating to real analyticity of special Lagrangian $m$-folds, including a construction of special Lagrangian $m$-folds by evolving real analytic $(m-1)$-submanifolds of $\mathbb{C}^m$. Section $\S\S$ discusses moment maps, and shows that if $N$ is a Lagrangian submanifold with symmetry group $G$ then the
moment map of $G$ is constant on $N$.

In §5 we study cohomogeneity one SL $m$-folds $N$ in $\mathbb{C}^m$, where the orbits of the symmetry group $G \subset \text{SU}(m) \times \mathbb{C}^m$ are of codimension one in $N$. Then $N$ is foliated by a 1-parameter family of $G$-orbits parametrized by $t \in \mathbb{R}$. We write the condition that $N$ be special Lagrangian as an o.d.e. upon $G$-orbits depending on $t$, and by solving this equation we find examples of SL $m$-folds in $\mathbb{C}^m$.

Section 6 considers special Lagrangian cones $N$ in $\mathbb{C}^m$. As cones are invariant under the group $\mathbb{R}^+$ of dilations of $\mathbb{C}^m$, by including dilations we can define the generalized symmetry group $G \subset \mathbb{R}^+ \times \text{SU}(m)$ of $N$. So as in §5 we can consider special Lagrangian cones on which the generalized symmetry group acts with cohomogeneity one.

As an example of this, in §7 we study SL cones in $\mathbb{C}^m$ invariant under a subgroup $G \cong U(1)^{m-2}$ of diagonal matrices in $\mathbb{C}^m$, for $m \geq 3$. We reduce the problem to an o.d.e. in $m$ complex variables $w_1(t), \ldots, w_m(t)$, and by solving this o.d.e. fairly explicitly, we prove the existence of a large family of distinct SL cones in $\mathbb{C}^m$ on $T^{m-1}$, and also of smaller families of SL cones on $S^2 \times T^{m-3}$ in $\mathbb{C}^m$ for $m \geq 4$, and of SL cones on $S^3 \times T$ in $\mathbb{C}^m$ for $m \geq 5$.

In §8 we specialize to the case $m = 3$, and consider $U(1)$-invariant SL cones in $\mathbb{C}^3$ in more detail. Finally, section 9 gives a new construction of SL $m$-folds in $\mathbb{C}^m$ starting with a special Lagrangian $m$-fold $L$ in $\mathbb{C}^m$ with ‘perpendicular symmetries’, that is, vector fields in $\text{su}(m) \times \mathbb{C}^m$ which are perpendicular to $L$ at every point.

We remark that Goldstein [5, Th. 1] has proved some results related to those of §7. He considers a compact toric Kähler–Einstein $n$-fold $N$ with positive scalar curvature and a $U(1)^n$-action preserving the structure. Then there is a unique, flat $U(1)^n$-orbit $L$ which is minimal Lagrangian. Furthermore, he shows that there is at least one subgroup $U(1)^{n-1}$ in $U(1)^n$ with a sequence of non-flat minimal Lagrangian $U(1)^{n-1}$-invariant tori $L_k$ converging to $L$.

When Goldstein’s results are applied to $U(1)^{m-1}$ acting on $\mathbb{CP}^{m-1}$ by isometries, they prove the existence of a family of $U(1)^{m-2}$-invariant minimal Lagrangian tori $T^{m-1}$ in $\mathbb{CP}^{m-1}$, close to the unique minimal Lagrangian $U(1)^{m-1}$-orbit. It can be shown that these lift to $U(1)^{m-2}$-invariant special Lagrangian cones in $\mathbb{C}^m$, which is what we study in §8.

There is also some overlap between the results of this paper, especially §8, and those of Castro and Urbano [2, 3] and Haskins [4]. In particular, Haskins studies $U(1)$-invariant SL cones in $\mathbb{C}^3$, so that nearly all of §8 is equivalent to results in [2], and also Theorem 6.4 below is essentially the same as [2, Remark 1, p. 81–2] and [4, Th. A]. I would like to thank these authors for sending me copies of their work.

Acknowledgements: The author would like to thank Nigel Hitchin, Mark Haskins, Karen Uhlenbeck, Ian McIntosh, Robert Bryant and Chuu-Lian Terng for helpful conversations, and the referee for careful proofreading and suggesting improvements.
2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [6].

**Definition 2.1** Let $(M,g)$ be a Riemannian manifold. An *oriented tangent* $k$-*plane* $V$ on $M$ is a vector subspace $V$ of some tangent space $T_x M$ to $M$ with $\dim V = k$, equipped with an orientation. If $V$ is an oriented tangent $k$-plane on $M$ then $g|_V$ is a Euclidean metric on $V$, so combining $g|_V$ with the orientation on $V$ gives a natural *volume form* $\text{vol}_V$ on $V$, which is a $k$-form on $V$. Now let $\varphi$ be a closed $k$-form on $M$. We say that $\varphi$ is a *calibration* on $M$ if for every oriented $k$-plane $V$ on $M$ we have $\varphi|_V \leq \text{vol}_V$. Here $\varphi|_V = \alpha \cdot \text{vol}_V$ for some $\alpha \in \mathbb{R}$ and $\varphi|_V \leq \text{vol}_V$ if $\alpha \leq 1$. Let $N$ be an oriented submanifold of $M$ with dimension $k$. Then each tangent space $T_x N$ for $x \in N$ is an oriented tangent $k$-plane. We say that $N$ is a *calibrated submanifold* or *$\varphi$-submanifold* if $\varphi|_{T_x N} = \text{vol}_{T_x N}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [6, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$.

**Definition 2.2** Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$ and complex structure $I$, and define a metric $g$, a real 2-form $\omega$ and a complex $m$-form $\Omega$ on $\mathbb{C}^m$ by

\[ g = |dz_1|^2 + \cdots + |dz_m|^2, \quad \omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m), \]

and \[ \Omega = dz_1 \wedge \cdots \wedge dz_m. \]

Then $\text{Re}\, \Omega$ and $\text{Im}\, \Omega$ are real $m$-forms on $\mathbb{C}^m$. Let $L$ be an oriented real submanifold of $\mathbb{C}^m$ of real dimension $m$, and let $\theta \in [0, 2\pi)$. We say that $L$ is a *special Lagrangian submanifold* of $\mathbb{C}^m$ with *phase* $e^{i\theta}$, if $L$ is calibrated with respect to $\cos \theta \text{ Re}\, \Omega + \sin \theta \text{ Im}\, \Omega$, in the sense of Definition 2.1.

We will often abbreviate ‘special Lagrangian’ by ‘SL’, and ‘$m$-dimensional submanifold’ by ‘$m$-fold’, so that we shall talk about SL $m$-folds in $\mathbb{C}^m$. Usually we take $\theta = 0$, so that $L$ has phase 1, and is calibrated with respect to $\text{Re}\, \Omega$. When we discuss special Lagrangian submanifolds without specifying a phase, we mean them to have phase 1.

Harvey and Lawson [6, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

**Proposition 2.3** Let $L$ be a real $m$-dimensional submanifold of $\mathbb{C}^m$. Then $L$ admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ with phase $e^{i\theta}$ if and only if $|\omega|_L \equiv 0$ and $|\sin \theta \text{ Re}\, \Omega - \cos \theta \text{ Im}\, \Omega|_L \equiv 0$.

Note that an $m$-dimensional submanifold $L$ in $\mathbb{C}^m$ is called *Lagrangian* if $|\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $(\sin \theta \text{ Re}\, \Omega - \cos \theta \text{ Im}\, \Omega)|_L \equiv 0$, which is how they get their name.
3 Real analyticity of SL submanifolds

In this section we collect together several results about special Lagrangian submanifolds in \( \mathbb{C}^m \) related to real analyticity. For simplicity we fix the phase of all special Lagrangian submanifolds to be 1. As special Lagrangian submanifolds in \( \mathbb{C}^m \) are calibrated, they are locally minimal. Harvey and Lawson [6, Th. III.2.7] use this to show that they are real analytic:

**Theorem 3.1** Let \( L \) be a special Lagrangian submanifold in \( \mathbb{C}^m \). Then \( L \) is real analytic wherever it is nonsingular.

Note that the restriction to nonsingular \( L \) is necessary here, as Harvey and Lawson [6, p. 97] give examples of singularities of SL submanifolds in \( \mathbb{C}^m \) which are not real analytic. Harvey and Lawson [6, Th. III.5.5] also use real analyticity in a different way, to prove the following result.

**Theorem 3.2** Suppose \( P \) is a real analytic \((m-1)\)-submanifold of \( \mathbb{C}^m \) with \( \omega|_P \equiv 0 \). Then there exists a locally unique special Lagrangian submanifold \( N \) of \( \mathbb{C}^m \) containing \( P \).

They assume \( P \) is real analytic because their proof uses the Cartan–Kähler Theorem, from the subject of exterior differential systems, and this only works in the real analytic category. One can think of the submanifold \( N \) as defined by a kind of Taylor series, which converges in a small neighbourhood of \( P \).

We will now show that \( N \) is the total space of a 1-parameter family \( \{ P_t : t \in (-\epsilon, \epsilon) \} \) of real analytic submanifolds of \( \mathbb{C}^m \) diffeomorphic to \( P \), which satisfy a first-order o.d.e. in \( t \), with initial data \( P_0 = P \). This will provide motivation for several of the constructions of special Lagrangian submanifolds in \( \mathbb{C}^m \) to be considered in this paper and its sequels.

**Theorem 3.3** Let \( P \) be a compact, orientable, real analytic \((m-1)\)-manifold, \( \chi \) a real analytic, nowhere vanishing section of \( \Lambda^{m-1}TP \), and \( \phi : P \to \mathbb{C}^m \) a real analytic embedding (immersion) such that \( \phi^*(\omega) \equiv 0 \) on \( P \). Then there exists \( \epsilon > 0 \) and a unique family \( \{ \phi_t : t \in (-\epsilon, \epsilon) \} \) of real analytic maps \( \phi_t : P \to \mathbb{C}^m \) with \( \phi_0 = \phi \), satisfying the equation

\[
\left( \frac{d\phi_t}{dt} \right)^c = (\phi_t)_*(\chi)^{b_1 \ldots b_{m-1}} (\text{Re } \Omega)_{b_1 \ldots b_{m-1} b_m} g^{b_m c}, \quad (1)
\]

using the index notation for (real) tensors on \( \mathbb{C}^m \), where \( g^{bc} \) is the inverse of the Euclidean metric on \( \mathbb{C}^m \). Define \( \Phi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m \) by \( \Phi(t, p) = \phi_t(p) \). Then \( N = \text{Image } \Phi \) is a nonsingular embedded (immersed) special Lagrangian submanifold of \( \mathbb{C}^m \).

**Proof.** Equation (1) is an evolution equation for the maps \( \phi_t : P \to \mathbb{C}^m \), with initial condition \( \phi_0 = \phi \). As \( P \) is compact and everything is real analytic, the existence of a unique solution for \( t \) in \( (-\epsilon, \epsilon) \) for some \( \epsilon > 0 \) follows from standard
techniques in partial differential equations. For instance, one can prove it by applying the Cauchy–Kowalevsky Theorem \[16, p. 234\] to an evolution equation for \((\phi_t, d\phi_t)\) derived from (1).

Thus the family \(\{\phi_t : t \in (-\epsilon, \epsilon)\}\) exists, and it remains to prove that \(N = \text{Image } \Phi\) is special Lagrangian. Now by Theorem 3.2 as \(\phi^* (\omega) \equiv 0\) and \(\phi\) is real analytic, there is a locally unique real analytic special Lagrangian submanifold \(N' \subset \mathbb{C}^m\) containing \(\phi(P)\). We shall show that \(N' = N\) locally.

To do this, observe that (1) also makes sense as an evolution equation for submanifolds of \(N'\). That is, we could look for a family \(\{\phi'_t : t \in (-\epsilon', \epsilon')\}\) of real analytic maps \(\phi'_t : P \to N'\) with \(\phi_0 = \phi\), satisfying the equation

\[
\frac{d\phi'_t}{dt} = (\phi'_t)_{*}(\chi)^{b_1 \ldots b_{m-1}}(\text{Re } \Omega|{N'})_{b_1 \ldots b_{m-1}b_m} (g|{N'})_{b_m c},
\]

using the index notation for tensors on \(N'\). It follows as above that for some \(\epsilon' > 0\) there exists a unique solution to this problem.

Let \(p \in P\) and \(t \in (-\epsilon', \epsilon')\), and set \(x = \phi'_t(p)\), so that \(x \in N'\). Treating \(\mathbb{C}^m\) and \((\mathbb{C}^m)^*\) as real vector spaces, we have orthogonal direct sums \(\mathbb{C}^m = T_x N' \oplus V\) and \(((\mathbb{C}^m)^*) = T^*_x N' \oplus V^*\), where \(V\) is the perpendicular subspace to \(T_x N'\). This induces a splitting \(\Lambda^m((\mathbb{C}^m)^*) = \bigoplus_{k=0}^{m} \Lambda^k T^*_x N' \otimes \Lambda^{m-k} V^*\).

Now \(\text{Re } \Omega \in \Lambda^m((\mathbb{C}^m)^*),\) and \(N'\) is calibrated with respect to \(\text{Re } \Omega\). This implies that the component of \(\text{Re } \Omega\) in \(\Lambda^{m-1} T^*_x N' \otimes V^*\) is zero, because this measures the change in \(\text{Re } \Omega|_{T_x N'}\) under small variations of the subspace \(T_x N'\), but \(\text{Re } \Omega|_{T_x N'}\) is maximum and therefore stationary.

Since \((\phi'_t)_{*}(\chi)|_p\) lies in \(\Lambda^{m-1} T_x N'\), it follows that

\[(\phi'_t)_{*}(\chi)^{b_1 \ldots b_{m-1}}|_p (\text{Re } \Omega)_{b_1 \ldots b_{m-1}b_m} \in T_x N' \subset (\mathbb{C}^m)^*,\]

as the component in \(V^*\) comes from the component of \(\text{Re } \Omega\) in \(\Lambda^{m-1} T^*_x N' \otimes V^*\), which is zero. Therefore

\[
(\phi'_t)_{*}(\chi)^{b_1 \ldots b_{m-1}}|_p (\text{Re } \Omega)_{b_1 \ldots b_{m-1}b_m} = 0.
\]

Because the splitting \((\mathbb{C}^m)^* = T^*_x N' \oplus V^*\) is orthogonal we have \(g^{bc} = (g|_{T_x N'})^{bc} + h^{bc}\) for some \(h \in S^2 V\). Contracting this with (1) shows that

\[
(\phi'_t)_{*}(\chi)^{b_1 \ldots b_{m-1}}(\text{Re } \Omega)_{b_1 \ldots b_{m-1}b_m} g^{b_m c} = 0,
\]

at \(p\), as the r.h.s. of (1) lies in \(T^*_x N'\), so its contraction with \(h\) is zero.

Equation (1) holds for all \(p \in P\) and \(t \in (-\epsilon', \epsilon')\). Therefore by (1) the \(\phi'_t\) also satisfy (1), and so \(\phi'_t = \phi_t\) by uniqueness. Hence \(\phi_t\) maps \(P\) to \(N'\), and \(\Phi\) maps \((-\epsilon, \epsilon) \times P\) to \(N'\), if \(\epsilon\) is sufficiently small.

Finally, suppose \(\phi = \phi_0\) is an embedding. Then \(\phi_t : P \to N'\) is also an embedding for small \(t\). But \(d\phi_t/dt\) is a normal vector field to \(\phi_t(P)\) in \(N'\), with
length \(|(\phi_t)_*(\chi)|\). As \(\chi\) is nonvanishing, this vector field is nonzero, so \(\Phi\) is an embedding for small \(\epsilon\), with Image \(\Phi\) an open subset of \(N'\), which is therefore special Lagrangian. If \(\phi\) is an immersion, then \(\Phi\) is a special Lagrangian immersion, in a similar way. \(\square\)

The condition that \(P\) be compact is not always necessary here. Whether \(P\) is compact or not, in a small neighbourhood of any \(p \in P\) the maps \(\phi_t\) always exist for \(t \in (-\epsilon, \epsilon)\) and some \(\epsilon > 0\), which may depend on \(p\). If \(P\) is compact we can choose an \(\epsilon > 0\) valid for all \(p\), but if \(P\) is noncompact there may not exist such an \(\epsilon\).

We can also relax the condition that \(\phi : P \to \mathbb{C}^m\) be an embedding or an immersion, and instead require only that \(\phi\) be real analytic. Then the conclusions of the theorem still hold, except that \(\Phi\) is no longer an embedding or an immersion, and Image \(\Phi\) will in general be a singular special Lagrangian submanifold of \(\mathbb{C}^m\). This can be used as a technique for constructing singular special Lagrangian submanifolds.

## 4 Symmetries and moment maps

Let \(\mathbb{C}^m\) have its usual metric \(g\) and Kähler form \(\omega\). Then the group of automorphisms of \(\mathbb{C}^m\) preserving \(g\) and \(\omega\) is \(U(m) \ltimes \mathbb{C}^m\), where \(\mathbb{C}^m\) acts by translations. Let \(G\) be a Lie subgroup of \(U(m) \ltimes \mathbb{C}^m\), with Lie algebra \(\mathfrak{g}\), and let \(\phi : \mathfrak{g} \to C^\infty(T\mathbb{C}^m)\) be the natural action of \(\mathfrak{g}\) on \(\mathbb{C}^m\) by vector fields.

Then a moment map for the action of \(G\) on \(\mathbb{C}^m\) is a smooth \(\mu : \mathbb{C}^m \to \mathfrak{g}^*\), such that

(a) \(\iota(\phi(x))\omega = x \cdot d\mu\) for all \(x \in \mathfrak{g}\), where ‘\(\iota\)’ is the pairing between \(\mathfrak{g}\) and \(\mathfrak{g}^*\),

(b) \(\mu\) is equivariant with respect to the \(G\)-action \(\Phi\) on \(\mathbb{C}^m\) and the coadjoint \(G\)-action on \(\mathfrak{g}^*\).

If \(G\) is compact or semisimple then a moment map \(\mu\) always exists, but in general there may be obstructions to the existence of \(\mu\).

The subsets \(\mu^{-1}(c)\) for \(c \in \mathfrak{g}^*\) are called level sets of the moment map. Define the centre \(Z(\mathfrak{g}^*)\) to be the vector subspace of \(\mathfrak{g}^*\) fixed by the coadjoint action of \(G\). Then, as \(\mu(\gamma \cdot z) = \text{Coad}(\gamma)\mu(z)\) for each \(z \in M\) and \(\gamma \in G\), we see that \(\mu^{-1}(c)\) is \(G\)-invariant if and only if \(c \in Z(\mathfrak{g}^*)\).

Here is a result characterizing \(G\)-orbits \(\mathcal{O}\) with \(\omega|_{\mathcal{O}} \equiv 0\).

**Proposition 4.1** Let \(G\) be a connected Lie subgroup of \(U(m) \ltimes \mathbb{C}^m\) with Lie algebra \(\mathfrak{g}\) and moment map \(\mu : \mathbb{C}^m \to \mathfrak{g}^*\), and let \(\mathcal{O}\) be an orbit of \(G\) in \(\mathbb{C}^m\). Then \(\omega|_{\mathcal{O}} \equiv 0\) if and only if \(\mathcal{O} \subseteq \mu^{-1}(c)\) for some \(c \in Z(\mathfrak{g}^*)\).

**Proof.** Let \(x, y \in \mathfrak{g}\), and let \(\phi(x), \phi(y)\) be the induced vector fields on \(\mathbb{C}^m\). Then, by definition of the moment map \(\mu\), we see that

\[
\omega(\phi(x), \phi(y)) = \phi(y) \cdot (x \cdot d\mu) = x \cdot (\phi(y) \cdot d\mu) = x \cdot L_{\phi(y)}\mu,
\]

where \(L_{\phi(y)}\) denotes the Lie derivative along \(\phi(y)\).
where \( \mathcal{L}_{\phi(y)} \) is the Lie derivative. As \( \mathcal{O} \) is a \( G \)-orbit, the vector fields \( \phi(x) \) for \( x \in g \) are tangent to \( \mathcal{O} \), and generate its whole tangent space. Thus, \( \omega|_{\mathcal{O}} \equiv 0 \) if and only if \( \omega(\phi(x), \phi(y)) = 0 \) on \( \mathcal{O} \) for all \( x, y \in g \).

By \((\mathcal{O})\) this holds if and only if \( \mathcal{L}_{\phi(y)}\mu = 0 \) on \( \mathcal{O} \) for all \( y \in g \). Since \( G \) is connected, this is true if and only if \( \mu \) is constant on \( \mathcal{O} \), so that \( \mathcal{O} \subseteq \mu^{-1}(c) \) for some \( c \in g^* \). But \( \mathcal{O} \) is \( G \)-invariant, so \( c \) is \( \text{Coad}(G) \)-invariant, and \( c \in Z(g^*) \). \( \square \)

The reason we are interested in moment maps is that \( G \)-invariant Lagrangian submanifolds in \( \mathbb{C}^m \) lie in level sets of the moment map \( \mu \) of \( G \). Thus, moment maps are a tool for studying Lagrangian (and hence special Lagrangian) submanifolds with symmetries.

**Proposition 4.2** Suppose \( N \) is a connected Lagrangian submanifold of \( \mathbb{C}^m \), and \( G \) a connected Lie subgroup of \( U(m) \ltimes \mathbb{C}^m \) preserving \( N \), with Lie algebra \( g \). Then \( G \) admits a moment map \( \mu \), and \( N \subseteq \mu^{-1}(c) \) for some \( c \in Z(g^*) \).

**Proof.** Let \( x \in g \). Then \( \iota(\phi(x))\omega \) is a closed 1-form on \( \mathbb{C}^m \), so there exists a smooth function \( f_x : \mathbb{C}^m \to \mathbb{R} \), unique up to addition of a constant, with \( df_x = \iota(\phi(x))\omega \). Since \( \omega|_{\mathcal{O}} \equiv 0 \) and \( \phi(x) \) is tangent to \( N \) we have \( df_x|_N \equiv 0 \), so \( f_x \) is constant on \( N \), as \( N \) is connected.

As the functions \( f_x \) are defined up to addition of a constant, we can choose \( f_x \) uniquely by requiring that \( f_x = 0 \) on \( N \). Clearly \( f_x \) is linear in \( x \). Hence there is a unique map \( \mu_0 : \mathbb{C}^m \to g^* \) with \( x \cdot \mu_0 = f_x \) for all \( x \in g \), and thus \( \iota(\phi(x))\omega = x \cdot d\mu_0 \) as in part (a) above. The \( G \)-equivariance in part (b) follows because \( N \) is \( G \)-invariant.

Thus we have constructed a particular moment map \( \mu_0 \) for \( G \), with the property that \( \mu_0 \equiv 0 \) on \( N \). Hence \( G \) admits a moment map. If \( \mu \) is any moment map for \( G \) then \( \mu - \mu_0 \) is a constant \( c \in Z(g^*) \), and so \( N \subseteq \mu^{-1}(c) \). \( \square \)

Now consider special Lagrangian \( m \)-folds \( N \) in \( \mathbb{C}^m \). Then instead of \( U(m) \ltimes \mathbb{C}^m \) we should use \( \text{SU}(m) \ltimes \mathbb{C}^m \), the group of automorphisms of \( \mathbb{C}^m \) preserving \( g, \omega \) and \( \Omega \).

**Definition 4.3** Let \( N \) be a special Lagrangian \( m \)-fold in \( \mathbb{C}^m \). Define the symmetry group \( \text{Sym}(N) \) of \( N \) to be the Lie subgroup of \( \text{SU}(m) \ltimes \mathbb{C}^m \) preserving \( N \). That is, the elements of \( \text{Sym}(N) \) are automorphisms of \( \mathbb{C}^m \) preserving \( g, \omega, \Omega \) and \( N \). Define the restricted symmetry group \( \text{Sym}^0(N) \) of \( N \) to be the connected component of \( \text{Sym}(N) \) containing the identity.

From Proposition 4.2 we get:

**Corollary 4.4** Let \( N \) be a connected special Lagrangian \( m \)-fold in \( \mathbb{C}^m \), and set \( G = \text{Sym}^0(N) \subseteq \text{SU}(m) \ltimes \mathbb{C}^m \). Then \( G \) admits a moment map \( \mu : \mathbb{C}^m \to g^* \), and \( N \subseteq \mu^{-1}(c) \) for some \( c \in Z(g^*) \).

### 4.1 Special Lagrangian \( m \)-folds with cohomogeneity one

Let \( N \) be a special Lagrangian submanifold of \( \mathbb{C}^m \). The symmetry group \( \text{Sym}(N) \) was defined above to be the Lie subgroup of \( \text{SU}(m) \ltimes \mathbb{C}^m \) preserving
Now it is a general principle that the easiest geometric objects to construct are those with large symmetry groups. It can be shown that all homogeneous special Lagrangian submanifolds are affine subspaces $\mathbb{R}^m$ in $\mathbb{C}^m$, which are not very interesting.

The next most symmetric kinds of special Lagrangian submanifold $N$ are those of cohomogeneity one, that is, where the orbits of the symmetry group are of codimension one in $N$. For these we can prove the following theorem.

**Theorem 4.5** Let $G$ be a connected Lie subgroup of $\text{SU}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$ and moment map $\mu : \mathbb{C}^m \to \mathfrak{g}^*$, let $\mathcal{O}$ be an oriented orbit of $G$ in $\mathbb{C}^m$ with $\dim \mathcal{O} = m - 1$, and suppose $\mathcal{O} \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. Then there exists a locally unique, $G$-invariant special Lagrangian submanifold $N$ in $\mathbb{C}^m$ containing $\mathcal{O}$. Furthermore $N \subset \mu^{-1}(c)$, and $N$ is fibred by $G$-orbits isomorphic to $\mathcal{O}$ near $\mathcal{O}$. Thus $N$ is locally diffeomorphic to $(-\epsilon, \epsilon) \times \mathcal{O}$, for some $\epsilon > 0$, and we can think of $N$ as a smooth curve of $G$-orbits.

**Proof.** Proposition 4.1 shows that $\omega|_{\mathcal{O}} \equiv 0$, and as $\mathcal{O}$ is a $G$-orbit it is a real analytic submanifold of $\mathbb{C}^m$, with dimension $m - 1$ by assumption. Therefore Theorem 3.3 shows that there exists a locally unique SL submanifold $N$ in $\mathbb{C}^m$ containing $\mathcal{O}$. But $\mathcal{O}$ is $G$-invariant and $G \subset \text{SU}(m) \ltimes \mathbb{C}^m$, so $N$ must also be (locally) $G$-invariant, by local uniqueness. Hence $N \subseteq \mu^{-1}(c)$, by Proposition 4.2.

We can also construct $N$ by an evolution equation, as in Theorem 3.3. Choose $x \in \mathcal{O}$, and let $H$ be the stabilizer of $x$ in $G$. Set $P = G/H$, and define $\phi : P \to \mathbb{C}^m$ by $\phi(\gamma H) = \gamma x$ for all $\gamma \in G$. Then $\phi$ is an immersion, with $\phi(P) = \mathcal{O}$. Clearly $P$ and $\phi$ are real analytic, and $\phi^*(\omega) \equiv 0$ on $P$.

As $P \cong \mathcal{O}$ is oriented and $G$ acts on it by isometries, we can choose a nonvanishing, $G$-invariant section $\chi$ of $\Lambda^{m-1}TP$. Suppose for the moment that $P$ is compact. Then Theorem 3.3 applies to give an embedding $\Phi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m$, whose image is an open subset of $N$. As $\chi$ is $G$-invariant and $\phi$ is $G$-equivariant, we see by uniqueness that $\Phi$ is equivariant under the actions of $G$ on $P$ and $\mathbb{C}^m$.

Thus $\Phi((t) \times P)$ is a $G$-orbit in $\mathbb{C}^m$ isomorphic to $\mathcal{O}$ for $t \in (-\epsilon, \epsilon)$, and $N$ is fibred by a smooth $1$-parameter family of $G$-orbits isomorphic to $\mathcal{O}$ near $\mathcal{O}$. This completes the proof, except that we assumed $P$ was compact to apply Theorem 3.3. This assumption is in fact unnecessary. As in the discussion after Theorem 3.3, for any $p \in P$ the $\phi_t$ exist near $p$ for $t \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$. We can then use $G$-equivariance to extend $\phi_t$ uniquely to all of $P$. □

This means that special Lagrangian submanifolds of cohomogeneity one in $\mathbb{C}^m$ are relatively easy to construct and classify. The strategy is to first identify all the suitable Lie subgroups $G$ in $\text{SU}(m) \ltimes \mathbb{C}^m$ which admit moment maps. Note that though not all subgroups of $\text{SU}(m) \ltimes \mathbb{C}^m$ admit moment maps, the symmetry group $\text{Sym}(N)$ of a Lagrangian submanifold always admits a moment map, and so subgroups without moment maps are excluded as they cannot preserve any special Lagrangian submanifolds.
Once we have chosen a Lie subgroup $G$ with moment map $\mu$, we then work out the types of $G$-orbit $O$ in $\mu^{-1}(c)$ for $c \in Z(g^*)$, and see if any have dimension $m - 1$. Clearly we must have $\dim G \geq m - 1$ to get any suitable orbits. But if $\dim G$ is too large then there won’t be any suitable orbits either.

We then find the cohomogeneity one $G$-invariant special Lagrangian $m$-folds in $\mathbb{C}^m$ by solving a first-order ordinary differential equation in $(m - 1)$-dimensional $G$-orbits. This can often be done explicitly, or failing this, a qualitative description of the solutions can be given.

Theorem 4.5 says only that this o.d.e. is soluble for small $t$. Solutions generally exist in some open interval $I$ in $\mathbb{R}$. As $t$ approaches the ends of the interval, two things can happen: the orbit $\phi_t(O)$ can go off to infinity, or it can collapse down to another $G$-orbit of smaller dimension. By including this $G$-orbit in $N$, one sometimes gets a closed, nonsingular submanifold in $\mathbb{C}^m$.

5 Examples of cohomogeneity one SL $m$-folds

We now give some examples of cohomogeneity one special Lagrangian $m$-folds in $\mathbb{C}^m$, as in §4.3. We begin with three examples in $\mathbb{C}^3$, the first taken from Harvey and Lawson [6, §III.3.A].

Example 5.1 Let $G \cong T^2$ be the group of diagonal matrices in SU(3), so that each $\gamma \in G$ acts on $\mathbb{C}^3$ by

$$\gamma : (z_1, z_2, z_3) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3)$$

for some $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ with $\theta_1 + \theta_2 + \theta_3 = 0$. As $G$ is abelian, $Z(g^*) = g^*$, and for any $c \in g^*$ the generic $G$-orbit in $\mu^{-1}(c)$ is a copy of $T^2$, and so 2-dimensional.

Thus we can apply Theorem 4.5 to find a family of $T^2$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$, by solving an ordinary differential equation. Let $a_1, a_2$ and $b$ be real numbers, and define a subset $L_{a_1, a_2, b}$ in $\mathbb{C}^3$ by

$$L_{a_1, a_2, b} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - |z_3|^2 = a_1, \quad |z_2|^2 - |z_3|^2 = a_2, \quad \text{Im}(z_1 z_2 z_3) = b\}.$$

Harvey and Lawson show that $L_{a_1, a_2, b}$ is a $T^2$-invariant SL 3-fold in $\mathbb{C}^3$, and it’s also easy to see that any connected, $T^2$-invariant SL 3-fold is a subset of some $L_{a_1, a_2, b}$.

Here the equations $|z_1|^2 - |z_3|^2 = a_1$ and $|z_2|^2 - |z_3|^2 = a_2$ are the moment maps of $G$, which must be constant on $L_{a_1, a_2, b}$ by Proposition 4.3. The third equation $\text{Im}(z_1 z_2 z_3) = b$ can also be interpreted as a kind of generalized moment map equation, associated to the 3-form $\text{Im } \Omega$.

The $L_{a_1, a_2, b}$ are not all nonsingular. In fact one can show:

- (i) $L_{0,0,0}$ has an isolated singular point at $(0,0,0)$.
- (ii) $L_{r^2,0,0}$ is singular on the circle $\{(z_1, 0, 0) : |z_1| = r\}$ for $r > 0$.
- (iii) $L_{0,r^2,0}$ is singular on the circle $\{(0, z_2, 0) : |z_2| = r\}$ for $r > 0$. 

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(iv) \( L_{r_2,-r_2,0} \) is singular on the circle \( \{(0,0,z_3) : |z_3| = r \} \) for \( r > 0 \).

(v) All other \( L_{a_1,a_2,b} \) are nonsingular, and diffeomorphic to \( \mathbb{R} \times T^2 \).

Observe that \( \mathbb{C}^3 \) is fibred by this family of special Lagrangian 3-folds; that is, there is exactly one passing through each point in \( \mathbb{C}^3 \). Here is a second example, adapted from Harvey and Lawson [6, §III.3.B]. It can also be obtained by applying Theorem 6.4 to the special Lagrangian cone \( \mathbb{R}^3 \) in \( \mathbb{C}^3 \).

**Example 5.2** Let \( G \) be the subgroup \( \text{SO}(3) \) of \( \text{SU}(3) \), embedded as \( 3 \times 3 \) real matrices in the obvious way. Then the moment map of \( G \) is

\[
\mu(z_1, z_2, z_3) = (\text{Im}(z_1ar{z}_2), \text{Im}(z_2ar{z}_3), \text{Im}(z_3\bar{z}_1)).
\]

As \( Z(g^*) = \{0\} \), any \( G \)-invariant special Lagrangian 3-fold lies in \( \mu^{-1}(0) \).

Now all points in \( \mu^{-1}(0) \) may be written as \((\lambda x_1, \lambda x_2, \lambda x_3)\), where \( \lambda \in \mathbb{C} \) and \( x_1, x_2, x_3 \) are real, and normalized so that \( x_1^2 + x_2^2 + x_3^2 = 1 \). Therefore the orbits of \( G \) in \( \mu^{-1}(0) \) are \( O_\lambda \) for \( \lambda \in \mathbb{C} \), where

\[
O_\lambda = \{(\lambda x_1, \lambda x_2, \lambda x_3) : x_j \in \mathbb{R}, \quad x_1^2 + x_2^2 + x_3^2 = 1 \}.
\]

Clearly \( O_0 \) is a point, and \( O_\lambda \cong S^2 \) if \( \lambda \neq 0 \), and \( O_\lambda = O_{-\lambda} \).

We can therefore interpret the o.d.e. on \( G \)-orbits in \( \mu^{-1}(0) \) discussed in Theorem 6.4 as an o.d.e. on \( \lambda \). Calculation shows that it is \( d\lambda/dt = \bar{\lambda}^2 \), where \( \lambda = \lambda(t) \). Hence we see that \( d(\lambda^3)/dt = 3|\lambda|^2 \), which is real, and so \( d(\text{Im}(\lambda^3))/dt = 0 \). Thus the integral curves of the o.d.e. are of the form \( \text{Im}(\lambda^3) = c \) for \( c \in \mathbb{R} \).

So for each \( c \in \mathbb{R} \), define

\[
N_c = \{(\lambda x_1, \lambda x_2, \lambda x_3) : x_j \in \mathbb{R}, \quad x_1^2 + x_2^2 + x_3^2 = 1, \quad \lambda \in \mathbb{C}, \quad \text{Im}(\lambda^3) = c \}.
\]

Then \( N_c \) is a special Lagrangian 3-fold in \( \mathbb{C}^3 \). When \( c = 0 \), it is a singular union of three copies of \( \mathbb{R}^3 \) intersecting at 0, and when \( c \neq 0 \) it is nonsingular, the disjoint union of three copies of \( \mathbb{R} \times S^2 \). Note also that \( N_c = N_{-c} \).

Here is a rather trivial example.

**Example 5.3** Let \( G \) be the subgroup \( U(1) \times \mathbb{R} \) of \( \text{SU}(3) \times \mathbb{C}^3 \), acting by

\[
(e^{i\theta}, x) : (z_1, z_2, z_3) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2, x + z_3),
\]

for \( \theta \in [0,2\pi) \) and \( x \in \mathbb{R} \). The moment map of \( G \) is

\[
\mu(z_1, z_2, z_3) = (|z_1|^2 - |z_2|^2, \text{Im} z_3),
\]

and \( Z(g^*) = g^* = \mathbb{R}^2 \), and \( G \)-orbits are copies of \( S^1 \times \mathbb{R} \) unless \( z_1 = z_2 = 0 \), when they are copies of \( \mathbb{R} \).

As in Example 5.1, we find the following. Let \( a, b, c \in \mathbb{R} \), and define

\[
N_{a,b,c} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - |z_2|^2 = a, \quad \text{Re}(z_1z_2) = b, \quad \text{Im}(z_3) = c \}.
\]
Then $N_{a,b,c}$ is a $G$-invariant special Lagrangian 3-fold in $\mathbb{C}^3$. If $a = b = 0$ then $N_{a,b,c}$ is the singular union of two copies of $\mathbb{R}^3$ intersecting in $\mathbb{R}$, and otherwise $N_{a,b,c}$ is nonsingular and diffeomorphic to $S^3 \times \mathbb{R}^2$.

Again, $\mathbb{C}^3$ is fibred by these $N_{a,b,c}$. Note that $N_{a,b,c}$ is actually the product of lower-dimensional SL submanifolds in $\mathbb{C}^2$ and $\mathbb{C}$. It turns out that these three examples represent all cohomogeneity one SL 3-folds in $\mathbb{C}^3$.

**Theorem 5.4** Every homogeneous special Lagrangian 3-fold in $\mathbb{C}^3$ is conjugate under $SU(3) \ltimes \mathbb{C}^3$ to the standard 3-plane $\mathbb{R}^3 \subset \mathbb{C}^3$. Every cohomogeneity one special Lagrangian 3-fold in $\mathbb{C}^3$ is conjugate under $SU(3) \ltimes \mathbb{C}^3$ to a subset of $\mathbb{R}^3$ or one of the 3-folds of Examples 5.1, 5.2 and 5.3.

We leave the proof as an exercise; one uses the classification of Lie groups to identify all Lie subgroups $G$ of $SU(3) \ltimes \mathbb{C}^3$, and show that either $G$ has no suitable 2-dimensional orbits in $\mu^{-1}(c)$ for $c \in Z(g^*)$, or else that the $G$-invariant submanifolds reduce to one of the cases in the theorem. Note that $Sym(\mathbb{R}^3)$ is $SO(3) \ltimes \mathbb{R}^3$, which has several Lie subgroups $G$ leading to subsets of $\mathbb{R}^3$.

Next we give some higher-dimensional examples. Here is an example generalizing Example 5.1, taken from [1] §III.3.A.

**Example 5.5** Let $G \cong T^{m-1}$ be the group of diagonal matrices in $SU(m)$, so that each $\gamma \in G$ acts on $\mathbb{C}^m$ by

$$
\gamma : (z_1, \ldots, z_m) \mapsto (e^{i\theta_1}z_1, \ldots, e^{i\theta_m}z_m)
$$

for some $\theta_1, \ldots, \theta_m \in \mathbb{R}$ with $\theta_1 + \cdots + \theta_m = 0$.

Let $a_1, \ldots, a_{m-1}$ and $b$ be real numbers. Define $L_{a_1, \ldots, a_{m-1}, b}$ by

$$
L_{a_1, \ldots, a_{m-1}, b} = \left\{ (z_1, \ldots, z_m) \in \mathbb{C}^m : |z_j|^2 - |z_m|^2 = a_j \text{ for } j = 1, \ldots, m-1, \right. \\
\left. \quad \text{and } \left\{ \begin{array}{ll}
\text{Re}(z_1 \ldots z_m) = b & \text{if } m \text{ is even} \\
\text{Im}(z_1 \ldots z_m) = b & \text{if } m \text{ is odd}
\end{array} \right. \right\}.
$$

Then Harvey and Lawson show that $L_{a_1, \ldots, a_{m-1}, b}$ is a special Lagrangian $m$-fold in $\mathbb{C}^m$. If $b \neq 0$ then $L_{a_1, \ldots, a_{m-1}, b}$ is nonsingular and diffeomorphic to $T^{m-1} \times \mathbb{R}$. When $b = 0$, it may be nonsingular or have various kinds of singularity, depending on the values of $a_1, \ldots, a_{m-1}$.

Again, $\mathbb{C}^m$ is fibred by these special Lagrangian $m$-folds. Here is another example of Harvey and Lawson [1] §III.3.B, generalizing Example 5.2. It can also be derived from Theorem 6.4 with $C = \mathbb{R}^m$ in $\mathbb{C}^m$.

**Example 5.6** Let $G$ be $SO(m)$ in $SU(m)$. For each $c \in \mathbb{R}$, define

$$
N_c = \left\{ (\lambda x_1, \ldots, \lambda x_m) : x_j \in \mathbb{R}, \quad x_1^2 + \cdots + x_m^2 = 1, \quad \lambda \in \mathbb{C}, \quad \text{Im}(\lambda^m) = c \right\}.
$$

Then $N_c$ is a special Lagrangian $m$-fold in $\mathbb{C}^m$. When $c = 0$, it is a singular union of $m$ copies of $\mathbb{R}^m$ intersecting at 0. If $m$ is even and $c \neq 0$, $N_c$ is a
nonsingular, disjoint union of $m/2$ copies of $\mathbb{R} \times S^{m-1}$. If $m$ is odd and $c \neq 0$, $N_c$ is a nonsingular, disjoint union of $m$ copies of $\mathbb{R} \times S^{m-1}$, and $N_c = N_{-c}$.

We conclude with a more complex example due to Marshall [15, §3.4].

**Example 5.7** The usual action of SU(2) on $\mathbb{C}^2$ induces an action of SU(2) on $S^3 \mathbb{C}^2$. Identifying $S^3 \mathbb{C}^2$ with $\mathbb{C}^4$ in an appropriate way, this defines a subgroup $G$ of SU(4) isomorphic to SU(2). Calculation shows that we may take the Lie algebra $\mathfrak{g}$ of $G$ to be spanned by

$$
\begin{pmatrix}
3i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & -3i
\end{pmatrix},
\begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
-\sqrt{3} & 0 & 2 & 0 \\
0 & -2 & 0 & \sqrt{3} \\
0 & 0 & -\sqrt{3} & 0
\end{pmatrix},
\begin{pmatrix}
0 & i\sqrt{3} & 0 & 0 \\
i\sqrt{3} & 0 & 2i & 0 \\
0 & 2i & 0 & i\sqrt{3} \\
0 & 0 & i\sqrt{3} & 0
\end{pmatrix}.
$$

For each $d \in \mathbb{R}$, define $N_d \subset \mathbb{C}^4$ by

$$
N_d = \left\{ (z_1, \ldots, z_4) \in \mathbb{C}^4 : \sqrt{3}(z_1 \bar{z}_2 + z_3 \bar{z}_4) + 2z_2 \bar{z}_3 = 0, \\
3|z_1|^2 + |z_2|^2 - |z_3|^2 - 3|z_4|^2 = 0, \\
\text{Im}\left(2\sqrt{3}(z_1 z_3^2 + z_2 z_4^2) - 9z_1 z_2 z_3 z_4 + \frac{9}{2}z_1^2 z_3^2 + \frac{3}{2}z_2^2 z_4^2 = d\right) \right\}.
$$

Then $N_d$ is a $G$-invariant SL 4-fold in $\mathbb{C}^4$. Here the first two equations defining $N_d$ say that the moment map $\mu$ is zero.

It can be shown that $N_0$ is the union of two cones on $S^3/\mathbb{Z}_3$, with one singular point at 0, where if we identify $S^3$ with the unit sphere in $\mathbb{C}^2$, then the $\mathbb{Z}_3$-action is generated by $(w_1, w_2) \mapsto (e^{2\pi i/3} w_1, e^{2\pi i/3} w_2)$. Similarly, if $d \neq 0$ then $N_d$ is nonsingular and diffeomorphic to $\mathbb{R} \times S^3/\mathbb{Z}_3$.

### 6 Special Lagrangian cones in $\mathbb{C}^m$

In §4 we studied symmetries of special Lagrangian $m$-folds in $\mathbb{C}^m$, which were required to preserve the metric $g$, Kähler form $\omega$ and complex volume form $\Omega$ on $\mathbb{C}^m$. However, there is a more general kind of automorphism of $\mathbb{C}^m$ which does not preserve $g$, $\omega$ and $\Omega$, but does preserve the idea of SL submanifolds in $\mathbb{C}^m$.

**Definition 6.1** Let $\mathbb{R}_+$ be the group of positive real numbers under multiplication, and let $t \in \mathbb{R}_+$ act on $\mathbb{C}^m$ by $(z_1, \ldots, z_m) \mapsto t(z_1, \ldots, tz_m)$. We call this action of $t$ on $\mathbb{C}^m$ a dilation. Let $N$ be a real $m$-dimensional submanifold of $\mathbb{C}^m$. Then $tN$ is also a real $m$-dimensional submanifold of $\mathbb{C}^m$ for each $t \in \mathbb{R}_+$. Clearly, $N$ is special Lagrangian if and only if $tN$ is, so that dilations preserve the idea of SL submanifolds in $\mathbb{C}^m$.

Combining dilations with the automorphisms $\text{SU}(m) \ltimes \mathbb{C}^m$ of $\mathbb{C}^m$ gives a group $(\mathbb{R}_+ \times \text{SU}(m)) \ltimes \mathbb{C}^m$ acting on $\mathbb{C}^m$ preserving SL submanifolds. If $N$ is a
special Lagrangian submanifold of \( \mathbb{C}^m \), define the \textit{generalized symmetry group} \( \text{GSym}(N) \) of \( N \) to be the Lie subgroup \( G \) of \( (\mathbb{R}_+ \times \text{SU}(m)) \ltimes \mathbb{C}^m \) preserving \( N \), and define the \textit{restricted generalized symmetry group} \( \text{GSym}^0(N) \) to be the identity component of \( \text{GSym}(N) \).

The symmetry group \( \text{Sym}(N) \) of \( N \) is a normal subgroup of \( \text{GSym}(N) \), and \( \text{GSym}(N)/\text{Sym}(N) \) is a subgroup of \( \mathbb{R}_+ \). It is convenient to distinguish between \( \text{Sym}(N) \) and \( \text{GSym}(N) \), because \( \text{Sym}^0(N) \) always admits a moment map, but in general \( \text{GSym}^0(N) \) has no moment map as it doesn’t preserve \( \omega \).

Submanifolds invariant under dilations are called \textit{cones}.

\[ \text{Definition 6.2} \] Let \( C \) be a (singular) submanifold of \( \mathbb{C}^m \). We call \( N \) a \textit{cone} in \( \mathbb{C}^m \), with vertex 0, if \( 0 \in C \) and \( tC = C \) for all \( t \in \mathbb{R}_+ \). That is, \( C \) is a cone if it is invariant under dilations. Let \( C \) be a cone, and define \( \Sigma = \{ z \in C : |z| = 1 \} = C \cap S^{2m-1} \). Then \( C = \{ tz : z \in \Sigma, t \in [0, \infty) \} \). We call \( \Sigma \) the \textit{link} of \( C \), and \( C \) the cone on \( \Sigma \).

A closed submanifold \( N \) in \( \mathbb{C}^m \) is called \textit{Asymptotically Conical}, or \( AC \) for short, if there exists a closed cone \( N_0 \) in \( \mathbb{C}^m \) with isolated singular point at 0, such that \( N \) is asymptotic to \( N_0 \) to order \( O(r^{-1}) \) as \( r \to \infty \), where \( r \) is the radius function in \( \mathbb{C}^m \). We call \( N_0 \) the \textit{asymptotic cone} of \( N \).

If \( N \) is a special Lagrangian cone, then \( \text{GSym}(N) = \mathbb{R}_+ \ltimes \text{Sym}(N) \). We will be particularly interested in conical and asymptotically conical SL submanifolds in \( \mathbb{C}^m \), as they can provide local models for singularities of SL \( m \)-folds in Calabi–Yau \( m \)-folds. Here is why. Let \( N \) be a nonsingular \( AC \) special Lagrangian \( m \)-fold in \( \mathbb{C}^m \), asymptotic to a singular cone \( N_0 \). If \( t > 0 \) then \( tN \) is also \( AC \), and \( tN \to N_0 \) as \( t \to 0^+ \).

Thus the singular SL submanifold \( N_0 \) is the limit of the family of nonsingular SL submanifolds \( \{ tN : t > 0 \} \). So \( AC \) special Lagrangian submanifolds provide local models for how singularities can develop in families of nonsingular SL submanifolds.

### 6.1 Special Lagrangian cones of cohomogeneity one

In §4.1, we studied special Lagrangian \( m \)-folds \( N \) in \( \mathbb{C}^m \) upon which \( \text{Sym}(N) \) acts with cohomogeneity one. We can also consider \( N \) upon which the \textit{generalized} symmetry group \( \text{GSym}(N) \) above acts with cohomogeneity one. In particular, if \( N \) is a cone then \( \text{GSym}(N) = \mathbb{R}_+ \ltimes \text{Sym}(N) \), so that \( \text{GSym}(N) \) has cohomogeneity one when \( \text{Sym}(N) \) has cohomogeneity two. Here is a result on cohomogeneity one cones, similar to Theorem 4.5.

\[ \text{Theorem 6.3} \] Let \( G \) be a connected Lie subgroup of \( \text{SU}(m) \) with Lie algebra \( \mathfrak{g} \) and moment map \( \mu : \mathbb{C}^m \to \mathfrak{g}^* \), with \( \mu(0) = 0 \). Let \( \mathcal{O} \) be an oriented orbit of \( \mathbb{R}_+ \ltimes G \) in \( \mathbb{C}^m \) with \( \dim \mathcal{O} = m-1 \), and suppose \( \mathcal{O} \subset \mu^{-1}(0) \). Then there exists a locally unique, \( \mathbb{R}_+ \ltimes G \)-invariant special Lagrangian cone \( N \) in \( \mathbb{C}^m \) containing \( \mathcal{O} \). Furthermore \( N \subset \mu^{-1}(0) \), and \( N \) is fibred by \( \mathbb{R}_+ \ltimes G \)-orbits isomorphic to \( \mathcal{O} \) near \( \mathcal{O} \). Thus \( N \) is locally diffeomorphic to \( (-\epsilon, \epsilon) \times \mathcal{O} \), for some \( \epsilon > 0 \), and we can think of \( N \) as a smooth curve of \( \mathbb{R}_+ \times G \)-orbits.
The proof is very similar to Theorem 4.5, so we will not give it, but here are a few comments. Since we are constructing cones in \( \mathbb{C}^m \) we exclude translations from our symmetry group, as they would move the vertex. Thus we take \( G \subset SU(m) \) rather than \( G \subset SU(m) \ltimes \mathbb{C}^m \). The moment map \( \mu \) for \( G \) always exists, and we specify it uniquely by requiring that \( \mu(0) = 0 \).

Suppose \( N \) is an \( (\mathbb{R}^+ \times G) \)-invariant special Lagrangian cone. Then \( N \subset \mu^{-1}(c) \) for \( c \in Z(\mathfrak{g}^*) \), by Proposition 4.2. But \( \mu(tz) = t^2 \mu(z) \) for \( t \in \mathbb{R}^+ \), which forces \( c = t^2 c \) as \( N = tN \), so \( c = 0 \). Thus \( N \subset \mu^{-1}(0) \). Conversely, following Proposition 4.1 one can show that if \( O \) is an \( \mathbb{R}^+ \times G \)-orbit then \( \omega|_O \equiv 0 \) if and only if \( O \subset \mu^{-1}(0) \), and this is used to apply Theorems 3.2 and 3.3.

### 6.2 AC special Lagrangian \( m \)-folds from cones

We shall now show that given any special Lagrangian cone in \( \mathbb{C}^m \), one can automatically construct a 1-parameter family of asymptotically conical SL \( m \)-folds in \( \mathbb{C}^m \) from it. This was first noticed by Castro and Urbano [2, Remark 1, p. 81–2] in the context of minimal Lagrangian submanifolds, proved independently by Haskins [7, Th. A], and also independently by the author.

**Theorem 6.4** Let \( C \) be a closed special Lagrangian cone in \( \mathbb{C}^m \) with isolated singular point 0, set \( \Sigma = \{ z \in \mathbb{C} : |z| = 1 \} \), and for each \( c > 0 \) define

\[
N_c = \left\{ \lambda z : z \in \Sigma, \lambda \in \mathbb{C}, \operatorname{Im}(\lambda^m) = cm, \arg(\lambda) \in (0, \pi/m) \right\} = \left\{ c(\sin(m \theta))^{-1/m} e^{i \theta} z : z \in \Sigma, \theta \in (0, \pi/m) \right\}.
\]

Then \( N_c \) is an immersed AC special Lagrangian \( m \)-fold in \( \mathbb{C}^m \) diffeomorphic to \( \Sigma \times \mathbb{R} \), and asymptotic to \( C \cup e^{i \pi/m} C \).

**Proof.** One can prove this quite simply by relating the tangent spaces \( T_p N_c \) of \( N_c \) to those of \( \Sigma \), and showing that each \( T_p N_c \) is special Lagrangian. But we will instead give a proof using Theorem 3.3.

Now \( \Sigma \) is a compact, nonsingular Riemannian \( (m - 1) \)-manifold, with a natural orientation. Let \( \chi \) be the unique positive section of \( \Lambda^{m-1} T \Sigma \) with \( |\chi| = 1 \). It follows easily from Theorem 3.3 that \( \Sigma \) and \( \chi \) are real analytic. Consider a 1-parameter family \( \{ \phi_t : t \in (-\epsilon, \epsilon) \} \) of maps \( \phi_t : \Sigma \to \mathbb{C}^m \) defined by \( \phi_t : z \mapsto \lambda(t) z \), where \( \lambda(t) : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\} \) is a differentiable function.

Calculation shows that the \( \phi_t \) satisfy the evolution equation (1) of Theorem 3.3 if and only if \( \lambda \) satisfies the o.d.e.

\[
\frac{d\lambda}{dt} = \lambda^{m-1}.
\]

But then \( d(\lambda^m)/dt = m\lambda^{m-1} \lambda = m|\lambda|^{2m-2} \), which is real. So \( \operatorname{Im}(\lambda^m) \) is constant along the integral curves of (1). In particular, if \( c > 0 \) then \( \{ \lambda \in \mathbb{C} : \operatorname{Im}(\lambda^m) = cm, \arg(\lambda) \in (0, \pi/m) \} \) is an integral curve of (1), and the result then follows quickly from Theorem 3.3. \( \square \)
7 \textit{U(1)}^{m-2}\text{-invariant cones in } \mathbb{C}^{m}

We shall now apply Theorem 6.3 to the example of \( G = \text{U}(1)^{m-2} \) in SU\( (m) \). The case \( m = 3 \) has already been analyzed by Mark Haskins [6, §3–§5], using somewhat different techniques.

Let \( m \geq 3 \) and \( a_1 \leq \cdots \leq a_m \) be integers, not all zero, with highest common factor 1, and with \( a_1 + \cdots + a_m = 0 \). Note that this implies that \( a_1 < 0 \) and \( a_m > 0 \). Define a subgroup \( G \subset \text{U}(1)^m \) to be
\[
\{(e^{i\alpha_1}, \ldots, e^{i\alpha_m}) \in \text{U}(1)^m : \alpha_j \in \mathbb{R}, \alpha_1 + \cdots + \alpha_m = 0, \ a_1 \alpha_1 + \cdots + a_m \alpha_m = 0\}.
\]

Then \( G \cong \text{U}(1)^{m-2} \), because the \( a_j \) are integers. If we instead allowed the \( a_j \) to be real numbers then \( G \) would be isomorphic to \( \text{U}(1)^k \times \mathbb{R}^{m-2-k} \) for some \( 0 \leq k \leq m - 2 \). Most of the analysis below would still work, but the resulting \( \text{SL} \) \( m \)-folds would not be as interesting.

Let \( G \) act on \( \mathbb{C}^m \) in the obvious way, by
\[
(e^{i\alpha_1}, \ldots, e^{i\alpha_m} : (z_1, \ldots, z_m) \mapsto (e^{i\alpha_1}z_1, \ldots, e^{i\alpha_m}z_m).
\]

Then \( G \subset \text{SU}(m) \). We shall apply Theorem 6.3 to construct \( \mathbb{R}_+ \times G \)-invariant \( \text{SL} \) \( m \)-folds \( N \) in \( \mathbb{C}^m \), which we will regard as the total space of a 1-parameter family of \( \mathbb{R}_+ \times G \)-orbits \( O_t \) for \( t \in (-\epsilon, \epsilon) \).

So, for \( t \in (-\epsilon, \epsilon) \) define \( \phi_t : \mathbb{R}_+ \times G \to \mathbb{C}^m \) by
\[
\phi_t : (r, (u^{i\alpha_1}, \ldots, u^{i\alpha_m})) \mapsto (re^{i\alpha_1}w_1(t), \ldots, re^{i\alpha_m}w_m(t))
\]
for \( r \in \mathbb{R}_+ \) and \( (e^{i\alpha_1}, \ldots, e^{i\alpha_m}) \in G \), where \( w_1, \ldots, w_m : (-\epsilon, \epsilon) \to \mathbb{C} \) are differentiable functions. Let \( O_t = \text{Im} \phi_t \), and let \( N \) be the union of the \( O_t \) for \( t \in (-\epsilon, \epsilon) \). We will shortly use equation (1) of Theorem 5.3 to derive an evolution equation for \( \phi_t \) and \( w_1, \ldots, w_m \), which will imply that \( N \) is special Lagrangian in \( \mathbb{C}^m \), with phase \( i^{m-2} \).

First, we consider what the conditions that \( O \subset \mu^{-1}(0) \) and \( \dim O = m - 1 \) in Theorem 6.3 mean for \( w_1, \ldots, w_m \). The Lie algebra of \( \text{U}(1)^m \) is \( \mathbb{R}^m \), and the Lie algebra of \( G \) is the subspace of \( (x_1, \ldots, x_m) \in \mathbb{R}^m \) such that \( x_1 + \cdots + x_m = 0 \) and \( a_1x_1 + \cdots + a_m x_m = 0 \). Thus, the condition that \( (w_1, \ldots, w_m) \in \mu^{-1}(0) \) is that \( x_1|w_1|^2 + \cdots + x_m|w_m|^2 = 0 \) whenever \( x_1, \ldots, x_m \in \mathbb{R}, x_1 + \cdots + x_m = 0 \) and \( a_1x_1 + \cdots + a_m x_m = 0 \).

Clearly, this is true if and only if
\[
|w_j|^2 = a_j u + v, \quad j = 1, \ldots, m, \quad \text{for some } u, v \in \mathbb{R}.
\]

This is also the condition that \( O_t \subset \mu^{-1}(0) \), and by the discussion after Theorem 5.3 this is equivalent to \( \omega|O_t| \equiv 0 \). Also, \( \dim O_t = m - 1 \) holds if no more than one of \( w_1(t), \ldots, w_m(t) \) are zero for all \( t \in (-\epsilon, \epsilon) \). For simplicity we make the stronger assumption that \( w_j(t) \) is nonzero for all \( j \) and \( t \).

Next, we use equation (1) to derive an o.d.e. for \( \phi_t \) and \( w_1, \ldots, w_m \). To do this we need an \( \mathbb{R}_+ \times G \)-invariant section \( \chi \) of \( \Lambda^{m-1}T(\mathbb{R}_+ \times G) \). In terms of the
natural local coordinates $r$ and $\alpha_1, \ldots, \alpha_m$ on $\mathbb{R}_+ \times U(1)^m$, calculation shows that we may take $\chi$ to be

$$\chi = \frac{2r}{m} \frac{\partial}{\partial r} \wedge \sum_{1 \leq j < k \leq m} (-1)^{j+k-1} (a_j - a_k) \partial_1 \wedge \cdots \wedge \partial_{j-1} \wedge \partial_{j+1} \wedge \cdots \wedge \partial_{k-1} \wedge \partial_{k+1} \wedge \cdots \wedge \partial_m,$$

where $\partial_j = \partial/\partial \alpha_j$. To apply (1), we need an expression for $(\phi_t)_*(\chi)$ at a point $(z_1, \ldots, z_m)$ in $\mathcal{O}_t$. Using the equations

$$ (\phi_t)_* \left( \frac{\partial}{\partial r} \right) = z_1 \frac{\partial}{\partial z_1} + \cdots + z_m \frac{\partial}{\partial z_m} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \cdots + \bar{z}_m \frac{\partial}{\partial \bar{z}_m} $$

and

$$ (\phi_t)_* \left( \partial_j \right) = iz_j \frac{\partial}{\partial z_j} - i \bar{z}_j \frac{\partial}{\partial \bar{z}_j} $$

we can do this, and the result is rather complicated.

But all we will actually need is the $(m-1,0)$ component of $(\phi_t)_*(\chi)$, throwing away all terms in $\partial/\partial \bar{z}_j$, and calculation shows that this is given by

$$ (\phi_t)_*(\chi)^{(m-1,0)} = 2i^{m-2} \sum_{j=1}^m (-1)^{j-1} a_j z_1 \cdots z_{j-1} z_{j+1} \cdots z_m \cdot \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_{j-1}} \wedge \frac{\partial}{\partial z_{j+1}} \wedge \cdots \wedge \frac{\partial}{\partial z_m}. $$

As $\Omega$ is an $(m,0)$-tensor, we see that the contraction of $(\phi_t)_*(\chi)$ with $\Omega$ is the same as that of $(\phi_t)_*(\chi)^{(m-1,0)}$ with $\Omega$. Hence, using the index notation for tensors on $\mathbb{C}^m$, we get

$$ (\phi_t)_*(\chi)^{b_1 \cdots b_{m-1} \Omega_{b_1 \cdots b_{m-1} b_m}} = 2i^{m-2} \sum_{j=1}^m a_j z_1 \cdots z_{j-1} z_{j+1} \cdots z_m (dz_j) b_m. $$

Multiplying by $(-i)^{m-2}$ and contracting with $g^{b_m c}$ gives

$$ (\phi_t)_*(\chi)^{b_1 \cdots b_{m-1} (-i)^{m-2} \Omega_{b_1 \cdots b_{m-1} b_m}} g^{b_m c} = 2 \sum_{j=1}^m a_j z_1 \cdots z_{j-1} z_{j+1} \cdots z_m \left( \frac{\partial}{\partial z_j} \right)^c. $$

Since $(\phi_t)_*(\chi)$ and $g$ are real tensors, taking real parts gives

$$ (\phi_t)_*(\chi)^{b_1 \cdots b_{m-1} \text{Re}((-i)^{m-2} \Omega)_{b_1 \cdots b_{m-1} b_m}} g^{b_m c} = \sum_{j=1}^m a_j z_1 \cdots z_{j-1} z_{j+1} \cdots z_m \left( \frac{\partial}{\partial z_j} \right)^c + \sum_{j=1}^m a_j z_1 \cdots z_{j-1} z_{j+1} \cdots z_m \left( \frac{\partial}{\partial \bar{z}_j} \right)^c. $$
Now by Theorem 3.3, a sufficient condition for \( N \) to be special Lagrangian with phase \( i^{m-2} \) is
\[
\left( \frac{d\phi_t}{dt} \right)^c = (\phi_t)_*(\chi^{b_1...b_{m-1}}(\text{Re}(i)^{m-2}\Omega)_{b_1...b_{m-1}b_m}g^{b_m}),
\]
where we have inserted the factor \((-i)^{m-2}\) in front of \( \Omega \) to get \( N \) of phase \( i^{m-2} \) rather than 1.

Setting \((z_1, \ldots, z_m)\) to be \( \phi_t(1,1, \ldots, 1) = (w_1, \ldots, w_m) \) and combining the last two equations, we find that a sufficient condition for \( N \) to be special Lagrangian with phase \( i^{m-2} \) is
\[
\sum_{j=1}^m dw_j \left( \frac{\partial}{\partial z_j} \right)^c + \sum_{j=1}^m d\bar{w}_j \left( \frac{\partial}{\partial \bar{z}_j} \right)^c = \sum_{j=1}^m a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m \left( \frac{\partial}{\partial z_j} \right)^c + \sum_{j=1}^m a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m \left( \frac{\partial}{\partial \bar{z}_j} \right)^c.
\]

Equating coefficients, this is true if
\[
\frac{dw_j}{dt} = a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m \quad \text{for} \quad j = 1, \ldots, m. \tag{10}
\]

Now we assumed above that (9) holds for \( w_1, \ldots, w_m \), so we should check that our evolution equation (10) preserves \( w_j \) of this form. From (10) we get
\[
\frac{d|w_j|^2}{dt} = w_j \frac{d\bar{w}_j}{dt} + \bar{w}_j \frac{dw_j}{dt} = a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m = 2a_j \text{Re}(w_1 \cdots w_m).
\]

Thus the evolution (10) does preserve \( w_1, \ldots, w_m \) of the form (9), and \( u, v \) are functions of \( t \) satisfying
\[
\frac{du}{dt} = 2 \text{Re}(w_1 \cdots w_m), \quad \frac{dv}{dt} = 0.
\]

So \( v \) is constant. Since \( a_1 + \cdots + a_m = 0 \) we have \( |w_1|^2 + \cdots + |w_m|^2 = mv \), and \( v > 0 \). As rescaling all the \( z_j \) by a positive constant leads to the same SL \( m \)-fold, we may as well fix \( v = 1 \).

We summarize our progress so far in the following theorem.

**Theorem 7.1** Suppose \( w_1, \ldots, w_m : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\} \) and \( u : (-\epsilon, \epsilon) \to \mathbb{R} \) are differentiable functions, satisfying
\[
\frac{dw_j}{dt} = a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m, \quad j = 1, \ldots, m, \tag{11}
\]
\[
\frac{du}{dt} = 2 \text{Re}(w_1 \cdots w_m), \quad \text{and} \tag{12}
\]
\[
|w_j|^2 = a_j u + 1, \quad j = 1, \ldots, m. \tag{13}
\]
If (11) and (12) hold for all $t \in (-\epsilon, \epsilon)$ and (13) holds for $t = 0$, then (13) holds for all $t$. Define a subset $N$ of $\mathbb{C}^m$ by

$$N = \left\{ (r e^{i\alpha_1} w_1(t), \ldots, r e^{i\alpha_m} w_m(t)) : r > 0, \quad t \in (-\epsilon, \epsilon), \quad \alpha_j \in \mathbb{R}, \quad \alpha_1 + \cdots + \alpha_m = 0, \quad a_1\alpha_1 + \cdots + a_m\alpha_m = 0 \right\}. \quad (14)$$

Then $N$ is a special Lagrangian submanifold in $\mathbb{C}^m$ with phase $i^{m-2}$.

7.1 Rewriting these equations

We will eventually solve equations (11)–(13) of Theorem 7.1 fairly explicitly. First, we find a way to rewrite them using fewer variables. Set

$$w_j(t) = e^{i\theta_j(t)} \sqrt{a_j u(t) + 1}$$

for differentiable functions $\theta_1, \ldots, \theta_m : (-\epsilon, \epsilon) \to \mathbb{R}$. Define

$$\theta = \theta_1 + \cdots + \theta_m \quad \text{and} \quad Q(u) = \prod_{j=1}^{m} (a_j u + 1).$$

Then calculation shows that (11)–(13) are equivalent to

$$\frac{du}{dt} = 2 \text{Re}(w_1 \cdots w_m) = 2Q(u)^{1/2} \cos \theta \quad \text{and} \quad (15)$$

$$\frac{d\theta_j}{dt} = -\frac{a_j Q(u)^{1/2} \sin \theta}{a_j u + 1}, \quad j = 1, \ldots, m. \quad (16)$$

Summing (16) from $j = 1$ to $m$ gives

$$\frac{d\theta}{dt} = -Q(u)^{1/2} \sin \theta \sum_{j=1}^{m} a_j \frac{1}{a_j u + 1}. \quad (17)$$

Now although the point $(w_1, \ldots, w_m)$ is determined by the $m + 1$ real variables $\theta_1, \ldots, \theta_m$ and $u$, we are actually only interested in the $\mathbb{R}_+ \times G$-orbit $O_t$ of $(w_1, \ldots, w_m)$. It is not difficult to show that

$$O_t = \left\{ (r e^{i\alpha_1} \sqrt{u_1 u + 1}, \ldots, r e^{i\alpha_m} \sqrt{a_m u + 1}) : r > 0, \quad \alpha_j \in \mathbb{R}, \quad \alpha_1 + \cdots + \alpha_m = \theta, \quad a_1\alpha_1 + \cdots + a_m\alpha_m = \psi \right\},$$

where $\psi = a_1\theta_1 + \cdots + a_m\theta_m$. Thus, $O_t$ depends only on the three real variables $\theta, \psi$ and $u$. Furthermore, (16) shows that $\psi$ evolves by

$$\frac{d\psi}{dt} = -Q(u)^{1/2} \sin \theta \sum_{j=1}^{m} a_j^2 \frac{1}{a_j u + 1}. \quad (18)$$

Thus we may rewrite Theorem 7.1 as follows:
Theorem 7.2

Let \( u, \theta, \psi : (-\epsilon, \epsilon) \to \mathbb{R} \) be differentiable functions satisfying

\[
\frac{du}{dt} = 2Q(u)^{1/2} \cos \theta,
\]

(17)

\[
\frac{d\theta}{dt} = -Q(u)^{1/2} \sin \theta \sum_{j=1}^{m} \frac{a_j}{a_j u + 1}
\]

(18)

and

\[
\frac{d\psi}{dt} = -Q(u)^{1/2} \sin \theta \sum_{j=1}^{m} \frac{a_j^2}{a_j u + 1}.
\]

(19)

such that \( a_j u(t) + 1 > 0 \) for \( j = 1, \ldots, m \) and \( t \in (-\epsilon, \epsilon) \). Define a subset \( N \) of \( \mathbb{C}^m \) to be

\[
\left\{ \left( r e^{i\alpha_1} \sqrt{a_1 u(t) + 1}, \ldots, r e^{i\alpha_m} \sqrt{a_m u(t) + 1} \right) : r > 0, \ t \in (-\epsilon, \epsilon), \ a_j \in \mathbb{R}, \ \alpha_1 + \cdots + \alpha_m = \theta(t), \ a_1 \alpha_1 + \cdots + a_m \alpha_m = \psi(t) \right\}.
\]

(20)

Then \( N \) is a special Lagrangian submanifold in \( \mathbb{C}^m \) with phase \( i^{m-2} \).

This is a significant simplification, as Theorem 7.1 was written in terms of an o.d.e. in \( m \) complex variables \( w_1, \ldots, w_m \), but we have reduced this to only 3 real variables \( u, \theta, \psi \). In fact we can show that \( u \) and \( \theta \) are dependent in a simple way, and so reduce the number of real variables to two.

Suppose for the moment that \( \sin \theta(0) \neq 0 \), and divide (17) by (18). This gives an expression for \( \frac{du}{d\theta} \), eliminating \( t \). Separating variables shows that

\[
\int_{u(0)}^{u(t)} \sum_{j=1}^{m} \frac{a_j}{a_j u + 1} du = -2 \int_{\theta(0)}^{\theta(t)} \cot \theta d\theta,
\]

which integrates explicitly to

\[
\log Q(u) = -2 \log \sin \theta + C
\]

for all \( t \in (-\epsilon, \epsilon) \), for some \( C \in \mathbb{R} \). So exponentiating gives \( Q(u) \sin^2 \theta = e^C > 0 \).

If on the other hand \( \sin \theta(0) = 0 \) then (18) shows that \( \theta \) is constant, so \( Q(u) \sin^2 \theta \equiv 0 \). In both cases we see that \( Q(u) \sin^2 \theta \) is constant, so its square root \( Q(u)^{1/2} \sin \theta \) is also constant, as it is continuous. Thus we have \( Q(u)^{1/2} \sin \theta = A \) for some \( A \in \mathbb{R} \).

This simplifies (18) and (19), as we can replace the factor \( Q(u)^{1/2} \sin \theta \) by \( A \). Also, from (17) we obtain

\[
\left( \frac{du}{dt} \right)^2 = 4Q(u) \cos^2 \theta = 4(Q(u) - Q(u) \sin^2 \theta) = 4(Q(u) - A^2).
\]

Thus we have proved:
Proposition 7.3 In the situation of Theorem 7.2 we have
\[ (Q(u))^{1/2} \sin \theta \equiv A \] (21)
for some \( A \in \mathbb{R} \) and all \( t \in (-\epsilon, \epsilon) \), and (17)-(19) are equivalent to
\[ \left( \frac{du}{dt} \right)^2 = 4(Q(u) - A^2), \] (22)
\[ \frac{d\theta}{dt} = -A \sum_{j=1}^{m} a_j u + 1 \quad \text{and} \quad \frac{d\psi}{dt} = -A \sum_{j=1}^{m} a_j^2 u + 1. \] (23)

The following lemma pins down the ranges of \( u \) and \( A \).

Lemma 7.4 In the situation above, \( u \) lies in \((-a_{m}^{-1}, -a_{1}^{-1})\) for all \( t \in (-\epsilon, \epsilon) \), and \( 0 \leq A^2 \leq Q(u) \leq 1 \). Also, \( A \) lies in \([-1, 1]\).

Proof. As by assumption \( a_1, \ldots, a_m \) are not all zero with \( a_1 \leq \cdots \leq a_m \) and \( a_1 + \cdots + a_m = 0 \), we see that \( a_1 < 0 \) and \( a_m > 0 \), so the interval \((-a_{m}^{-1}, -a_{1}^{-1})\) is well-defined. Since \( a_j u + 1 > 0 \) for \( j = 1, \ldots, m \), we have \( a_1 u + 1 > 0 \) and \( a_m u + 1 > 0 \). Therefore \( u \) is confined to \((-a_{m}^{-1}, -a_{1}^{-1})\).

Now \( Q'(0) = \sum_{j=1}^{m} a_j = 0 \), so \( Q \) has a turning point at 0. As the roots of \( Q \) are \(-a_{1}^{-1}, \ldots, -a_{m}^{-1}\), they are all real, and there are none in \((-a_{m}^{-1}, -a_{1}^{-1})\). So 0 is the only turning point of \( Q \) in this interval. Thus \( Q \leq 1 \) in \((-a_{m}^{-1}, -a_{1}^{-1})\), as \( Q(0) = 1 \). Equation (21) shows that \( Q(u) \geq A^2 \geq 0 \) for \( t \in (-\epsilon, \epsilon) \). As \( Q(u) \leq 1 \), this gives \( A^2 \leq 1 \), so \( A \in [-1, 1] \).

As by changing the sign of one of the complex coordinates \( z_j \) we change the sign of \( A \), we lose little by restricting our attention to \( A \in [0,1] \). This is equivalent to supposing that \( \sin \theta(0) \geq 0 \). To describe the SL \( m \)-folds \( N \) of Theorem 7.2 in more detail, we shall divide into the three cases

(a) \( A = 0 \),
(b) \( A = 1 \), and
(c) \( A \in (0,1) \).

We deal with each case separately.

7.2 Case (a): \( A = 0 \)

Suppose \( A = 0 \) in Theorem 7.2 and Proposition 7.3. Then (21) shows that \( \theta \) and \( \psi \) are constant, and (21) that \( \sin \theta = 0 \), so \( \theta \equiv n\pi \) for some integer \( n \), and (17) becomes \( \frac{du}{dt} = 2(-1)^n Q(u)^{1/2} \). So \( u \) is monotone increasing or decreasing in \( t \), and fills out some open interval in \( \mathbb{R} \).

The possible range of \( u \) is \((-a_{m}^{-1}, -a_{1}^{-1})\). Fixing \( \theta = \psi = 0 \) and letting \( u \) take its maximum range, Theorem 7.2 gives
Proposition 7.5 Let $a_1 \leq \cdots \leq a_m$ be integers, not all zero, with $a_1 + \cdots + a_m = 0$. Define a subset $N$ of $\mathbb{C}^m$ to be
\[
\left\{ \left( re^{i\alpha_1} \sqrt{a_1 u+1}, \ldots, re^{i\alpha_m} \sqrt{a_m u+1} \right) : r > 0, \quad u \in (-a_m^{-1}, -a_1^{-1}) \right\},
\]
\[
\left\{ \left( r e^{i\alpha_j} \sqrt{a_j u+1} \right) : r \geq 0, \quad u \in (-a_m^{-1}, -a_1^{-1}) \right\},
\]
Then $N$ is a nonsingular SL submanifold in $\mathbb{C}^m$ with phase $i^{m-2}$.

This special Lagrangian $m$-fold $N$ is not closed in $\mathbb{C}^m$. Its closure $\bar{N}$ is given by replacing the conditions $r > 0$ and $u \in (-a_m^{-1}, -a_1^{-1})$ in (24) by $r \geq 0$ and $u \in (-a_m^{-1}, -a_1^{-1})$. Thus $\bar{N} \setminus N$ is the disjoint union of 0 and two $\mathbb{R}_+ \times G$-orbits $O_+$, $O_-$ corresponding to $u = -a_1^{-1}$ and $u = -a_m^{-1}$.

Suppose $O_\pm$ have dimension $m-1$. Then $\bar{N}$ is a submanifold with boundary, with an isolated singular point at 0. Furthermore, by Theorem 6.3 there must exist locally unique, cohomogeneity one special Lagrangian cones with phase $i^{m-2}$ containing $O_+$ and $O_-$. That is, we can extend $N$ as a special Lagrangian submanifold beyond the boundary components $O_\pm$.

The condition that $\dim O_+ = m-1$ turns out to be that $a_1 < a_2$, and the condition that $\dim O_- = m-1$ turns out to be that $a_m < a_m$. It is then not difficult to prove the following result.

Proposition 7.6 Let $a_1 < a_2 < \cdots < a_{m-1} < a_m$ be integers, not all zero, with $a_1 + \cdots + a_m = 0$. Define a subset $N$ of $\mathbb{C}^m$ to be
\[
\left\{ \left( \pm re^{i\alpha_1} \sqrt{a_1 u+1}, re^{i\alpha_2} \sqrt{a_2 u+1}, \ldots, re^{i\alpha_{m-1}} \sqrt{a_{m-1} u+1} \right), \quad r \geq 0, \quad u \in [-a_m^{-1}, a_1^{-1}] \right\},
\]
\[
\left\{ \left( \pm re^{i\alpha_j} \sqrt{a_j u+1} \right), \quad r \geq 0, \quad u \in [-a_m^{-1}, a_1^{-1}] \right\}
\]
Then $N$ is a closed, embedded SL cone in $\mathbb{C}^m$ with phase $i^{m-2}$, with an isolated singular point at 0.

The significant idea involved here is that when $u = -a_1^{-1}$, the first complex coordinate $re^{i\alpha_1} \sqrt{a_1 u+1}$ in (24) becomes zero. The natural way to extend $N$ beyond $O_+$ is to change the sign of the square root $\sqrt{a_1 u+1}$. Similarly, when $u = -a_m^{-1}$ the last complex coordinate becomes zero, and we can extend $N$ beyond $O_-$ by changing the sign of $\sqrt{a_m u+1}$. The $m$-fold $N$ of (24) is actually a closed loop of $\mathbb{R}^+ \times G$-orbits, and so is a cone on a torus $T^{m-1}$.

We have not yet considered the possibilities that $O_+$ or $O_-$ have dimension less than $m-1$. Then we cannot use Theorem 6.3 to extend $N$ past $O_+$ or $O_-$. However, it turns out that if $\dim O_+ = m-2$, then $N$ is already nonsingular without boundary at $O_+$, and similarly for $O_-$. The local picture is of a family of circles collapsing to a point in $\mathbb{R}^2$. However, if $\dim O_+ \leq m-3$ then $N$ is unavoidably singular on $O_+$.

The condition that $\dim O_+ = m-2$ is that $a_1 = a_2 < a_3$, and the condition that $\dim O_- = m-2$ is that $a_{m-2} < a_{m-1} = a_m$. When $\dim O_+ = m-1$
and \( \dim \mathcal{O}_- = m - 2 \), we can prove the following result by the same method as Proposition 7.6. 

**Proposition 7.7** Let \( a_1 < a_2 < \cdots < a_{m-2} < a_{m-1} = a_m \) be integers, not all zero, with \( a_1 + \cdots + a_m = 0 \). Define a subset \( N \) of \( \mathbb{C}^m \) to be
\[
\left\{ \left( \pm e^{i\alpha_1} \sqrt{a_1 u + 1}, \ldots, \pm e^{i\alpha_m} \sqrt{a_m u + 1} \right) : r \geq 0, \quad \alpha_j \in \mathbb{R}, \quad \alpha_1 + \cdots + \alpha_m = 0, \quad a_1 \alpha_1 + \cdots + a_m \alpha_m = 0 \right\}.
\]

Then \( N \) is a closed, embedded SL cone in \( \mathbb{C}^m \) with phase \( i^{m-2} \), with an isolated singular point at 0.

In this case \( N \) is a closed interval of \( \mathbb{R}_+ \times G \)-orbits rather than a loop, and is topologically a cone on \( S^2 \times T^{m-3} \). When \( m = 3 \) we must take \( a_1 = -2 \) and \( a_2 = a_3 = 1 \), and we just get a copy of \( \mathbb{R}^3 \) in \( \mathbb{C}^3 \). But for \( m \geq 4 \), the proposition gives new, interesting SL \( m \)-folds in \( \mathbb{C}^m \).

Similarly, when \( \dim \mathcal{O}_+ = \dim \mathcal{O}_- = m - 2 \), we prove:

**Proposition 7.8** Let \( a_1 = a_2 < a_3 < \cdots < a_{m-2} < a_{m-1} = a_m \) be integers, not all zero, with \( a_1 + \cdots + a_m = 0 \). Define a subset \( N \) of \( \mathbb{C}^m \) to be
\[
\left\{ (r e^{i\alpha_1} \sqrt{a_1 u + 1}, \ldots, r e^{i\alpha_m} \sqrt{a_m u + 1}) : r \geq 0, \quad u \in [-a_m^{-1}, -a_1^{-1}], \quad \alpha_j \in \mathbb{R}, \quad \alpha_1 + \cdots + \alpha_m = 0, \quad a_1 \alpha_1 + \cdots + a_m \alpha_m = 0 \right\}.
\]

Then \( N \) is a closed, embedded SL cone in \( \mathbb{C}^m \) with phase \( i^{m-2} \), with an isolated singular point at 0.

Again, \( N \) is a closed interval of \( \mathbb{R}_+ \times G \)-orbits rather than a loop, and is topologically a cone on \( S^3 \times T^{m-4} \). The conditions on \( a_1, \ldots, a_m \) do not hold when \( m = 3 \). When \( m = 4 \) we must take \( a_1 = a_2 = -1 \) and \( a_3 = a_4 = 1 \), and we just get a copy of \( \mathbb{R}^4 \) in \( \mathbb{C}^4 \). But for \( m \geq 5 \), the proposition yields new, interesting SL \( m \)-folds in \( \mathbb{C}^m \).

All these examples can also be constructed using the ‘perpendicular symmetry’ construction to be described in \( \S 6 \). Specifically, in Proposition 7.3 we set \( n = m \) and \( c = 0 \), and define \( G \) as above. The proposition then yields a special Lagrangian submanifold \( N_0 \) in \( \mathbb{C}^m \), of which the \( m \)-folds \( N \) constructed above will be subsets.

**7.3 Case (b): \( A = 1 \)**

Next suppose that \( A = 1 \) in Theorem 7.2 and Proposition 7.3. As \( A^2 = Q(u) \leq 1 \) this gives \( Q(u) \equiv 1 \), which forces \( u \equiv 0 \), as \( Q(0) = 1 \) is the strict maximum of \( Q \) in the permitted interval \((-a_m^{-1}, -a_1^{-1})\). Then \( \{21\} \) gives \( \sin \theta \equiv 1 \), so that
\[ \theta \equiv \pi/2. \]

The second equation of (23) then becomes

\[ \frac{d\psi}{dt} = -\sum_{j=1}^{m} a_j^2. \]

Hence solutions to (17)–(19) exist for all \( t \in \mathbb{R} \), and are given by

\[ u(t) = 0, \quad \theta(t) = \frac{\pi}{2} \quad \text{and} \quad \psi(t) = \psi(0) - t \sum_{j=1}^{m} a_j^2, \]

for \( \psi(0) \in \mathbb{R} \).

Now \( \psi \) takes all values in \( \mathbb{R} \). Thus the equation

\[ a_1 \alpha_1 + \cdots + a_m \alpha_m = \psi(t) \]

in (20) is actually no restriction. So the SL \( m \)-fold \( N \) of (20) reduces to

\[ N = \left\{ (r e^{i\alpha_1}, \ldots, r e^{i\alpha_m}) : r > 0, \quad \alpha_j \in \mathbb{R}, \quad \alpha_1 + \cdots + \alpha_m = \frac{\pi}{2} \right\}. \] (26)

This \( N \) is entirely independent of \( a_1, \ldots, a_m \). It has generalized symmetry group \( G_{\text{Sym}}(N) = \mathbb{R}_+ \times U(1)^{m-1} \), which acts transitively, and symmetry group \( \text{Sym}(N) = U(1)^{m-1} \), which acts with cohomogeneity one.

Now we have already studied SL submanifolds of \( \mathbb{C}^m \) on which \( U(1)^{m-1} \) acts with cohomogeneity one in Example 5.5. In fact \( N \) is half of the SL \( m \)-fold \( L_0, \ldots, 0 \) of Example 5.5, rotated to give it phase \( i^{m-2} \) rather than 1.

### 7.4 Case (c): \( A \in (0, 1) \), local treatment

We shall discuss case (c) above from two points of view. Firstly, when \( \cos \theta > 0 \) for \( t \in (-\epsilon, \epsilon) \) we will find a more explicit expression for the manifold \( N \) of Theorem 7.2 by eliminating \( t \), and writing \( \theta \) and \( \psi \) as functions of \( u \). Then in \( \S 7.5 \) we will discuss the behaviour of equations (17)–(19) for \( t \in \mathbb{R} \) rather than \( (-\epsilon, \epsilon) \), and show that they admit periodic solutions.

We would like to write the SL \( m \)-fold \( N \) of Theorem 7.2 in as simple and explicit a way as possible. One way of doing this is to eliminate \( t \), and write everything instead as a function of the variable \( u \). Now \( \frac{du}{dt} \) has the same sign as \( \cos \theta \) by (17). Thus, if \( \cos \theta \) changes sign in \( (-\epsilon, \epsilon) \) then we cannot write \( t \) as a function of \( u \), but if \( \cos \theta \) has constant sign then we can.

Let us assume that \( \cos \theta > 0 \) for all \( t \in (-\epsilon, \epsilon) \). Then (22) gives

\[ \int_{u(0)}^{u(t)} \frac{du}{2 \sqrt{Q(u) - A^2}} = \int_0^t dt = t. \] (27)

This equation defines \( u \) implicitly as a function of \( t \). The integral on the left hand side is called an elliptic integral if \( m = 3 \) or 4, and a hyperelliptic integral if \( m \geq 5 \).

From (23) and the expression above for \( \frac{du}{dt} \) we get

\[ \frac{d\psi}{du} = -\frac{A}{2 \sqrt{Q(u) - A^2}} \sum_{j=1}^{m} \frac{a_j^2}{a_j u + 1}. \]

Integrating this gives an expression for \( \psi \) in terms of \( u \). Setting \( u_0 = u(0) \) and \( u_{\pm \epsilon} = u(\pm \epsilon) \), we have proved:
Theorem 7.9  Under the assumptions above, the SL $m$-fold $N$ of Theorem 7.2 may be written as

\[
\left\{ \left( re^{i\alpha_1} \sqrt{a_1 u + 1}, \ldots, re^{i\alpha_m} \sqrt{a_m u + 1} \right) : r > 0, \ u \in (u_{-\epsilon}, u_{\epsilon}), \right\}
\]

where $\alpha_j \in \mathbb{R}$, $\alpha_1 + \cdots + \alpha_m = \theta(u)$, $a_1 \alpha_1 + \cdots + a_m \alpha_m = \psi(u)$,

where $\theta(u)$ and $\psi(u)$ are given by $\theta(u) = \sin^{-1}(AQ(u)^{-1/2})$ and

\[
\psi(u) = \psi(u_0) - \frac{A}{2} \int_{u_0}^u \frac{dv}{\sqrt{Q(v) - A^2}} \sum_{j=1}^m \frac{a_j^2}{a_j v + 1}.
\] (28)

This is a reasonably explicit expression for $N$. The integral defining $\psi$ probably cannot be simplified any further without making special assumptions about $a_1, \ldots, a_m$.

7.5 Case (c): $A \in (0, 1)$, global behaviour

Next we study the global behaviour of solutions to equations (17)–(19) of Theorem 7.2 when $A \in (0, 1)$. We begin with a preliminary lemma on the range of $u$.

Lemma 7.10  Suppose $A \in (0, 1)$. Then there exist constants $\alpha < 0, \beta > 0$ and $\gamma > 0$ with $Q(\alpha) = Q(\beta) = A^2$ and $Q(u) \geq A^2$ for $u \in (\alpha, \beta)$, such that for all $t$ for which solutions $u, \theta$ exist to (17)–(18), we have $u(t) \in [\alpha, \beta]$.

Proof. From the proof of Lemma 7.4 we know that $Q$ strictly increases from 0 to 1 in $[-a_m^{-1}, 0]$, and so as $A \in (0, 1)$ there is a unique $\alpha \in (-a_m^{-1}, 0)$ with $Q(\alpha) = A^2$. Similarly, there is a unique $\beta \in (0, -a_1^{-1})$ with $Q(\beta) = A^2$. Clearly $Q(u) > A^2$ for $u \in (\alpha, \beta)$.

Also, if $u$ lies in $(-a_m^{-1}, -a_1^{-1})$ then $Q(u) \geq A^2$ if and only if $u \in [\alpha, \beta]$. But we know from Lemma 7.4 that if $u, \theta$ are solutions to (17)–(18) then $u$ is confined to $(-a_m^{-1}, -a_1^{-1})$, and $Q(u) \geq A^2$. Thus $u(t) \in [\alpha, \beta]$ for all $t$ for which the solutions exist, as we have to prove.

As $u$ is confined to $[\alpha, \beta]$, there exists $K > 0$ with $a_j u + 1 \leq K$ for $j = 1, \ldots, m$ and all $t$ for which the solution exists. Thus

\[ 0 < A^2 \leq Q(u) = (a_j u + 1) \prod_{\substack{i \in S \setminus \{j\}}} (a_i u + 1) \leq (a_j u + 1)K^{m-1}, \]

so that $a_j u(t) + 1 \geq \gamma = A^2K^{1-m} > 0$ for $j = 1, \ldots, m$. □

Now we can show that solutions exist for all $t \in \mathbb{R}$, and $u, \theta$ are periodic.
Proposition 7.11 Let \( u(0), \theta(0) \) and \( \psi(0) \) be given, with \( a_j u(0) + 1 > 0 \) for \( j = 1, \ldots, m \) and \( A = Q(u(0))^{1/2} \sin \theta(0) \in (0, 1) \). Then there exist unique solutions \( u(t), \theta(t) \) and \( \psi(t) \) to equations (17)–(19) of Theorem 7.2 for all \( t \in \mathbb{R} \), with these values at \( t = 0 \). Furthermore \( u \) and \( \theta \) are nonconstant and periodic with period \( T > 0 \), and there exists \( \Psi > 0 \) with \( \psi(t+T) = \psi(t) - \Psi \) for all \( t \in \mathbb{R} \).

Proof. The only way for solutions of (17)–(19) to become singular is for some \( n \) with period \( T > 0 \) and \( \theta \) and \( \psi \) to be given, with \( u(t) > 0 \), \( \theta(t) \), and \( \psi(t) \) is periodic, with period \( T > 0 \), and there exists \( \Psi > 0 \) with \( \psi(t+T) = \psi(t) - \Psi \) for all \( t \in \mathbb{R} \).

Next we show that if \( \Psi \) is a rational multiple \( a/b \) of 2\( \pi \) then the family of \( \mathbb{R}_+ \times G \)-orbits \( \mathcal{O}_t \) is periodic, with period \( bT \), and \( N \) is a cone on \( T^{m-1} \).

Proposition 7.12 In the situation of Proposition 7.11, suppose that \( \Psi = 2\pi q \) for \( q \in \mathbb{Q} \). Define a subset \( N \) of \( \mathbb{C}^m \) to be

\[
\left\{ \left( \text{exp}\left(i\alpha_1 u(t) + 1\right), \ldots, \text{exp}\left(i\alpha_m u(t) + 1\right) \right) : r \geq 0, \ t \in \mathbb{R}, \ \alpha_j \in \mathbb{R}, \ \alpha_1 + \cdots + \alpha_m = \theta(t), \ a_1 \alpha_1 + \cdots + a_m \alpha_m = \psi(t) \right\}.
\]

Then \( N \) is a closed, embedded SL cone in \( \mathbb{C}^m \) with phase \( i^{m-2} \), which is topologically a cone on \( T^{m-1} \), and has just one singular point at 0.

Proof. Let \( q = a/b \), for \( a, b \) positive integers with \( \text{hcf}(a, b) = 1 \). Then as \( u(t+T) = u(t), \ \theta(t+T) = \theta(t) \) and \( \psi(t+T) = \psi(t) - \Psi \), we have \( u(t+bT) = u(t), \ \theta(t+bT) = \theta(t) \) and \( \psi(t+bT) = \psi(t) - 2\pi a \). Now the definition (30) of the
SL m-fold $N$ constructed in Theorem 7.2 actually depends only on $e^{i\theta}$ and $e^{i\psi}$ rather than on $\theta$ and $\psi$, as $a_1, \ldots, a_m$ are integers. Thus, adding an integer multiple of $2\pi$ to $\psi$ makes no difference to $N$.

This means that the the 1-parameter family of $\mathbb{R}_+ \times G$-orbits $O_t$ which make up $N$ satisfies $O_{t+bT} = O_t$ for all $t$, and is periodic, with period $bT$. The definition of $N$ given in (29) differs from (20) in that $r > 0$ rather than $r > 0$, and $t \in \mathbb{R}$ rather than $t \in (-\epsilon, \epsilon)$. We allow $r = 0$ to add the point 0 to $N$, which makes $N$ closed. We allow $t \in \mathbb{R}$ because $u, \theta$ and $\psi$ exist for all $t \in \mathbb{R}$ by Proposition 7.11. In fact, as $t$ has period $bT$, we would get the same $N$ if we replaced $t \in \mathbb{R}$ by $t \in [0, bT)$.

So $N$ is a closed loop of $\mathbb{R}_+ \times G$-orbits $O_t$, together with the point zero. It is clear that $N$ is closed and, at least as an immersed submanifold, it is topologically a cone on $T^{m-1}$, with just one singular point at 0. So we need only show that $N$ is embedded. If two $\mathbb{R}_+ \times G$-orbits $O_{t_1}, O_{t_2}$ intersect, then they are the same. But from Theorem 6.3, an $\mathbb{R}_+ \times G$-orbit locally determines $N$ uniquely. Thus, our loop of $\mathbb{R}_+ \times G$-orbits cannot cross itself, and must be embedded.

The proposition is potentially interesting because closed special Lagrangian cones in $\mathbb{C}^m$ with isolated singular points are natural models for singularities of SL $m$-folds in Calabi–Yau $m$-folds. We will now investigate the range of $\Psi$, and hence show that the proposition yields very many such cones.

By the reasoning used to prove (28), we can show that

$$\Psi = \int_{\alpha}^{\beta} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_j v + 1}. \quad (30)$$

As Lemma 7.10 defines $\alpha, \beta$ in terms of $Q(u)$ and $A$, we see that $\Psi$ depends only on $a_1, \ldots, a_m$ and $A$, and not on the initial data $u(0), \theta(0)$ and $\psi(0)$.

Up to now we have regarded $A$ as a function of $u(0), \theta(0)$ and $\psi(0)$. We now change our point of view. Lemma 7.10 defines $\alpha, \beta$ depending on $A \in (0, 1)$. Given any $A \in (0, 1)$, set $u(0) = \alpha, \theta(0) = \pi/2$ and $\psi(0) = 0$. This is a valid set of initial data, and yields this value of $A$. Thus $A$ can take any value in $(0, 1)$, and we can regard $u(0), \theta(0), \psi(0)$ and $\Psi$ as functions of $A$. We now calculate the limit of $\Psi$ as $A$ approaches 0 or 1.

**Proposition 7.13** In the situation above, $\Psi : (0, 1) \to (0, \infty)$ is a real analytic function of $A$, and satisfies $\Psi(A) \to \pi(a_m - a_1)$ as $A \to 0$ and $\Psi(A) \to \pi(2 \sum_{j=1}^{m} a_j^2)^{1/2}$ as $A \to 1$.

**Proof.** It is obvious from (30) and the definition of $\alpha$ and $\beta$ that $\Psi$ is real analytic. As $A \to 0$, we have $\alpha \to -1/a_m$ and $\beta \to -1/a_1$. Also, as $A \to 0$ the factor $(A^{-2}Q(v) - 1)^{-1/2}$ in (30) tends to zero, except near $\alpha$ and $\beta$. Hence, as $A \to 0$, the integrand in (30) gets large near $\alpha \approx -1/a_m$ and $\beta \approx -1/a_1$, and very close to zero in between.

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So to understand $\Psi$ as $A \to 0$, it is enough to study the integral near $\alpha$ and $\beta$. We shall model it at $\alpha$. Suppose $a_{m-k} < a_{m-k+1} = \cdots = a_m$, so that $a_m$ has multiplicity $k$. Then near $v = -1/a_m$ we have

\[ Q(v) \approx C(v + a_m^{-1})^k, \quad \text{where} \quad C = a_m^k \prod_{j=1}^{m-k} (1 - a_j/a_m). \]

Since $A^2 = Q(\alpha)$ this gives $A^2 \approx C(\alpha + a_m^{-1})^k$, so that $\alpha \approx A^{2/k}C^{-1/k} - a_m^{-1}$.

Therefore, when $v \approx \alpha$ we have

\[ A^{-2}Q(v) - 1 \approx A^{-2}(v + a_m^{-1})^k - 1 \quad \text{and} \quad \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx \frac{ka_m}{v + a_m}, \]

taking only the highest-order terms. Thus, when $A$ is small we see that

\[ \int_{A^{-2}C^{-1/k} - a_m^{-1}}^{0} \frac{dv}{\sqrt{A^{-2}C(v + a_m^{-1})^k - 1}} \cdot \frac{ka_m}{v + a_m} \approx \int_{0}^{\infty} \frac{2a_m \, dw}{w^2 + 1} = \pi a_m, \]

where we have approximated the second integral by replacing the upper limit $\sqrt{A^{-2}Ca_m^k - 1}$ by $\infty$. Hence, for small $A$ we have

\[ \int_{0}^{\infty} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx \pi a_m, \]

and similarly

\[ \int_{0}^{\beta} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx -\pi a_1, \]

so that $\Psi(A) \to \pi(a_m - a_1)$ as $A \to 0$.

Next consider the behaviour of $\Psi$ as $A \to 1$. When $A$ is close to 1, $u$ is small and $\theta$ is close to $\pi/2$. So write $\theta = \frac{\pi}{2} + \phi$, for $\phi$ small. Then, setting $Q(u) \approx 1$,

\[ \cos \theta \approx -\phi, \quad \sin \theta \approx 1, \quad \sum_{j=1}^{m} \frac{a_j}{a_ju + 1} \approx -u \sum_{j=1}^{m} a_j^2 \quad \text{and} \quad \sum_{j=1}^{m} \frac{a_j^2}{a_ju + 1} \approx \sum_{j=1}^{m} a_j^2, \]

and

\[ \int_{0}^{\infty} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx \int_{0}^{\infty} \frac{ka_m}{v + a_m} \cdot \frac{2a_m \, dw}{w^2 + 1} = \pi a_m, \]

where we have used the approximation $\sqrt{A^{-2}Ca_m^k - 1}$ by $\infty$. Hence, for small $A$ we have

\[ \int_{0}^{\infty} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx \pi a_m, \]

and similarly

\[ \int_{0}^{\beta} \frac{dv}{\sqrt{A^{-2}Q(v) - 1}} \sum_{j=1}^{m} \frac{a_j^2}{a_jv + 1} \approx -\pi a_1, \]

so that $\Psi(A) \to \pi(a_m - a_1)$ as $A \to 0$.
taking only the highest order terms, equations (17)–(19) become
\[ \frac{du}{dt} \approx -2\phi, \quad \frac{d\phi}{dt} \approx u \sum_{j=1}^{m} a_j^2, \quad \text{and} \quad \frac{d\psi}{dt} \approx -\sum_{j=1}^{m} a_j^2. \]

The first two of these equations show that \( u \) and \( \phi \) undergo approximately simple harmonic oscillations with period \( T = 2\pi(2\sum_{j=1}^{m} a_j^2)^{-1/2} \). Then the third equation shows that \( \Psi \approx -\frac{d\psi}{dt}T \), as \( \frac{d\psi}{dt} \) is approximately constant, and so \( \Psi \rightarrow \pi(2\sum_{j=1}^{m} a_j^2)^{1/2} \) as \( A \rightarrow 1 \). \( \square \)

Now the limits \( \pi(a_m - a_1) \) and \( \pi(2\sum_{j=1}^{m} a_j^2)^{1/2} \) of \( \Psi(A) \) as \( A \rightarrow 0, 1 \) satisfy
\[ \left[ \pi(2\sum_{j=1}^{m} a_j^2)^{1/2} \right]^2 - \left[ \pi(a_m - a_1) \right]^2 = \pi^2(a_1 + a_m)^2 + 2\pi^2(a_2^2 + \cdots + a_{m-1}^2). \]

Thus \( \pi(a_m - a_1) \leq \pi(2\sum_{j=1}^{m} a_j^2)^{1/2} \), with equality if and only if \( a_1 + a_m = 0 \) and \( a_2 = \cdots = a_{m-1} = 0 \). As the \( a_j \) are integers with highest common factor 1, and \( a_1 \leq \cdots \leq a_m \), these conditions imply that \( a_1 = -1, a_2 = \cdots = a_{m-1} = 0 \) and \( a_m = 1 \).

Therefore we have two cases:

(i) \( (a_1, \ldots, a_m) \neq (-1, 0, \ldots, 0, 1) \), and \( \lim_{A \rightarrow 0} \Psi(A) < \lim_{A \rightarrow 1} \Psi(A) \), or

(ii) \( (a_1, \ldots, a_m) = (-1, 0, \ldots, 0, 1) \), and \( \lim_{A \rightarrow 0} \Psi(A) = \lim_{A \rightarrow 1} \Psi(A) = 2\pi \).

In case (i), as \( \lim_{A \rightarrow 0} \Psi(A) < \lim_{A \rightarrow 1} \Psi(A) \) we see that \( \Psi \) is not constant, and as it is real analytic it can have only finitely many stationary points in \((0, 1)\). So we deduce:

**Corollary 7.14** Suppose \( (a_1, \ldots, a_m) \neq (-1, 0, \ldots, 0, 1) \). Then for a countable dense subset of \( A \in (0, 1) \) we have \( \Psi(A) \in 2\pi\mathbb{Q} \).

In case (ii), we can solve equations (11)–(13) completely. For as \( a_2 = \cdots = a_{m-1} = 0 \) we see from (11) that \( w_2, \ldots, w_{m-1} \) are constant, and from (13) that \( |w_2| = \cdots = |w_{m-1}| = 1 \). Applying a diagonal matrix in \( SU(m) \), we may choose \( w_2 = \cdots = w_{m-1} = 1 \). Then (11) and (13) reduce to
\[ \frac{dw_1}{dt} = -\bar{w}_m, \quad \frac{dw_m}{dt} = \bar{w}_1 \quad \text{and} \quad |w_1|^2 + |w_m|^2 = 2, \]

which have solutions
\[ w_1 = Be^{it} + Ce^{-it} \quad \text{and} \quad w_m = i\bar{B}e^{-it} - i\bar{C}e^{it}, \]

for \( B, C \in \mathbb{C} \) with \( |B|^2 + |C|^2 = 1 \). It is easy to show that \( A = 2\text{Im}(\bar{B}C) \in [-1, 1] \), and that \( \Psi(A) = 2\pi \) for all \( A \).
The special Lagrangian $m$-fold $N$ of (14) is thus

\[
N = \left\{ \left( re^{i\alpha_1}(Be^{it} + Ce^{-it}), re^{i\alpha_2}, \ldots, re^{i\alpha_{m-1}}, re^{i\alpha_m} \right) : \\
r > 0, \ t \in \mathbb{R}, \ \alpha_j \in \mathbb{R}, \ \alpha_1 + \cdots + \alpha_m = 0, \ \alpha_1 = \alpha_m \right\}.
\]

Now this is the result of applying the SU($m$) transformation

\[
(z_1, \ldots, z_m) \mapsto (Bz_1 - iCz_m, z_2, \ldots, z_{m-1}, -i\bar{C}z_1 + \bar{B}z_m)
\]

of $\mathbb{C}^m$ to the special Lagrangian cone

\[
N' = \left\{ \left( re^{i(\alpha_1+t)}, re^{i\alpha_2}, \ldots, re^{i\alpha_{m-1}}, ire^{i(\alpha_m-t)} \right) : \\
r > 0, \ t \in \mathbb{R}, \ \alpha_j \in \mathbb{R}, \ \alpha_1 + \cdots + \alpha_m = 0, \ \alpha_1 = \alpha_m \right\}.
\]

But this is identical to the cone defined in (26), which we saw in §7.3 has transitive generalized symmetry group $G_{\text{Sym}}(N) = \mathbb{R}_+ \times \text{U}(1)^{m-1}$, and symmetry group $\text{Sym}(N) = \text{U}(1)^{m-1}$.

Here is how to interpret this. Since $\Psi(A) = 2\pi$ for all $A$, we would expect Proposition 7.12 to yield a 1-parameter family of distinct $\text{U}(1)^{m-2}$-invariant SL cones on $T^{m-1}$ in $\mathbb{C}^m$, parametrized by $A \in (0, 1)$, and with the same symmetry group $\text{U}(1)^{m-1}$. But in fact these SL cones are all isomorphic under transformations in SU($m$), and have symmetry group $\text{U}(1)^{m-2}$ rather than $\text{U}(1)^{m-1}$.

One consequence of this is that there are no SL cones on $T^{m-1}$ in $\mathbb{C}^m$ with $\text{Sym}^0(N)$ equal to this particular symmetry group $G = \text{U}(1)^{m-2}$ in SU($m$), since any SL cone symmetric under this group $\text{U}(1)^{m-2}$ is also symmetric under a larger group $\text{U}(1)^{m-1}$ in SU($m$).

Drawing together much of the work above, in particular Proposition 7.12 and Corollary 7.14, we have the main result of this section.

**Theorem 7.15** Let $G$ be a Lie subgroup of SU($m$) isomorphic to $\text{U}(1)^{m-2}$, for some $m \geq 3$. If $G$ is conjugate in SU($m$) to the group

\[
\left\{ \left( z_1, \ldots, z_m \right) \mapsto \left( e^{i\alpha_1}z_1, \ldots, e^{i\alpha_m}z_m \right) : \\
\alpha_j \in \mathbb{R}, \ \alpha_1 + \cdots + \alpha_m = 0, \ \alpha_1 = \alpha_m \right\},
\]

then every $G$-invariant SL cone in $\mathbb{C}^m$ locally has symmetry group $\text{U}(1)^{m-1}$.

Otherwise there exists a countably infinite family of distinct, closed, embedded SL cones $N$ in $\mathbb{C}^m$ with $\text{Sym}^0(N) = G$, each of which is topologically a cone on $T^{m-1}$, with just one singular point at 0.

Here is what we mean by saying the construction yields a countable family of distinct cones. Fixing a subgroup $G \cong \text{U}(1)^{m-2}$ in SU($m$), we consider two $G$-invariant SL cones $N_1, N_2$ to be equivalent if there exists a $G$-equivariant isometric isomorphism of $\mathbb{C}^m$ identifying $N_1$ and $N_2$. The theorem says that for
there are countably many distinct equivalence classes of $G$-equivariant SL cones.

More generally, we could ask about the classification of the SL cones constructed in the second part of the theorem for all $U(1)^{m-2}$-subgroups $G$, up to isometric isomorphisms of $\mathbb{C}^m$. Let $N_1, N_2$ be two such cones with groups $G_1, G_2$, and suppose $N_1$ and $N_2$ are identified by an isometric isomorphism of $\mathbb{C}^m$. Then $G_1$ and $G_2$ are conjugate in $SU(m)$, since $G_j = Sym^0(N_j)$.

Thus, for each of the countable number of conjugacy classes of $U(1)^{m-2}$-subgroups $G$ in $SU(m)$ not conjugate to $G$, we have found a countable family of distinct $G$-invariant special Lagrangian $T^{m-1}$-cones in $\mathbb{C}^m$, up to isometric isomorphisms of $\mathbb{C}^m$. One reason these cones are interesting is as local models for singularities of SL $m$-folds in Calabi–Yau $m$-folds.

To see how big the whole family is, here is a crude ‘parameter count’. Up to isomorphism, $N$ depends on integers $a_1 \leq \cdots \leq a_m$ with $a_1 + \cdots + a_m = 0$ and $\gcd(a_1, \ldots, a_m) = 1$ and $A \in (0, 1)$ with $\Psi(A) = 2\pi q$, for $q \in \mathbb{Q}$. The most obvious thing to do is to set $q = a/b$ with $\gcd(a, b) = 1$, and say that $N$ depends on the $m + 1$ integers $a_1, \ldots, a_{m-1}, a$ and $b$.

However, this is probably not the best point of view. Instead, we should drop the condition $\gcd(a_1, \ldots, a_m) = 1$, and replace the $a_j$ by $\tilde{a}_j = a_jb$ for $j = 1, \ldots, m$. With these new values we get $\Psi(A) = 2\pi a$, so that $\Psi$ lies in $2\pi \mathbb{Z}$ rather than $2\pi \mathbb{Q}$, and we can say that $N$ depends on the $m$ integers $\tilde{a}_1, \ldots, \tilde{a}_{m-1}, a$, which have highest common factor $1$.

A partial version of the case $m = 3$ of Theorem 7.15 was first due to Mark Haskins [7, Th. C]. Haskins does not study the periodicity conditions in the case $A \in (0, 1)$, but only when $A = 0$. The author’s treatment was completed independently somewhat later, and uses different methods.

### 7.6 Relation to integrable systems

We saw above that equation (11) has very nice behaviour — various quantities are conserved, and one can say a lot about the solutions, even writing them explicitly using elliptic integrals. The reason for this is that (11) is completely integrable, as we will now show.

An introduction to integrable systems is given in Hitchin, Segal and Ward [14]. From [14, Def. 6.1, p. 49] and [14, Def. 2.7, p. 61], we may define a completely integrable Hamiltonian system to be a $2m$-dimensional symplectic manifold $(M, \omega)$ with a Hamiltonian $H : M \to \mathbb{R}$, such that there exist $m$ conserved quantities $p_1, \ldots, p_m : H \to \mathbb{R}$ which are independent almost everywhere and satisfy

$$\{H, p_i\} \equiv \{p_i, p_j\} \equiv 0 \quad \text{for } i, j = 1, \ldots, m, \quad (32)$$

where $\{, \}$ is the Poisson bracket on $M$.

Choose $a_1, \ldots, a_m \in \mathbb{R} \setminus \{0\}$. Let $M$ be $\mathbb{C}^m$ with coordinates $(w_1, \ldots, w_m)$,
and define a symplectic form $\omega$ and a Hamiltonian function $H$ on $M$ by
\[
\omega = \frac{i}{2} \sum_{j=1}^{m} a_j^{-1} dw_j \wedge d\bar{w}_j \quad \text{and} \quad H : (w_1, \ldots, w_m) \mapsto 2 \Im(w_1 \cdots w_m).
\]

The equations of motion of this Hamiltonian system are easily shown to be
\[
\frac{dw_j}{dt} = a_j w_1 \cdots w_{j-1} w_{j+1} \cdots w_m, \quad j = 1, \ldots, m,
\]
as in (11). Define functions $p_1, \ldots, p_m : M \to \mathbb{R}$ by
\[
p_j = a_m |w_j|^2 - a_j |w_m|^2 \quad \text{for} \quad j = 1, \ldots, m - 1, \quad \text{and} \quad p_m = \Im(w_1 \cdots w_m).
\]

It is easy to show that $p_1, \ldots, p_m$ are conserved, independent almost everywhere, and satisfy (32). Thus (11) is indeed a completely integrable Hamiltonian system.

This proof assumes that $a_1, \ldots, a_m$ are nonzero, but not that $a_1 + \cdots + a_m = 0$, which is the case in (11). If some $a_j$ is zero then $\frac{dw_j}{dt} = 0$, so $w_j$ is constant. Fixing all the $w_j$ with $a_j = 0$, we can regard (11) as an o.d.e. in the remaining $w_k$, which is then integrable as above.

8 U(1)-invariant SL cones in $\mathbb{C}^3$

We now specialize to the case $m = 3$ in the situation of §7, so that we are studying SL cones $N$ in $\mathbb{C}^3$ invariant under a U(1) subgroup of SU(3). In this case we can improve the treatment of §7 in several ways. In particular, we will define the group $G \cong U(1)$ in a neater way, we will write down conformal coordinates on $N \cap S^5$, and we will solve equations (17)–(19) explicitly in terms of Jacobi elliptic functions.

The material of this section has already been studied by several authors, from different points of view. Probably the first were Castro and Urbano [1], who constructed examples of minimal Lagrangian tori in $\mathbb{C}P^2$ using integrable systems methods. Special Lagrangian $T^2$-cones in $\mathbb{C}^3$ can be reconstructed from Castro and Urbano’s results.

Later, Haskins [7, §3–5] studied U(1)-invariant special Lagrangian cones in $\mathbb{C}^3$, and most of this section overlaps with his work. In particular, the author learnt the material of §8.2 from his papers.

From Theorem 7.1 we get:

**Theorem 8.1** Let $a_1, a_2, a_3 \in \mathbb{R}$ be not all zero, with $a_1 + a_2 + a_3 = 0$. Suppose $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$ and $u : \mathbb{R} \to \mathbb{R}$ satisfy
\[
\frac{dw_1}{dt} = a_1 w_2 w_3, \quad \frac{dw_2}{dt} = a_2 w_3 w_1, \quad \frac{dw_3}{dt} = a_3 w_1 w_2, \quad (33)
\]
\[
\frac{du}{dt} = 2 \Re(w_1 w_2 w_3), \quad (34)
\]
\[
|w_j|^2 = a_j u + 1 \quad \text{for} \quad j = 1, 2, 3. \quad (35)
\]
If \( \mathbf{34} \) and \( \mathbf{35} \) hold for all \( t \) and \( \mathbf{33} \) holds for \( t = 0 \), then \( \mathbf{36} \) holds for all \( t \). Define a subset \( N \) of \( \mathbb{C}^3 \) by

\[
N = \left\{ \left( re^{i\alpha_1}w_1(t), re^{i\alpha_2}w_2(t), re^{i\alpha_3}w_3(t) \right) : r > 0, \ t \in \mathbb{R}, \right. \\
\left. \alpha_j \in \mathbb{R}, \ \alpha_1 + \alpha_2 + \alpha_3 = 0, \ a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0 \right\}. \tag{36}
\]

Then \( N \) is a special Lagrangian 3-fold in \( \mathbb{C}^3 \) with phase \( i \).

Here we have replaced the domain \( (-\epsilon, \epsilon) \) of \( w_j \) and \( u \) by \( \mathbb{R} \), as the results of \( \mathbf{34} \) imply that solutions of \( \mathbf{34} \) exist for all \( t \in \mathbb{R} \). We will find it convenient later to rewrite the theorem in terms of \( b_1, b_2, b_3 \), where

\[
b_1 = \frac{1}{\sqrt{3}}(a_2 - a_3), \quad b_2 = \frac{1}{\sqrt{3}}(a_3 - a_1) \quad \text{and} \quad b_3 = \frac{1}{\sqrt{3}}(a_1 - a_2).
\]

Then \( b_1 + b_2 + b_3 = 0 \), and as \( a_1 + a_2 + a_3 = 0 \) we find that

\[
a_1 = \frac{1}{\sqrt{3}}(b_3 - b_2), \quad a_2 = \frac{1}{\sqrt{3}}(b_1 - b_3) \quad \text{and} \quad a_3 = \frac{1}{\sqrt{3}}(b_2 - b_1). \tag{37}
\]

It is easy to show that \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \) satisfy \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \) and \( a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0 \) if and only if \( \alpha_j = b_j s \) for some \( s \in \mathbb{R} \). Also, \( \mathbf{34} \) is equivalent to \( |w_1|^2 + |w_2|^2 + |w_3|^2 = 3 \) and \( b_1|w_1|^2 + b_2|w_2|^2 + b_3|w_3|^2 = 0 \). Thus, Theorem \( \mathbf{31} \) becomes:

**Theorem 8.2** Let \( b_1, b_2, b_3 \in \mathbb{R} \) be not all zero, with \( b_1 + b_2 + b_3 = 0 \). Define \( a_1, a_2, a_3 \) by \( \mathbf{37} \). Suppose \( w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C} \setminus \{0\} \) satisfy

\[
\frac{dw_1}{dt} = a_1 w_2 w_3, \quad \frac{dw_2}{dt} = a_2 w_3 w_1 \quad \text{and} \quad \frac{dw_3}{dt} = a_3 w_1 w_2, \tag{38}
\]

\[|w_1|^2 + |w_2|^2 + |w_3|^2 = 3 \quad \text{and} \quad b_1|w_1|^2 + b_2|w_2|^2 + b_3|w_3|^2 = 0. \tag{39}\]

If \( \mathbf{33} \) holds for all \( t \) and \( \mathbf{38} \) holds for \( t = 0 \), then \( \mathbf{36} \) holds for all \( t \). Define a subset \( N \) of \( \mathbb{C}^3 \) by

\[
N = \left\{ \left( re^{ib_1s}w_1(t), \ldots, re^{ib_3s}w_3(t) \right) : r > 0, \ s \in \mathbb{R}, \ t \in \mathbb{R} \right\}. \tag{40}
\]

Then \( N \) is a special Lagrangian 3-fold in \( \mathbb{C}^3 \) with phase \( i \).

We shall show that \( (s, t) \) are conformal coordinates on the unit sphere \( N \cap \mathcal{S}^5 \) in \( N \), and that the corresponding map \( \Phi : \mathbb{R}^3 \to \mathcal{S}^5 \) is harmonic.

**Proposition 8.3** In the situation of Theorem 8.2, define \( \Phi : \mathbb{R}^2 \to \mathcal{S}^5 \) by

\[
\Phi : (s, t) \mapsto \frac{1}{\sqrt{3}}(e^{ib_1s}w_1(t), e^{ib_2s}w_2(t), e^{ib_3s}w_3(t)), \tag{41}
\]

where \( \mathcal{S}^5 \) is the unit sphere in \( \mathbb{C}^3 \). Then \( \Phi \) is a conformal harmonic map.
Proof. From \(\{38\}\) and \(\{41\}\) we see that
\[
\frac{\partial \Phi}{\partial s} = \frac{1}{\sqrt{3}}(ib_1e^{ib_1s}w_1, ib_2e^{ib_2s}w_2, ib_3e^{ib_3s}w_3) \quad \text{and} \quad \frac{\partial \Phi}{\partial t} = \frac{1}{\sqrt{3}}(a_1e^{ib_1s}w_2w_3, a_2e^{ib_2s}w_3w_1, a_3e^{ib_3s}w_1w_2).
\]

Now \(\Phi\) is conformal if and only if \(\frac{\partial \Phi}{\partial s}\) and \(\frac{\partial \Phi}{\partial t}\) are orthogonal and the same length. But
\[
g\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial t}\right) = -\frac{1}{3}[a_1b_1 + a_2b_2 + a_3b_3] \Im(w_1w_2w_3) = 0,
\]
as \(a_1b_1 + a_2b_2 + a_3b_3 = 0\), so they are orthogonal. Also
\[
\left|\frac{\partial \Phi}{\partial s}\right|^2 = \frac{1}{3}(b_1^2|w_1|^2 + b_2^2|w_2|^2 + b_3^2|w_3|^2) \quad \text{and} \quad \left|\frac{\partial \Phi}{\partial t}\right|^2 = \frac{1}{3}(a_1^2|w_2|^2|w_3|^2 + a_2^2|w_3|^2|w_1|^2 + a_3^2|w_1|^2|w_2|^2).
\]

One can then prove from equations \(\{37\}\), \(\{38\}\) and \(b_1 + b_2 + b_3 = 0\) that \(\left|\frac{\partial \Phi}{\partial s}\right|^2 = \left|\frac{\partial \Phi}{\partial t}\right|^2\), and thus \(\Phi\) is conformal.

Now \(\mathcal{N}\) is a cone in \(\mathbb{C}^3\), and is minimal because any calibrated submanifold is minimal. Therefore the intersection \(\mathcal{N} \cap S^5\) is minimal in \(S^5\). But \(\mathcal{N} \cap S^5\) is the image of \(\Phi\). Hence \(\Phi : \mathbb{R}^2 \to S^5\) is a conformal map with minimal image. But it is well known in the field of harmonic maps that a conformal map from a Riemann surface is harmonic if and only if it has minimal image. Thus \(\Phi\) is harmonic.

Using the method of \(\S 7.1\) to rewrite the \(w_j\) in terms of real variables \(u, \theta, \psi\), as in Theorem \(7.2\) and Proposition \(7.3\) we get

**Theorem 8.4** In the situation of Theorem 8.2 the functions \(w_1, w_2, w_3\) may be written \(w_j = e^{i\theta_j}\sqrt{a_ju + 1}\), for \(u, \theta_1, \theta_2, \theta_3 : \mathbb{R} \to \mathbb{R}\). Define
\[
Q(u) = (a_1u + 1)(a_2u + 1)(a_3u + 1),
\]
\[
\theta = \theta_1 + \theta_2 + \theta_3 \quad \text{and} \quad \psi = a_1\theta_1 + a_2\theta_2 + a_3\theta_3.
\]
Then \(Q(u)^{1/2}\sin \theta \equiv A\) for some \(A \in \mathbb{R}\), and \(u, \theta_j, \theta \) and \(\psi\) satisfy
\[
\left(\frac{du}{dt}\right)^2 = 4(Q(u) - A^2), \quad \frac{d\theta_j}{dt} = -\frac{a_jA}{a_ju + 1}, \quad \frac{d\theta}{dt} = -A\sum_{j=1}^{3} \frac{a_ju + 1}{a_ju + 1}, \quad \frac{d\psi}{dt} = -A\sum_{j=1}^{3} \frac{a_j^2}{a_ju + 1}.
\]
(42)

The phase special Lagrangian 3-fold \(N\) of \(\{40\}\) may also be written
\[
N = \left\{ (re^{i\alpha_1} \sqrt{a_1u(t) + 1}, \ldots, re^{i\alpha_3} \sqrt{a_3u(t) + 1}) : r > 0, \ t \in \mathbb{R} \right\}
\]
\[
\alpha_j \in \mathbb{R}, \quad \alpha_1 + \alpha_2 + \alpha_3 = \theta(t), \ a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \psi(t) \}
\]

33
Our next theorem follows from the case $m = 3$ of Theorem 7.15.

**Theorem 8.5** Let $b_1, b_2, b_3$ be distinct integers with highest common factor 1 and $b_1 + b_2 + b_3 = 0$, and define a subgroup $G \cong U(1)$ of SU(3) to be the set of transformations

$$(z_1, z_2, z_3) \mapsto (e^{ib_1s}z_1, e^{ib_2s}z_2, e^{ib_3s}z_3)$$

for $s \in \mathbb{R}$. Then there exists a countably infinite family of distinct, closed, embedded special Lagrangian cones $N$ in $\mathbb{C}^3$ with $\text{Sym}^0(N) = G$, each of which is topologically a cone on $T^2$, and has just one singular point at 0.

Here we have tidied the theorem up by defining the group $G$ using $b_1, b_2, b_3$ instead of $a_1, a_2, a_3$, and requiring the $b_j$ to be integers rather than the $a_j$. The condition that $G$ should not be conjugate in SU(3) to the group (31) turns out to be that $b_1, b_2, b_3$ are distinct.

The sense in which the cones are distinct was explained after Theorem 7.15. Two sets of integers $b_1, b_2, b_3$ and $b'_1, b'_2, b'_3$ produce isomorphic families of SL cones in $\mathbb{C}^3$ if and only if the corresponding groups $G, G'$ are conjugate in SU(3). This happens if and only if $b'_j = b_{\sigma(j)}$ for $j = 1, 2, 3$ or $b'_j = -b_{\sigma(j)}$ for $j = 1, 2, 3$, for some $\sigma \in S_3$.

8.1 The Jacobi elliptic functions

We now give a brief introduction to the Jacobi elliptic functions, which we will use in §8.2 to solve (42) explicitly. The following material can be found in Chandrasekharan [3, Ch. VII]. For each $k \in [0, 1]$, the Jacobi elliptic functions $\text{sn}(t, k)$, $\text{cn}(t, k)$, $\text{dn}(t, k)$ with modulus $k$ are the unique solutions to the o.d.e.s

\begin{align*}
\left( \frac{d}{dt} \text{sn}(t, k) \right)^2 &= (1 - \text{sn}^2(t, k))(1 - k^2 \text{sn}^2(t, k)), \\
\left( \frac{d}{dt} \text{cn}(t, k) \right)^2 &= (1 - \text{cn}^2(t, k))(1 - k^2 + k^2 \text{cn}^2(t, k)), \\
\left( \frac{d}{dt} \text{dn}(t, k) \right)^2 &= -(1 - \text{dn}^2(t, k))(1 - k^2 - \text{dn}^2(t, k)),
\end{align*}

with initial conditions

\begin{align*}
\text{sn}(0, k) &= 0, & \text{cn}(0, k) &= 1, & \text{dn}(0, k) &= 1, \\
\frac{d}{dt} \text{sn}(0, k) &= 1, & \frac{d}{dt} \text{cn}(0, k) &= 0, & \frac{d}{dt} \text{dn}(0, k) &= 0.
\end{align*}

They satisfy the identities

$$\text{sn}^2(t, k) + \text{cn}^2(t, k) = 1 \quad \text{and} \quad k^2 \text{sn}^2(t, k) + \text{dn}^2(t, k) = 1,$$

and the differential equations

\begin{align*}
\frac{d}{dt} \text{sn}(t, k) &= \text{cn}(t, k) \text{dn}(t, k), & \frac{d}{dt} \text{cn}(t, k) &= -\text{sn}(t, k) \text{dn}(t, k) \\
\text{and} & & \frac{d}{dt} \text{dn}(t, k) &= -k^2 \text{sn}(t, k) \text{cn}(t, k).
\end{align*}
When \( k = 0 \) or 1 they reduce to trigonometric functions:

\[
\begin{align*}
\text{sn}(t, 0) &= \sin t, \quad \text{cn}(t, 0) = \cos t, \quad \text{dn}(t, 0) = 1, \\
\text{sn}(t, 1) &= \tanh t, \quad \text{cn}(t, 1) = \text{dn}(t, 1) = \text{sech} t.
\end{align*}
\]

(49)

For \( k \in [0, 1) \) the Jacobi elliptic functions are periodic in \( t \), with \( \text{sn}(t, k) \) and \( \text{cn}(t, k) \) of period \( 4K(k) \) and \( \text{dn}(t, k) \) of period \( 2K(k) \), where

\[
K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}.
\]

(50)

### 8.2 Explicit solution using Jacobi elliptic functions

Following Haskins [7, §4], we shall solve (42) fairly explicitly. The answer depends on the order of \( a_1, a_2, a_3 \) and 0. For simplicity, we shall suppose that \( a_2 \leq a_1 < c < a_3 \). The solutions for the other possible orders may be obtained by permutations of 1, 2, 3, and sign changes, in an obvious way.

Now \( Q(u) - A^2 \) has three real roots \( \gamma_1, \gamma_2, \gamma_3 \), which may be ordered so that \( \gamma_1 \geq \gamma_2 \geq 0 > \gamma_3 \). Then \( Q(u) - A^2 = a_1 a_2 a_3 (u - \gamma_1)(u - \gamma_2)(u - \gamma_3) \), and the first equation of (42) becomes

\[
\left( \frac{du}{dt} \right)^2 = 4a_1 a_2 a_3 (u - \gamma_1)(u - \gamma_2)(u - \gamma_3).
\]

As in [7, Prop. 4.2] the solutions are \( u(t) = \gamma_3 + (\gamma_2 - \gamma_3) \text{sn}^2(at + c, b) \), where \( a^2 = a_1 a_2 a_3 (\gamma_1 - \gamma_3), \quad b^2 = (\gamma_2 - \gamma_3)/(\gamma_1 - \gamma_3), \quad c \in \mathbb{R} \), and \( \text{sn}(\cdot, \cdot) \) is the Jacobi snoidal function. This can easily be verified using (43).

Substituting into the other three equations of (42) gives explicit expressions for \( \frac{d\theta_j}{dt}, \frac{d\psi}{dt} \) and \( \frac{dw_2}{dt} \), so we obtain \( \theta_j, \psi \) and \( \psi \) by integration. We have proved:

**Proposition 8.6** Under the assumptions above, the solutions of (42) are

\[
\begin{align*}
\theta_j(t) &= \theta_j(0) - A \int_0^t \frac{dt}{\gamma_3 + \frac{a_3}{a_1} + (\gamma_2 - \gamma_3) \text{sn}^2(at + c, b)}, \quad (52)
\end{align*}
\]

where \( a^2 = a_1 a_2 a_3 (\gamma_1 - \gamma_3), \quad b^2 = (\gamma_2 - \gamma_3)/(\gamma_1 - \gamma_3) \) and \( c \in \mathbb{R} \).

When \( A = 0 \) we can write \( w_1, w_2, w_3 \) explicitly in terms of \( \text{sn}, \text{cn} \) and \( \text{dn} \). By (42) we may take \( \theta_j \equiv 0 \) so that \( w_1, w_2, w_3 \) are real. The roots \( \gamma_1, \gamma_2, \gamma_3 \) of \( Q(u) - A^2 \) are \(-1/a_1, -1/a_2, -1/a_3 \), and the orders \( a_2 < a_2 < a_3 \) and \( \gamma_1 \geq \gamma_2 \geq 0 > \gamma_3 \) imply that \( \gamma_j = -1/a_j \). Thus, combining (43) with (51) gives explicit expressions for \( w_2^2 \), which by (52) and the definition of \( b^2 \) above reduce to

\[
\begin{align*}
w_1^2 &= \frac{a_3 - a_1}{a_3} \text{dn}^2(at, b), \quad w_2^2 = \frac{a_3 - a_2}{a_3} \text{cn}^2(at, b), \quad w_3^2 = \frac{a_2 - a_3}{a_2} \text{sn}^2(at, b),
\end{align*}
\]

putting \( c = 0 \) for simplicity. Hence, from Theorem 8.2 we deduce:
Theorem 8.7 Let \( b_1, b_2, b_3 \in \mathbb{Z} \) satisfy \( b_2 > b_3 > 0 > b_1 \) and \( b_1 + b_2 + b_3 = 0 \). Define \( a_1, a_2, a_3 \) by (37), and \( a > 0 \) and \( b \in (0, 1) \) by
\[
a^2 = a_2(a_1 - a_3) \quad \text{and} \quad b^2 = \frac{a_1(a_2 - a_3)}{a_2(a_1 - a_3)}. \tag{53}
\]
Define a subset \( N \) of \( \mathbb{C}^3 \) by
\[
N = \left\{ \left( re^{ib_1s} \left( \frac{a_3-a_1}{a_3} \right)^{1/2} \text{dn}(at, b), re^{ib_2s} \left( \frac{a_3-a_2}{a_3} \right)^{1/2} \text{cn}(at, b), \right. \right.
\]
\[
\left. re^{ib_3s} \left( \frac{a_2-a_3}{a_2} \right)^{1/2} \text{sn}(at, b) : r > 0, \quad s, t \in \mathbb{R} \right\}. \tag{54}
\]
Then \( N \) is a special Lagrangian cone on \( T^2 \) in \( \mathbb{C}^3 \) with phase \( i \). Furthermore, \( (s, t) \) are conformal coordinates on \( N \cap S^3 \).

Here the expressions for \( a^2 \) and \( b^2 \) come from Proposition 8.4 by putting \( \gamma_j = -1/a_j \), and \( b_2 > b_3 \geq 0 > b_1 \) is equivalent to the condition \( a_2 \leq a_1 < 0 < a_3 \) above. The additional assumption \( b_3 > 0 \) ensures that \( b \in (0, 1) \), as \( b = 1 \) if and only if \( b_3 = 0 \). We have taken \( b_1, b_2, b_3 \in \mathbb{Z} \) to make the \( s \) coordinate periodic, with period \( 2\pi \). We know from (53) that \( \text{sn}(at, b), \text{cn}(at, b) \) and \( \text{dn}(at, b) \) are periodic in \( t \), as \( b \in (0, 1) \). Thus \( N \) is indeed a cone on \( T^2 \).

8.3 Relation with harmonic tori in \( \mathbb{C}P^2 \) and \( S^5 \)

In Proposition 8.2 we showed that each of the SL 3-folds \( N \) in \( \mathbb{C}^3 \) constructed in Theorem 8.2 is the cone on the image of a conformal harmonic map \( \Phi : \mathbb{R}^2 \to S^5 \) defined by (41). Then in 8.2 we showed that the functions \( w_1, w_2, w_3 \) may be written explicitly in terms of the Jacobi elliptic functions.

Thus we have constructed a family of explicit conformal harmonic maps \( \Phi : \mathbb{R}^2 \to S^5 \). Furthermore, as \( N \) is Lagrangian, one can show that if \( \pi : S^5 \to \mathbb{C}P^2 \) is the Hopf projection then \( \pi \circ \Phi \) is conformal and harmonic, so we also have a family of explicit conformal harmonic maps \( \Psi : \mathbb{R}^2 \to \mathbb{C}P^2 \).

Now harmonic maps from Riemann surfaces into spheres and projective spaces are an integrable system, and have been intensively studied in the integrable systems literature. For an introduction to the subject, see Fordy and Wood [4], in particular the articles by Bolton and Woodward [4, p. 59–82], McIntosh [4, p. 205–220] and Burstall and Pedit [4, p. 221–272].

Therefore our examples can be analyzed from the integrable systems point of view. We postpone this analysis to the sequel [13]. In [13, §5] we shall realize the SL 3-folds of Theorem 8.2 as special cases of a more general construction of special Lagrangian cones in \( \mathbb{C}^3 \), which involves two commuting o.d.e.s. Generic cones arising from this construction have only discrete symmetry groups.

Then in [13, §6] we work through the integrable systems framework for the corresponding family of harmonic maps \( \Psi : \mathbb{R}^2 \to \mathbb{C}P^2 \), showing that they are generically superconformal of finite type, and determining their harmonic sequences, Toda solutions, algebras of polynomial Killing fields, and spectral curves. From the integrable systems point of view, Theorem 8.5 is interesting because it constructs a large family of superconformal harmonic tori in \( \mathbb{C}P^2 \).
9 Construction by ‘perpendicular symmetries’

Now we explain a new construction of special Lagrangian submanifolds in \(\mathbb{C}^m\) beginning with a special Lagrangian \(m\)-fold \(L\) with ‘perpendicular symmetries’, that is, vector fields in \(\mathfrak{su}(m) \ltimes \mathbb{C}^m\) which are perpendicular to \(L\) at every point.

**Theorem 9.1** Let \(G\) be a \(k\)-dimensional abelian Lie subgroup of \(\text{SU}(m) \ltimes \mathbb{C}^m\) with Lie algebra \(\mathfrak{g}\), acting on \(\mathbb{C}^m\) with moment map \(\mu\), and let \(\phi : \mathfrak{g} \to C^\infty(T\mathbb{C}^m)\) be the corresponding action of \(\mathfrak{g}\) on \(\mathbb{C}^m\) by vector fields. Suppose \(L\) is an SL submanifold of \(\mathbb{C}^m\), such that \(\phi(x)\) is normal to \(L\) at \(L\) for every \(x\) in \(\mathfrak{g}\). For each \(c \in \mathfrak{g}\), define \(N_c = G \cdot (L \cap \mu^{-1}(c))\). Then \(N_c\) is special Lagrangian in \(\mathbb{C}^m\), with phase \(i\) if \(k\) is even, and phase \(i\) if \(k\) is odd.

**Proof.** We shall show that for each nonsingular point \(z\) in \(N_c\), the tangent plane \(T_z N_c\) is special Lagrangian. But as \(N_c\) is \(G\)-invariant and \(G \subset \text{SU}(m) \ltimes \mathbb{C}^m\), it is enough to verify this for one point in each orbit of \(G\) in \(N_c\). Thus we can restrict our attention to \(z \in L \cap \mu^{-1}(c)\). Then the condition for \(z\) to be a nonsingular point of \(N_c\) is that \(z\) is a nonsingular point of \(L\), and the vector fields \(\phi(\mathfrak{g})\) are linearly independent at \(z\).

Suppose these conditions hold. Choose a basis \(x_1, \ldots, x_k\) of \(\mathfrak{g}\) such that \(\phi(x_1)_{z}, \ldots, \phi(x_k)_{z}\) are orthonormal, which is possible by linear independence of \(\phi(\mathfrak{g})\) at \(z\). Now \(\phi(x_j)_{z}\) is normal to \(T_z L\), which is a special Lagrangian plane in \(\mathbb{C}^m\). Therefore \(I(\phi(x_j)_{z})\) lies in \(T_z L\), where \(I\) is the complex structure on \(\mathbb{C}^m\). Hence \(I(\phi(x_1)_{z}), \ldots, I(\phi(x_k)_{z})\) are orthonormal in \(T_z L\). Extend them to an orthonormal basis of \(T_z L\) with vectors \(v_1, \ldots, v_{m-k}\), so that

\[
T_z L = \left\langle I(\phi(x_1)_{z}), \ldots, I(\phi(x_k)_{z}), v_1, \ldots, v_{m-k}\right\rangle. \tag{55}
\]

Now the level sets of the moment map \(\mu\) of \(G\) are orthogonal to \(I(\phi(x_j)_{z})\) for all \(j\). Hence \(T_z (L \cap \mu^{-1}(c))\) is the subspace of \(T_z L\) orthogonal to \(I(\phi(x_j)_{z})\) for \(j = 1, \ldots, k\). Thus \(T_z (L \cap \mu^{-1}(c)) = \langle v_1, \ldots, v_{m-k}\rangle\). But \(N_c = G \cdot (L \cap \mu^{-1}(c))\), and so \(T_z N_c\) is the span of \(T_z (L \cap \mu^{-1}(c))\) and \(\phi(\mathfrak{g})_{z}\). Hence

\[
T_z N_c = \langle \phi(x_1)_{z}, \ldots, \phi(x_k)_{z}, v_1, \ldots, v_{m-k}\rangle. \tag{56}
\]

Comparing (55) and (56) and remembering that the bases are orthonormal, we see that in effect we have orthogonal direct sums \(T_z L = \mathbb{R}^k \oplus \mathbb{R}^{m-k}\) and \(T_z N_c = I(\mathbb{R}^k) \oplus \mathbb{R}^{m-k}\). It is easy to see that as \(T_z L\) is an SL plane with phase 1, this implies that \(T_z N_c\) is an SL plane with phase \(i^k\) or \(-i^k\), depending on the orientation chosen for \(N_c\). Thus, if \(k\) is even then \(N_c\) is special Lagrangian with phase 1, and if \(k\) is odd then \(N_c\) is special Lagrangian with phase \(i\), with the appropriate orientation.

The assumption that \(G\) is abelian was not actually used in the proof; but it is implied by the hypotheses, which is why we put it in. In the situation of the theorem, suppose \(\mathfrak{g}\) is not abelian, and let \(x, y \in \mathfrak{g}\). Then \(I(\phi(x)),\)
\( I(\phi(y)) \) are vector fields on \( \mathbb{C}^m \) tangent to \( L \) at \( L \). Hence the Lie bracket 
\[ [I(\phi(x)), I(\phi(y))] \] is also tangent to \( L \) at \( L \). But \( \phi(x), \phi(y) \) are holomorphic, 
and so 
\[ [I(\phi(x)), I(\phi(y))] = -[\phi(x), \phi(y)] = -\phi([x,y]). \]

Therefore \( \phi([x,y]) \) is tangent to \( L \), but it is also perpendicular to \( L \). So 
\( \phi([x,y]) = 0 \) on \( L \). This forces \( \phi([x,y]) = 0 \) on \( \mathbb{C}^m \), since otherwise \( L \) lies in 
some affine \( \mathbb{C}^{m'} \subset \mathbb{C}^m \) for \( m' < m \), which contradicts \( L \) being Lagrangian. If 
\( \phi \) is effective then we have shown that \([x,y] = 0\) for all \( x,y \in g \), so that \( g \) is 
abelian, and thus \( G \) is abelian as it is connected.

We now characterize the possibilities for \( G \) and \( L \) in Theorem 9.1.

**Theorem 9.2** Let \( G \) be a connected abelian Lie subgroup of \( SU(m) \times \mathbb{C}^m \) acting 
on the affine space \( \mathbb{C}^m \), let \( g \) be the Lie algebra of \( G \), and let \( \phi : g \to C^\infty(T\mathbb{C}^m) \) 
be the corresponding action of \( g \) on \( \mathbb{C}^m \) by vector fields.

Then there exists an \( SL \) submanifold \( L \) of \( \mathbb{C}^m \) such that \( \phi(x) \) is normal to 
\( L \) at \( L \) for every \( x \) in \( g \), if and only if the following conditions hold:

1. There exists a \( G \)-invariant affine isomorphism \( \mathbb{C}^m \cong \mathbb{C}^{a_1} \times \cdots \times \mathbb{C}^{a_{n+2}} \);
2. \( L \) is a subset of the product manifold \( L_1 \times L_2 \times \cdots \times L_{n+2} \), where \( L_j \) is a 
special Lagrangian submanifold of \( \mathbb{C}^{a_j} \);
3. For \( j = 1, \ldots, n \), \( L_j \) is a cone in \( \mathbb{C}^{a_j} \), and each \( \gamma \in G \) acts on \( \mathbb{C}^{a_j} \) by 
multiplication by \( e^{i\theta_j} \);
4. \( L_{n+1} = \mathbb{R}^{a_{n+1}} \in \mathbb{C}^{a_{n+1}} \), and \( G \) acts on \( \mathbb{C}^{a_{n+1}} \) by translations in the 
direction of \( I(\mathbb{R}^{a_{n+1}}) \); and
5. \( G \) acts trivially on \( \mathbb{C}^{a_{n+2}} \).

**Proof.** For simplicity, we first suppose that \( G \) lies in the subgroup \( SU(m) \) of \( SU(m) \times \mathbb{C}^m \), and treat \( \mathbb{C}^m \) as a vector space rather than an affine space. Then 
\( g \) is an abelian Lie subalgebra of \( su(m) \), which we may regard as a vector space of 
commuting matrices. By standard results in linear algebra, we may decompose 
the complex vector space \( \mathbb{C}^m \) into a direct sum of *eigenspaces* of the action of \( g \).

Actually, one usually considers the eigenspaces of a single matrix, rather than 
of a vector space of commuting matrices. But the eigenspace decomposition of 
\( \mathbb{C}^m \) under a generic element of \( g \) is the same as its decomposition under \( g \), so 
the two points of view are equivalent.

Let us write the eigenspace decomposition as
\[ \mathbb{C}^m = \mathbb{C}^{a_1} \oplus \cdots \oplus \mathbb{C}^{a_n} \oplus V, \tag{57} \]
where \( \mathbb{C}^{a_1}, \ldots, \mathbb{C}^{a_n} \) are nonzero eigenspaces of \( g \) in \( \mathbb{C}^m \) with distinct, nonzero 
eigenvalues in \( ig^* \), and \( V \) is the zero eigenspace of \( g \). The decomposition \( (57) \) 
is unique up to the order of the subspaces \( \mathbb{C}^{a_1}, \ldots, \mathbb{C}^{a_n} \), and is orthogonal 
as \( g \subset u(m) \).

Each \( x \in g \) acts on \( \mathbb{C}^{a_j} \) by multiplication by \( it\theta_j \) for some \( \theta_j \in \mathbb{R} \), and is 
zero on \( V \). Now \( G \) is connected and abelian, so \( G \cong \mathbb{R}^k \), and \( \exp : g \to G \) is
surjective. Hence we can write each $\gamma \in G$ as $\exp(x)$ for $x \in \mathfrak{g}$, and so $\gamma$ acts on $\mathbb{C}^{a_j}$ by multiplication by $e^{i\theta_j}$, and as the identity on $V$. Putting $a_{n+1} = 0$ and $\mathbb{C}^{a_{n+2}} = V$, we have shown that (57) satisfies (i), (v) and the second part of (iii).

Now consider the general case with $G \subset SU(m) \ltimes \mathbb{C}^m$. By projecting $G$ from $SU(m) \ltimes \mathbb{C}^m$ to $SU(m)$ we can reduce to the previous case, and decompose $\mathbb{C}^m$ into eigenspaces. However, we now have to allow for $\mathfrak{g}$ to act by translations in each factor, as well as by $\mathfrak{su}(m)$ rotations.

Since the $\mathfrak{su}(m)$ part of $\mathfrak{g}$ acts on $\mathbb{C}^{a_1}, \ldots, \mathbb{C}^{a_n}$ with nonzero eigenvalue, and $\mathfrak{g}$ is abelian, by moving the origin in $\mathbb{C}^{a_j}$ we can eliminate the translation part, so that $G$ acts on $\mathbb{C}^{a_j}$ by multiplication by $e^{i\theta_j}$ for $j = 1, \ldots, n$, as in (iii). Moving the origin is allowed, as we seek only an affine isomorphism $\mathbb{C}^m \cong \mathbb{C}^{a_1} \times \cdots \times \mathbb{C}^{a_{n+2}}$, rather than a vector space isomorphism.

However, moving the origin in $V$ in (55) has no effect on the $V$ translation-component of the action of $\mathfrak{g}$, because this is the zero eigenspace of the $\mathfrak{su}(m)$ part of $\mathfrak{g}$. So we cannot eliminate translations in the $V$ directions by choosing the origin appropriately. Instead, define $\mathbb{C}^{a_{n+1}}$ to be the complex vector subspace of $V$ generated by the $V$ translation-components of $\mathfrak{g}$, and let $\mathbb{C}^{a_{n+2}}$ be the orthogonal complement to $\mathbb{C}^{a_{n+1}}$ in $V$.

Then we have an affine isomorphism $\mathbb{C}^m \cong \mathbb{C}^{a_1} \times \cdots \times \mathbb{C}^{a_{n+2}}$ such that each $\gamma \in G$ acts on $\mathbb{C}^{a_j}$ by multiplication by $e^{i\theta_j}$ for $j = 1, \ldots, n$, $G$ acts by translations on $\mathbb{C}^{a_{n+1}}$, and $G$ acts trivially on $\mathbb{C}^{a_{n+2}}$, so that (i), (v) and parts of (iii) and (iv) are satisfied.

Now we have put the $G$-action in a standard form, we prove the ‘only if’ part of the theorem. Suppose $L$ is special Lagrangian in $\mathbb{C}^m$, and $\phi(x)$ is normal to $L$ at $L$ for every $x$ in $\mathfrak{g}$. The key idea we shall use is that for each $x$ in $\mathfrak{g}$, as $\phi(x)$ is normal to $L$ and $L$ is Lagrangian, the vector field $I(\phi(x))$ is tangent to $L$ at $L$. By exponentiating $I(\phi(x))$ we get a 1-parameter family of diffeomorphisms of $\mathbb{C}^m$, which locally preserve $L$. That is, for each $z$ in the interior of $L$, there exists $\epsilon > 0$ such that $\exp(t I(\phi(x))) z \in L$ for $t \in (-\epsilon, \epsilon)$.

As $\mathfrak{g}$ is abelian and the vector fields $\phi(\mathfrak{g})$ are holomorphic, we see that

$$\exp(I(\phi(x))) \circ \exp(I(\phi(y))) = \exp(I(\phi(x + y)))$$

for all $x, y \in \mathfrak{g}$. Thus the $\exp(I(\phi(x)))$ form an abelian Lie group $\exp(I(\phi(\mathfrak{g})))$ of diffeomorphisms of $\mathbb{C}^m$, isomorphic to $\mathbb{R}^k$. We use this to extend $L$ to a globally invariant submanifold $L'$. Define

$$L' = \bigcup_{x \in \mathfrak{g}} \exp(I(\phi(x))) (L).$$

Then it is not difficult to show that $L'$ is a special Lagrangian submanifold of $\mathbb{C}^m$ containing $L$, invariant under $\exp(I(\phi(\mathfrak{g})))$.

One way to prove this is to use real analyticity, and the results of [4]. Each connected component $L'_i$ of the interior of $L'$ contains a connected component $L_i$ of the interior of $L$. As $L_i$ is real analytic and $L'_i$ is the orbit of $L_i$ under a
Lie group, \( L' \) is also real analytic. So as the special Lagrangian condition holds on a nonempty open subset \( L_i \) of \( L' \), it holds on all of \( L' \).

As \( L' \) is Lagrangian we have \( \omega|_{L'} \equiv 0 \). But \( \exp(I(\phi(x)))(L') = L' \) for \( x \in g \). Hence

\[
\exp(I(\phi(x)))^*(\omega)|_{L'} \equiv 0 \quad \text{for all } x \in g. \tag{58}
\]

Write \( \omega = \sum_{j=1}^{n+2} \omega^j \), where \( \omega^j \) is the projection of \( \omega \) to \( C^{a_j} \). Let \( x \in g \) act on \( C^{a_j} \) by multiplication by \( i \theta_j \) for \( j = 1, \ldots, n \). Then

\[
\exp(I(\phi(x)))^*(\omega) = e^{-2\theta_1} \omega^1 + \cdots + e^{-2\theta_n} \omega^n + \omega^{n+1} + \omega^{n+2}. \tag{59}
\]

Combining (58) and (59) for all \( x \in g \), and remembering that the eigenvalues of \( g \) on \( C^{a_1}, \ldots, C^{a_n} \) are distinct and nonzero, we see that

\[
\omega^j|_{L'} \equiv 0 \quad \text{for } j = 1, \ldots, n, \quad \omega^{n+1}|_{L'} + \omega^{n+2}|_{L'} \equiv 0.
\]

By considering the tangent spaces of \( L' \) we find that \( L' \) admits a local product structure, and deduce that \( L' \subseteq L_1 \times \cdots \times L_n \times N \), where \( L_j \) is a Lagrangian submanifold of \( C^{a_j} \) for \( j = 1, \ldots, n \), and \( N \) is Lagrangian in \( C^{a_{n+1}} \times C^{a_{n+2}} \).

Let the \( L_j \) and \( N \) be as small as possible such that \( L' \subseteq L_1 \times \cdots \times L_n \times N \). This defines the \( L_j \) and \( N \) uniquely. As \( L' \) is special Lagrangian, it follows that \( L_1, \ldots, L_n \) and \( N \) are actually special Lagrangian in \( C^{a_1}, \ldots, C^{a_n} \) and \( C^{a_{n+1}} \times C^{a_{n+2}} \), with some phases; we can fix the phases to be 1 by choosing the holomorphic volume forms on \( C^{a_j} \) appropriately.

Since \( L_1, \ldots, L_n \) and \( N \) are defined uniquely using \( L' \), which is invariant under \( \exp(I(\phi(g))) \), the \( L_1, \ldots, L_n \) and \( N \) must also be invariant under \( \exp(I(\phi(g))) \). But \( \exp(I(\phi(x))) \) multiplies by \( e^{-\theta_j} \) in \( C^{a_j} \) for some \( \theta_j \in \mathbb{R} \) for \( j = 1, \ldots, n \), and \( \theta_j \) can take any value in \( \mathbb{R} \) as \( x \) varies. Therefore \( L_j \) is invariant under all dilations of \( C^{a_j} \) for \( j = 1, \ldots, n \), and is a cone by Definition 6.2.

This proves part (iii) of Theorem 9.2. But the ‘if’ part follows very easily, given the discussion above, so the proof is complete. \( \square \)
The theorem tells us that to apply Theorem 9.1, we need examples of SL cones \( L_j \) in \( \mathbb{C}^a_j \). Now the most obvious SL cone in \( \mathbb{C}^a_j \) is \( \mathbb{R}^a_j \). If we take \( L_j = \mathbb{R}^a_j \) for all \( j \) then \( L' \) is just \( \mathbb{R}^m \) in \( \mathbb{C}^m \), and we easily prove:

**Proposition 9.3** Let \( 2 \leq n \leq m \), and let \( U(1)^{n-1} \times \mathbb{R}^{m-n} \) act on \( \mathbb{C}^m \) by

\[
(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}, x_{n+1}, \ldots, x_m) : (z_1, \ldots, z_m) \mapsto (e^{i\theta_1}z_1, \ldots, e^{i\theta_{n-1}}z_{n-1}, e^{-i(\theta_1+\cdots+\theta_{n-1})}z_n, z_{n+1} + ix_{n+1}, \ldots, z_m + ix_m),
\]

for \( \theta_1, \ldots, \theta_{n-1} \in [0, 2\pi) \) and \( x_{n+1}, \ldots, x_m \in \mathbb{R} \). Let \( G \) be any connected Lie subgroup of \( U(1)^{n-1} \times \mathbb{R}^{m-n} \), and \( L \) be \( \mathbb{R}^m \) in \( \mathbb{C}^m \). Then Theorem 9.1 applies to \( G \) and \( L \), and constructs a family of \( G \)-invariant SL submanifolds \( N_c \) in \( \mathbb{C}^m \) with phase 1 or \( i \), depending on \( c \in \mathfrak{g}^* \).

This gives many families of SL submanifolds in \( \mathbb{C}^m \), which can be written down very explicitly. Here is an example with \( G = U(1) \).

**Example 9.4** Let \( a_1, \ldots, a_m \) be integers with \( \text{hcf}(a_1, \ldots, a_m) = 1 \) and \( a_1 + \cdots + a_m = 0 \), and let \( G \) be \( U(1) \) acting on \( \mathbb{C}^m \) by

\[
e^{i\theta} : (z_1, \ldots, z_m) \mapsto (e^{ia_1\theta}z_1, \ldots, e^{ia_m\theta}z_m).
\]

Then \( G \) lies in \( \text{SU}(m) \). The moment map of this \( G \)-action is

\[
\mu : (z_1, \ldots, z_m) \mapsto a_1|z_1|^2 + \cdots + a_m|z_m|^2.
\]

Take \( L \) to be \( \mathbb{R}^m \) in \( \mathbb{C}^m \), and apply Proposition 9.3 and Theorem 9.1. We find that for each \( c \in \mathbb{R} \), the subset \( N_{c}^{a_1, \ldots, a_m} \) in \( \mathbb{C}^m \) given by

\[
N_{c}^{a_1, \ldots, a_m} \equiv \{ (e^{ia_1\theta}x_1, \ldots, e^{ia_m\theta}x_m) : \theta \in [0, 2\pi), \quad x_1, \ldots, x_m \in \mathbb{R}, \quad a_1x_1^2 + \cdots + a_mx_m^2 = c \}
\]

is an SL \( m \)-fold with phase \( i \). Define

\[
\mathcal{H}_{c}^{a_1, \ldots, a_m} = \{ (x_1, \ldots, x_m) : x_j \in \mathbb{R}, \quad a_1x_1^2 + \cdots + a_mx_m^2 = c \}.
\]

Then \( N_{c}^{a_1, \ldots, a_m} \) is the image of \( \mathcal{S}^1 \times \mathcal{H}_{c}^{a_1, \ldots, a_m} \) under the map

\[
\Phi : (e^{i\theta}, (x_1, \ldots, x_m)) \mapsto (e^{ia_1\theta}x_1, \ldots, e^{ia_m\theta}x_m).
\]

When \( c \neq 0 \) this is an *immersion*, so that \( N_{c_1}^{a_1, \ldots, a_m} \) is a nonsingular immersed \( m \)-submanifold. Also \( N_{c}^{a_1, \ldots, a_m} \) is a *cone* with an isolated singular point at 0, and is otherwise nonsingular as an immersed submanifold.

However, \( \Phi \) is generally not injective. Generically \( \Phi \) is 2:1, since

\[
\Phi(e^{i\theta}, (x_1, \ldots, x_m)) = \Phi(-e^{i\theta}, ((-1)^{a_1}x_1, \ldots, (-1)^{a_m}x_m)).
\]
Thus we may regard $N_c^{a_1,\ldots,a_m}$ as an immersion of $(S^1 \times H_{a_1^{a_2,\ldots,a_m}})/\mathbb{Z}_2$ in $\mathbb{C}^m$. When two or more of the $x_j$ vanish $\Phi$ may become $2k : 1$ for $k > 1$, and then $N_c^{a_1,\ldots,a_m}$ is singular as an embedded submanifold.

Here is some more detail on the topology of this example when $m = 3$.

**Example 9.5** Let $a_1, a_2$ be positive, coprime integers and $a_3 = -a_1 - a_2$, let $c \in \mathbb{R}$, and let $N_c^{a_1,a_2,a_3}$ and other notation be as in Example 9.4. Then if $c > 0$ then $\Phi$ is $2:1$ everywhere, and $N_c^{a_1,a_2,a_3}$ is an embedded 3-fold diffeomorphic to $(T^2 \times \mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts freely on $T^2 \times \mathbb{R}$. If $a_3$ is even then $\mathbb{Z}_2$ acts trivially on $\mathbb{R}$, and $N_c^{a_1,a_2,a_3}$ is diffeomorphic to $T^2 \times \mathbb{R}$. If $a_3$ is odd, then $N_c^{a_1,a_2,a_3}$ is the total space of a nontrivial real line bundle over the Klein bottle, and has only one end modelled on $T^2 \times (0, \infty)$.

When $c = 0$, there are two cases: if $a_3$ is even then $N_0^{a_1,a_2,a_3}$ is the union of two opposite, embedded $T^2$-cones, meeting at $0$, and if $a_3$ is odd then $N_0^{a_1,a_2,a_3}$ is just one embedded $T^2$-cone. The difference is that $H_0^{a_1,a_2,a_3} \setminus \{0\}$ separates into two cones with $x_3 > 0$ and $x_3 < 0$, and whether $a_3$ is even or odd determines whether the identity fixes these cones or swaps them.

When $c < 0$, if $a_3$ is even then $N_c^{a_1,a_2,a_3}$ is the union of two immersed copies of $S^1 \times \mathbb{R}^2$, and if $a_3$ is odd then $N_c^{a_1,a_2,a_3}$ is just one immersed $S^1 \times \mathbb{R}^2$. Note also that $\Phi$ maps $-2a_3$ points of the form $(e^{i\theta}, (0,0,x_3))$ to one point in $\mathbb{C}^3$, so $N_c^{a_1,a_2,a_3}$ is singular as an embedded submanifold along an $S^1$ in $\mathbb{C}^3$.

It can be shown that $N_1^{1,1,-2}$ is isomorphic to $L_{c/2,c/2,0}$ of Example 5.1 under a change of coordinates, and has a $T^2$ symmetry group. But in general $N_c^{a_1,a_2,a_3}$ is a new example, not of cohomogeneity one for $c \neq 0$.

Here is another example in $\mathbb{C}^3$.

**Example 9.6** Let $G$ be $\mathbb{R}$, acting on $\mathbb{C}^3$ by

$$t : (z_1, z_2, z_3) \mapsto (e^{it}z_1, e^{-it}z_2, z_3 + it).$$

Then $G$ lies in $\text{SU}(3)$. The moment map of this $G$-action is

$$\mu : (z_1, z_2, z_3) \mapsto |z_1|^2 - |z_2|^2 + 2 \text{Re } z_3.$$  

Applying Proposition 9.3 and Theorem 9.1 with $c = 0$ shows that

$$N = \left\{ (e^{it}x_1, e^{-it}x_2, x_3 + it) : t, x_1, x_2, x_3 \in \mathbb{R}, \quad x_1^2 - x_2^2 + 2x_3 = 0 \right\}$$

is an SL 3-fold in $\mathbb{C}^3$ with phase $i$, which is nonsingular and diffeomorphic to $\mathbb{R}^3$. One can picture $N$ as being a bit like a helicoid in $\mathbb{R}^3$.

In our last example we construct a family of SL 4-folds $N_c$ in $\mathbb{C}^4$ out of an SL cone $L_1$ in $\mathbb{C}^3$, using the ideas of Theorems 7.3 and 7.2.

**Example 9.7** Let $G$ be $U(1)$ acting on $\mathbb{C}^4$ by

$$e^{i\theta} : (z_1, \ldots, z_4) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3, e^{-3i\theta}z_4).$$

$$e^{i\theta} : (z_1, \ldots, z_4) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3, e^{-3i\theta}z_4).$$

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Then $G \subset SU(4)$, and has moment map
$$\mu : (z_1, \ldots, z_4) \mapsto |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2.$$ 

Let $L_1$ be an SL cone in $\mathbb{C}^3$. Then applying Theorems 9.1 and 9.2 to $G$ and $L = L_1 \times \mathbb{R}$ in $\mathbb{C}^3 \times \mathbb{C}$, we find that for each $c \in \mathbb{R}$,
$$N_c = \left\{ (e^{i\theta}x_1, e^{i\theta}x_2, e^{i\theta}x_3, e^{-3i\theta}x_4) : \theta \in [0, 2\pi), \right. \\
(x_1, x_2, x_3) \in L_1, \quad x_4 \in \mathbb{R}, \quad |x_1|^2 + |x_2|^2 + |x_3|^2 - 3x_4^2 = c \left. \right\}$$
is an SL 4-fold in $\mathbb{C}^4$ with phase $i$.

Now let $L_1$ be a cone on a compact Riemann surface $\Sigma$, with an isolated singular point at 0, and suppose for simplicity that $L_1 \cap e^{2\pi i/3}L_1 = \{0\}$. Then for $c > 0$ we find that $N_c$ is a nonsingular, embedded SL 4-fold diffeomorphic to $\Sigma \times S^1 \times \mathbb{R}$. Similarly, $N_0$ is a cone on $2 \Sigma \times S^1$ with an isolated singularity at 0, and if $c < 0$ then $N_c$ is singular on an $S^1$ in $\mathbb{C}^4$.

References


