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Stability of patterns with arbitrary period for a Ginzburg-Landau equation with a mean field

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We consider the following system of equations:

$$\begin{cases} A_t = A_{xx} + A - A^3 - AB, \quad t > 0, 0 < x < \frac{L}{2}, \\ B_t = \sigma B_{xx} + \mu(A^2)_{xx}, \quad t > 0, 0 < x < \frac{L}{2}, \\ A_x(0) = A_x\left(\frac{L}{2}\right) = 0, \\ B_x(0) = B_x\left(\frac{L}{2}\right) = 0, \end{cases}$$

where the spatial average $\langle B \rangle = 0$ and $\mu > \sigma > 0$. This system plays an important role as a Ginzburg-Landau equation with a mean field in several areas of the applied sciences and the steady-states of this system extend to periodic steady-states with period L on the real line which are observed in experiments. Our approach is by combining methods of nonlinear functional analysis such as nonlocal eigenvalue problems and the variational characterization of eigenvalues with Jacobi elliptic integrals. This enables us to give a complete classification of all stable steady-states for any positive L.

1 Introduction

In this paper, we study the existence and stability of periodic solutions for the following amplitude equations:

$$\begin{cases} A_t = A_{xx} + A - |A|^2 A - AB, \quad t > 0, 0 < x < \frac{L}{2}, \\ B_t = \sigma B_{xx} + \mu(|A|^2)_{xx}, \quad t > 0, 0 < x < \frac{L}{2}, \\ A_x(0) = A_x\left(\frac{L}{2}\right) = 0, \\ B_x(0) = B_x\left(\frac{L}{2}\right) = 0 \end{cases} \tag{1.1}$$

for all $\mu > \sigma > 0$. We assume throughout the paper that $L > 0$ is arbitrary and

$$\langle B \rangle = \frac{1}{L} \int_{-L/2}^{L/2} B(x)\, dx = 0.$$

Amplitude equations of the form (1.1) play a role in various areas of the applied sciences. In Coullet & Iooss [3], the system (1.1) was derived in the study of secondary stability of a one-dimensional cellular pattern. In Cox & Matthews [4] the system (1.1) was derived from various models arising in thermosolutal convection, rotating convection, or magnetoconvection.

Systems similar to (1.1) play an important role in binary fluid convection [16, 17]. They even arise in the modeling of sand banks and sand waves [11]. For a survey on hydrodynamics applications see Fauve [9]. For applications in biology or chemistry we refer to the survey paper by Cross & Hohenberg [5].

In the system (1.1), A may be complex. In this case one can decompose $A = R\exp(i\theta)$ with functions R and θ representing the amplitude and the phase of A, respectively. An explicit analytical treatment of the general case is very complicated. In this paper we therefore restrict our attention to the invariant subspace in which A **is real.** Both cases are physically relevant. The coefficients of the equation for A in general are complex. However, they are real when the partial differential equation from which the amplitude equation has been derived has a reflection symmetry for the spatial variable [14].

It is important to understand (1.1) for finite L since periodic solutions of any period play an important role for the system (1.1) posed on the real line. In general, the period will depend on physical parameters which are represented by the constants σ and μ in the system. The case of finite L is important to understand patterns which are not well separated, i.e. whose distance is not large in relation with the decay rate.

We now reduce system (1.1) to its final form. By setting $\tau = \frac{1}{\sigma}$, $\mu' = \frac{\mu}{\sigma}$, the system (1.1) can be written as follows:

$$\begin{cases} A_t = A_{xx} + A - |A|^2 A - AB, \quad t > 0,\, 0 < x < \frac{L}{2}, \\ \tau B_t = B_{xx} + \mu'(|A|^2)_{xx}, \quad t > 0,\, 0 < x < \frac{L}{2}, \\ A_x(0) = A_x\left(\frac{L}{2}\right) = 0, \\ B_x(0) = B_x\left(\frac{L}{2}\right) = 0, \quad \langle B \rangle = 0, \end{cases} \tag{1.2}$$

where

$$\tau > 0, \quad \mu' > 1.$$

Let us now suppose that we have a periodic steady-state of (1.2) with $A(x) > 0$ for all $0 < x < L/2$ and period L. Using Rolle's theorem, one sees that there exists an $x_0 \in \left(-\frac{L}{2}, \frac{L}{2}\right)$ such that $A'(x_0) = 0$ and by periodicity we also have $A'(x_0 + L) = 0$ and $A(x_0 + L) = A(x_0)$. Further, by the structure of (1.2), after a translation such that x_0 becomes zero any steady-state with $A > 0$ having period L can be represented by an even function on $\left(-\frac{L}{2}, \frac{L}{2}\right)$ with Neumann boundary conditions, or, equivalently, by a function on $\left(0, \frac{L}{2}\right)$ with Neumann boundary conditions.

On the other hand, for a function defined on the interval $\left(0, \frac{L}{2}\right)$ with Neumann boundary conditions, by even and periodic continuation we get a periodic steady-state on the real line with period L.

We will use this representation of a periodic function with period L by a function on the interval $(0, L/2)$ with Neumann boundary conditions throughout the paper. Thus, from now on, we study **solutions on** $\left(0, \frac{L}{2}\right)$ **with Neumann boundary conditions**.

We study these solutions by combining methods of nonlinear functional analysis such as nonlocal eigenvalue problems and the variational characterization of eigenvalues with Jacobi elliptic integrals. Using this rigorous approach, we give a complete classification of existence and stability of all steady-states defined for all $L > 0$.

By the remarks above, the steady-states of (1.2) satisfies

$$\begin{cases} A_{xx} + A - A^3 - AB = 0, \ 0 < x < \frac{L}{2}, \\ B_{xx} + \mu'(A^2)_{xx} = 0, \ 0 < x < \frac{L}{2}, \\ A_x(0) = A_x\left(\frac{L}{2}\right) = 0, \\ B_x(0) = B_x\left(\frac{L}{2}\right) = 0, \quad \langle B \rangle = 0. \end{cases} \tag{1.3}$$

Let $(A(x), B(x))$ be a steady state of (1.2). $(A(x), B(x))$ is said to be **linearly stable** if the following linearized eigenvalue problem

$$\begin{cases} \phi_{xx} + (1-B)\phi - 3A^2\phi - A\psi = \lambda_L \phi, \quad 0 < x < \frac{L}{2}, \\ \psi_{xx} + 2\mu'(A\phi)_{xx} = \tau\lambda_L\psi, \quad 0 < x < \frac{L}{2}, \\ \phi_x(0) = \phi_x(\frac{L}{2}) = 0, \psi_x(0) = \psi_x(\frac{L}{2}) = 0, < \psi >= 0, \\ \lambda_L \in \mathscr{C} \end{cases} \tag{1.4}$$

admits only eigenvalues with negative real parts. It is **linearly unstable** if (1.4) admits an eigenvalue with positive real part. It is **neutrally stable** if (1.4) admits an eigenvalue with zero real part, and all other eigenvalues have negative real parts. (In the appendix, we shall prove that all eigenvalues of (1.4) are **real**.)

We now state our main results. First we have

Theorem 1 *Let* $L > 0$ *and* $\mu' > 1$ *be fixed. Assume that* $A(x) > 0$ *and* $A^{'}(x) < 0$ *for* $0 < x < \frac{L}{2}$. *Then there exist two numbers* $\mu_1(L) > \mu_2(L) > 1$ *(to be given explicitly in* (3.26)) *such that the following holds.*

(1) If $\mu' > \mu_1(L)$, *all solutions of* (1.3) *are* **constant**.

(2) If $\mu' = \mu_1(L)$, *there exists exactly* **one** *solution of* (1.3).

(3) If $\mu_2(L) < \mu' < \mu_1(L)$, *there exist exactly* **two** *solutions of* (1.3).

(4) If $1 < \mu' \leqslant \mu_2(L)$, *there exists exactly* **one** *solution of* (1.3).

The next theorem classifies the instability of large classes of steady-state solutions.

Theorem 2 *All solutions of* (1.3)

(1) for which A *changes sign or*

(2) for which A_x *changes sign*

are **linearly unstable** *steady-states of the corresponding parabolic system (1.2).*

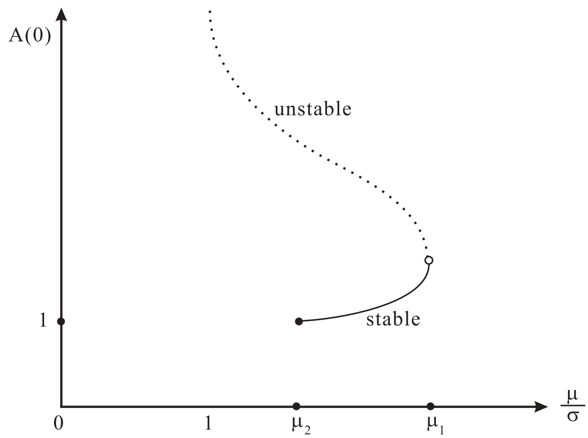

FIGURE 1. Bifurcation Curve.

Thus it only remains to study the stability of solutions of (1.3) for which A is positive and strictly monotone for $0 < x < \frac{L}{2}$ or for which A is constant. They are all given by Theorem 1. We have

Theorem 3 *For the solutions of* (1.3) *given in Theorem 1, we have*

(1) If $\mu' = \mu_1(L)$*, there exists exactly* **one** *solution of* (1.3)*. This solution is* **neutrally stable**.

(2) If $\mu_2(L) < \mu' < \mu_1(L)$*, there exist exactly* **two** *solutions of* (1.3)*. The one with small amplitude* $A(0)$ *is* **linearly stable** *and the one with large amplitude* $A(0)$ *is* **linearly unstable**.

(3) If $1 < \mu' \leqslant \mu_2(L)$*, there exists exactly* **one** *solution of* (1.3) *and this solution is* **linearly unstable**.

The explicit values of $\mu_1(L), \mu_2(L)$ are given in § 3. In particular,

$$\mu_2(L) = 1 + \frac{2\pi^2}{L^2} \tag{1.5}$$

and $\mu_1(L)$ is given in terms of elliptic integrals. Both $\mu_1(L)$ and $\mu_2(L)$ are functions of L – the domain size – only. Thus we have rigorously established the bifurcation picture of Figure 1 (see also Figure 3 of Matthews & Cox [13]).

Remarks

(1) The constant solution $A \equiv 1$ is unstable.

(2) The even and periodic extension to the real line of a strictly monotone solution of (1.3) has **minimal period** L. This explains the important role which the solutions of Theorem 1 play in the understanding of periodic solutions on the real line.

(3) Since solutions with sign-changing A or A' are unstable by Theorem 2 we do not pursue them any further and assume that $A > 0$ and $A' < 0$.

Previous numerical and analytical studies of these amplitude equation include Matthews & Cox [13] (numerical simulation, asymptotic expansion and bifurcation theory, in particular the use of Jacobi elliptic integrals to describe the shape of solutions and a numerical study of stability) and Norbury *et al.* [15] (rigorous study of the limit when the minimal period is large enough). In [15] the resulting steady-states are pulses or spikes and nonlocal eigenvalue problems are used, but no Jacobi elliptic integrals. In particular, we showed [15] that there are two large single pulse solutions, where one is stable and the other one is unstable.

Various other terms are used in the literature in the second equation of (1.1). One type, where the term $(|A|^2)_{xx}$ in the B-equation is replaced by $(|A|^2)_x$ has been considered by several authors, see Riecke [18] and the references therein. In that case the basic patterns are traveling pulses which arise in the convection of binary fluids. In Winterbottom *et al.* [24] a complex Ginzburg-Landau equation is coupled to a pseudoscalar. The stability of traveling and standing waves is studied by asymptotic expansions for large wavelengths and by numerical methods. In Doelman *et al.* [8] a real Ginzburg-Landau system with another type of diffusive mode B is studied on the real line using geometric singular perturbation theory and hypergeometric functions. Both these papers consider dynamic phenomena, whereas we consider only static patterns. On the hand, our method can analytically cover the case of finite minimal period in contrast to both papers.

The organization of the paper is as follows. In § 2, we perform a scaling argument and reduce the existence problem to an algebraic equation for consistency. In § 3, we use elliptic integrals to rigorously solve the algebraic equation for consistency and prove Theorem 1. In § 4, the spectrum of a linearized operator is analyzed and a key identity is derived. In § 5, we study a crucial nonlocal eigenvalue problem. § 6 contains the proof of Theorem 3, the main result on stability. In § 7, we prove Theorem 2 by showing the instability of all other solutions, invoking the variational characterization of eigenvalues. Section 8 is the conclusion section, where our results are summarized and an outlook is given. In the Appendix, the linear operator is first derived and then reduced to a self-adjoint nonlocal eigenvalue problem which is given in the reduction lemma, Lemma 13.

2 Scaling and consistency algebraic equation

In this section, we rescale the steady-state equation (1.3) and reduce it to a single ordinary differential equation coupled with an algebraic equation for consistency.

Integrating the second equation of (1.3) twice, we derive

$$B(x) = -\mu' A^2(x) + \mu' \langle A^2 \rangle, \qquad \text{where } \langle A^2 \rangle = \frac{2}{L} \int_I A^2(x)dx \tag{2.1}$$

and $I = \left(0, \frac{L}{2}\right)$. Substituting (2.1) into the first equation of (1.3), we obtain

$$\begin{cases} A_{xx} - aA + bA^3 = 0, \quad 0 < x < \frac{L}{2}, \\ A_x(0) = A_x\left(\frac{L}{2}\right) = 0, \\ A(x) > 0,\ A_x(x) < 0 \quad \text{for } 0 < x < \frac{L}{2}. \end{cases} \tag{2.2}$$

where

$$a = \mu'\langle A^2\rangle - 1, \quad b = \mu' - 1. \tag{2.3}$$

We consider a as a real and b as a positive parameter. Now we solve (2.2) with consistency condition (2.3). Note that a positive solution of (2.2) exists if and only if $a > 0$ (since $b > 0$). So we consider (2.2) in the subcritical case.

In this case, we let $\beta = \sqrt{a}$ and

$$A(x) = \frac{\beta}{\sqrt{\mu' - 1}}\, w_l(y), \tag{2.4}$$

where

$$y = \beta x, \quad \frac{\beta L}{2} = l. \tag{2.5}$$

Then w_l solves the following boundary value problem:

$$\begin{cases} w_{l,yy} - w_l + w_l^3 = 0, & w_{l,y}(0) = w_{l,y}(l) = 0, \\ w_{l,y}(y) < 0, \quad w_l(y) > 0 & \text{for } 0 < y < l. \end{cases} \tag{2.6}$$

In Lemma 5 we will show that (2.6) has a unique solution if $l > \pi/\sqrt{2}$ and no solution if $l \leqslant \pi/\sqrt{2}$.

Substituting (2.4) into (2.3), it is easy to see that β will have to satisfy the following consistency equation:

$$\beta^2 - \frac{2\mu'}{L(\mu' - 1)}\beta \int_0^{\frac{\beta L}{2}} w_l^2 dy + 1 = 0. \tag{2.7}$$

We write (2.7) in terms of the new length $l = \frac{\beta L}{2}$:

$$g(l) \equiv l^2 - tl \int_0^l w_l^2 dy + \frac{L^2}{4} = 0, \text{ where } t = \frac{\mu'}{\mu' - 1}. \tag{2.8}$$

In (2.8), t is a fixed parameter, L is a given parameter (which corresponds to the domain size), and l is the unknown. Note that the unknown l appears in the equation (2.8) in three ways: explicitly, as the upper boundary of the integral and as the index of the function w_l. The last two dependencies are new compared to the singular limit case $L \geqslant 1$ and they make this study more difficult than the singular limit case.

If there exists a solution l to (2.8), then, by taking

$$A(x) = \frac{2l}{L\sqrt{\mu' - 1}}\, w_l\left(\frac{2lx}{L}\right), \tag{2.9}$$

we obtain a solution of (2.2). Conversely, by taking (2.9), l must satisfy (2.8). Thus we have reduced our problem to showing existence of (2.6) (see Lemma 5) and solving the algebraic equation (2.8) for consistency.

3 Solving the consistency equation

In this section, we show that (2.6) has a unique solution if $l > \pi/\sqrt{2}$ and no solution if $l \leqslant \pi/\sqrt{2}$ (see Lemma 5) and we will solve the consistency equation (2.8). Our idea is to represent the solution of the ordinary differential equation (2.6) by Jacobi elliptic integrals. Then we use their properties to show our result of solvability of (2.6) and to solve the algebraic equation (2.8).

Let $w_l(0) = M$ and $w_l(l) = m$. Recall that $0 < m < M$. From (2.6), we have

$$(w_{l,y})^2 = w_l^2 - \frac{1}{2}w_l^4 - M^2 + \frac{1}{2}M^4 = \frac{1}{2}(w_l^2 - m^2)(M^2 - w_l^2), \tag{3.1}$$

$$-m^2 + \frac{1}{2}m^4 = -M^2 + \frac{1}{2}M^4. \tag{3.2}$$

From (3.2), we deduce that

$$M^2 + m^2 = 2. \tag{3.3}$$

Note that

$$l = \int_m^M \frac{dw_l}{\sqrt{\frac{1}{2}(w_l^2 - m^2)(M^2 - w_l^2)}} \tag{3.4}$$

and

$$\int_0^l w_l^2 dy = \int_m^M \frac{w_l^2 dw_l}{\sqrt{\frac{1}{2}(w_l^2 - m^2)(M^2 - w_l^2)}}. \tag{3.5}$$

We now represent the values of the function w_l by a phase function φ such that the relations (3.4), (3.5) can be expressed by Jacobi elliptic integrals.

Let

$$\frac{M^2 + m^2}{2} - w_l^2 = -\frac{M^2 - m^2}{2}\cos(2\varphi).$$

Then it is easy to see that (3.4) and (3.5) become

$$l = \sqrt{2 - k^2}K(k), \quad \int_0^l w_l^2\, dy = \frac{2}{\sqrt{2 - k^2}}E(k), \tag{3.6}$$

where

$$\frac{1}{M} = \sqrt{1 - \frac{k^2}{2}} \tag{3.7}$$

and $E(k)$ and $K(k)$ are Jacobi elliptic integrals:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi.$$

For the properties of elliptic integrals, we refer the reader to Abramowitz & Stegun [1] and Byrd & Friedman [2]. We list the following for later use:

$$E(0) = \frac{\pi}{2}, \quad K(0) = \frac{\pi}{2}, \quad \lim_{k \to 1} E(k) = 1, \quad \lim_{k \to 1} K(k) = +\infty, \tag{3.8}$$

$$k'K(k) < E(k) < \left(1 - \frac{k^2}{2}\right)K(k) < K(k), \tag{3.9}$$

$$\frac{dK(k)}{dk} = \frac{E(k) - (k')^2 K(k)}{k(k')^2}, \quad \frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}, \tag{3.10}$$

where

$$0 < k < 1 \quad \text{and} \quad k' = \sqrt{1-k^2}.$$

By (3.6), our consistency equation (2.8) can be rewritten in terms of the new variable $k \in (0,1)$:

$$\left(1 - \frac{k^2}{2}\right) K^2(k) - tE(k)K(k) + \frac{L^2}{8} = 0. \tag{3.11}$$

Thus the existence problem is reduced to solving (3.11) for k.

We begin with

Lemma 4 *For* $0 < k < 1$*, we have*

$$\frac{dk}{dl} > 0, \tag{3.12}$$

$$\frac{d}{dk}\left[(1 - \frac{k^2}{2})K^2(k)\right] > 0, \tag{3.13}$$

$$\frac{d^2}{dk^2}\left[(1 - \frac{k^2}{2})K^2(k)\right] > 0, \tag{3.14}$$

$$\frac{d}{dk}\left[E(k)K(k)\right] > 0, \tag{3.15}$$

$$\frac{d^2}{dk^2}\left[E(k)K(k)\right] < 0, \tag{3.16}$$

$$\frac{d^2}{dk^2}\left[\frac{(1 - \frac{k^2}{2})K(k)}{E(k)}\right] > 0, \tag{3.17}$$

$$\frac{d^2}{dk^2}\left[\frac{1}{E(k)K(k)}\right] > 0. \tag{3.18}$$

Proof The proof is mainly based on the inequality (3.9).

Using (3.10), we arrive at

$$\frac{d}{dk}\left[(1 - \frac{k^2}{2})K^2\right] = \frac{K}{k(k')^2}\left[(1+(k')^2)E - 2(k')^2K\right] > 0$$

by (3.9). Thus (3.13) is proved. Since $l^2 = (2-k^2)K^2$, (3.12) follows from (3.13).

Equation (3.14) is more difficult to prove. In fact, by lengthy computations, we have

$$\frac{d^2}{dk^2}\left[(1 - \frac{k^2}{2})K^2\right] = \frac{1}{k^2(k')^4}\left[(1+(k')^2)E^2 + EK(2 - 7(k')^2 - (k')^4) + (k')^2K^2(5(k')^2 - 1)\right]. \tag{3.19}$$

It is easy to see that if $5(k')^2 - 1 \leqslant 0$, then

$$(1+(k')^2)E^2 + EK(2 - 7(k')^2 - (k')^4) + (k')^2K^2(5(k')^2 - 1) > E^2 - (k')^2K^2 > 0.$$

Now we assume that $5(k')^2 - 1 > 0$. Then, using (3.9), we have

$$\begin{aligned}
&(1+(k')^2)E^2 + EK(2-7(k')^2-(k')^4) + (k')^2K^2(5(k')^2-1)\\
&\quad = (1+(k')^2)E(E-k'K) + (k')^2(5(k')^2-1)K\left(K - \frac{E}{1-\frac{k^2}{2}}\right)\\
&\qquad + EK\left[k'(1+(k')^2) + 2 - 7(k')^2 - (k')^4 + \frac{1}{1-\frac{k^2}{2}}(k')^2(5(k')^2-1)\right]. \qquad (3.20)
\end{aligned}$$

For the last term in (3.20), we have

$$\begin{aligned}
&k'(1+(k')^2) + 2 - 7(k')^2 - (k')^4 + \frac{1}{1-\frac{k^2}{2}}(k')^2(5(k')^2-1)\\
&\quad > k^2(2-5(k')^2) + (k'-(k')^2)(1+(k')^2) + \frac{k^2}{2}(k')^2(5(k')^2-1) > \frac{5}{2}k^4 > 0.
\end{aligned}$$

Thus (3.14) is proved.

By using (3.10), it is easy to see that

$$\frac{d}{dk}[EK] = \frac{E^2 - (k')^2K^2}{k(k')^2},$$

which is positive by (3.9). This proves (3.15).

We calculate

$$\frac{d^2}{dk^2}(EK) = \frac{1}{k^2(k')^2}\left[2(E-K)^2 - (E^2-(k')^2K^2)\frac{1-\frac{2}{3}k^2}{(k')^2}\right].$$

Note that

$$\begin{aligned}
&(E^2-(k')^2K^2)\left(1-\frac{2}{3}k^2\right) - 2(k')^2(E-K)^2\\
&\quad > (k')^2(E^2-K^2)(1-\frac{2}{3}k^2) - 2(k')^2(E-K)^2\\
&\quad = (k')^2(E-K)\left[(E+K)(1-\frac{2}{3}k^2) - 2(E-K)\right]\\
&\quad = (k')^2(E-K)\left[K(3-\frac{2}{3}k^2) - (1+\frac{2}{3}k^2)E\right] > 0
\end{aligned}$$

by (3.9), which proves (3.16).

Finally, (3.17) and (3.18) follow from (3.13)–(3.16) by simple calculus. □

Lemma 5 *If* $l > \pi/\sqrt{2}$ *the equation* (2.6) *has a unique solution. If* $l \leqslant \pi/\sqrt{2}$ *the equation* (2.6) *has no solution.*

Proof The inequality (3.12) implies that to every k with $0 < k < 1$ there belongs exactly one $l > 0$. By definition $0 < k < 1$ parameterizes all solutions of (2.6) with $1 < M < \sqrt{2}$. Now (3.6) and (3.8) imply that the solutions are also parameterized by $\pi/\sqrt{2} < l < +\infty$.

For any $l \leqslant \pi/\sqrt{2}$ the function $w_l = 1$ satisfies all requirements of (2.6) except $w_{l,y}(y) < 0$ for $0 < y < l$ and therefore it is not a solution. □

Remarks 1. Note that in Lemma 5 the functions with $w(0) = M$, $M > \sqrt{2}$ do not solve (2.6) since they do not satisfy $w(y) > 0$ for all $0 < y < l$. By Theorem 2, this would lead to an unstable solution of (1.3). Therefore we do not study this type of solution any further.

2. The solution with $w(0) = \sqrt{2}$ corresponds to $l = +\infty$ and it does not solve (2.6) either, since there is no $0 < l < +\infty$ with $w'(l) = 0$. Therefore this leads to a homoclinic connection which is defined on the whole line.

3. The solutions of (2.6) have a pitchfork bifurcation at $l > \pi/\sqrt{2}$.

Now we rewrite (3.11) as follows:

$$t = f(k) \equiv \frac{L^2}{8E(k)K(k)} + \frac{(1-\frac{k^2}{2})K(k)}{E(k)}, \tag{3.21}$$

where $f(k)$ is defined on $(0,1)$. By Lemma 4, we have

$$\frac{d^2}{dk^2} f(k) > 0 \tag{3.22}$$

and, by (3.8), we obtain

$$f(0) = \frac{L^2}{8E(0)K(0)} + \frac{K(0)}{E(0)} = \frac{L^2}{2\pi^2} + 1, \tag{3.23}$$

and

$$\lim_{k\to 1, k<1} f(k) = +\infty. \tag{3.24}$$

By (3.22), (3.23) and (3.24), the function $f(k)$ is strictly convex and attains a unique minimum at a point $k_0 \in (0,1)$ such that

$$f'(k_0) = \frac{L^2}{8}\frac{d}{dk}\left(\frac{1}{2E(k)K(k)}\right)\bigg|_{k=k_0} + \frac{d}{dk}\left(\frac{(1-\frac{k^2}{2})K}{E}\right)\bigg|_{k=k_0} = 0.$$

Furthermore, by strict convexity, for $k < k_0, f'(k) < 0$ and for $k > k_0, f'(k) > 0$.

Let us denote

$$t_1(L) = \min_{k\in(0,1)} f(k), \quad t_2(L) = f(0) = \frac{L^2}{2\pi^2} + 1. \tag{3.25}$$

Correspondingly, we define

$$\mu_1(L) = \frac{t_1}{t_1 - 1}, \quad \mu_2(L) = \frac{t_2}{t_2 - 1} = 1 + \frac{2\pi^2}{L^2}. \tag{3.26}$$

Summarizing these results, we obtain

Lemma 6 *Let L be fixed and $t_1 < t_2$ be given in* (3.25). *Then we have*

(a) Problem (3.11) *has a solution if and only if $t \geqslant t_1$.*

(b) For $t = t_1$, problem (3.11) *has a unique solution k_0 and we have $f'(k_0) = 0$.*

(c) For $t_1 < t < t_2$*, problem* (3.11) *has two solutions* $k_1 < k_2$*. Moreover, we have*

$$f'(k_1) < 0, \quad f'(k_2) > 0. \tag{3.27}$$

(d) For $t \geqslant t_2$*, problem* (3.11) *has a unique solution* k_0 *and we have* $f'(k_0) > 0$.

Going back to (2.8), we express the results of Lemma 6 in terms of L and μ'.

Lemma 7 *Let L be fixed and* $\mu_1 > \mu_2$ *be given in* (3.26)*. Then we have*

(a) Problem (2.8) *has a solution if and only if* $1 < \mu' \leqslant \mu_1$.

(b) For $\mu' = \mu_1$*, problem* (2.8) *has a unique solution* l_0 *and we have* $\frac{dg}{dl}(l_0) = 0$.

(c) For $\mu_2 < \mu' < \mu_1$*, problem* (2.8) *has two solutions* $l_1 < l_2$*. Moreover, we have*

$$\frac{dg}{dl}(l_1) < 0, \quad \frac{dg}{dl}(l_2) > 0, \tag{3.28}$$

where $g(l)$ *was defined in (2.8).*

(d) For $1 < \mu' \leqslant \mu_2$*, problem* (2.8) *has a unique solution* l_0 *and we have* $\frac{dg}{dl}(l_0) > 0$.

Theorem 1 now follows from Lemma 7.

Thus we have rigorously derived a complete picture of the existence of solutions with $A > 0$ on an interval of arbitrary length l.

4 Spectral analysis and a key identity

Let w_l be the unique solution of (2.6). We define the linear operator:

$$\mathscr{L}[\phi] = \phi_{yy} - \phi + 3w_l^2\phi, \ \phi \in \mathscr{X}_l,$$

where

$$\mathscr{X}_l = \left\{ \phi \in H^1(0,l) \ \middle| \ \phi_y(0) = \phi_y(l) = 0 \right\}. \tag{4.1}$$

In this section, we analyze the spectrum of $\mathscr{L}$. The following lemma will be useful in the study of the stability of solutions of (1.3).

Lemma 8 *Consider the following eigenvalue problem:*

$$\begin{cases} \mathscr{L}\phi = \lambda\phi, & 0 < y < l, \\ \phi_y(0) = \phi_y(l) = 0. \end{cases} \tag{4.2}$$

Then its eigenvalues λ_i *can be arranged in such a way that*

$$\lambda_1 > 0, \quad \lambda_j < 0, \quad j = 2, \ldots. \tag{4.3}$$

Moreover, the eigenfunction corresponding to λ_1 *(denoted by* Φ_1*) can be made positive.*

Proof Let the eigenvalues of $\mathscr{L}$ be arranged by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$. It is well-known that $\lambda_1 > \lambda_2$ and that the eigenfunction corresponding to λ_1 can be made positive. By using the equation for w_l, we get

$$-\lambda_1 = \min_{\int_0^l \phi^2\,dy=1} \left(\int_0^l [|\phi_y|^2 + \phi^2 - 3w_l^2\phi^2]\,dy \right)$$
$$\leqslant \left(\int_0^l w_l^2\,dy \right)^{-1} \left(\int_0^l [|w_{l,y}|^2 + w_l^2 - 3w_l^2 w_l^2]\,dy \right) < 0. \tag{4.4}$$

Next we claim that $\lambda_2 \leqslant 0$. This follows from a classical argument (see Theorem 2.11 of Lin & Ni [12]). For the sake of completeness, we include a proof here. By the variational characterization of λ_2, we have

$$-\lambda_2 = \sup_{v\in H^1(0,l/2)} \inf_{\phi\in H^1(0,l/2),\phi\not\equiv 0} \left[\frac{\int_0^l((\phi_y)^2 + \phi^2 - 3w_l^2\phi^2)\,dy}{\int_0^l \phi^2\,dy} \middle| v \not\equiv 0, \int_0^1 \phi v\,dy = 0 \right]. \tag{4.5}$$

On the other hand, w_l has least energy, that is

$$E[w_l] = \inf_{u\not\equiv 0, u\in H^1((0,l/2))} E[u],$$

where

$$E[u] = \frac{\int_0^l((u_y)^2 + u^2)\,dy}{(\int_0^l u^4\,dy)^{\frac{1}{2}}}. \tag{4.6}$$

Let

$$h(t) = E[w_l + t\phi], \quad \phi \in H^1(0,l).$$

Then $h(t)$ attains its minimum at $t = 0$ and hence

$$h''(0) = 2\left[\int_0^l (|\phi_y|^2 + \phi^2)\,dy - 3\int_0^l w_l^2\phi^2\,dy + 2\frac{(\int_0^l w_l^3\phi\,dy)^2}{\int_0^l w_l^4\,dy} \right] \frac{1}{\left(\int_0^l w_l^4\,dy\right)^{1/2}} \geqslant 0.$$

By (4.5), we see that

$$-\lambda_2 \geqslant \inf_{\int_0^l \phi w_l^3=0,\phi\not\equiv 0} \left[\int_0^l (|\phi_y|^2 + \phi^2)\,dy - 3\int_0^l w_l^2\phi^2\,dy + 2\frac{(\int_0^l w_l^3\phi\,dy)^2}{\int_0^l w_l^4\,dy} \right] \frac{1}{\int_0^l \phi^2\,dy} \geqslant 0.$$

Finally we claim that $\lambda_2 < 0$. But this follows from the proof of uniqueness of w_l, see Lemma 5. □

By Lemma 8, $\mathscr{L}^{-1}$ exists and hence $\mathscr{L}^{-1}w_l$ is well-defined. Our next goal in this section is to compute the integral $\int_0^l w_l \mathscr{L}^{-1} w_l\,dy$ and thus to derive the following key identity.

Lemma 9 *We have*

$$\int_0^l w_l(\mathscr{L}^{-1}w_l)\,dy = \frac{1}{4}\int_0^l w_l^2\,dy + \frac{1}{4} l \frac{d}{dl}\int_0^l w_l^2\,dy = \frac{1}{4}\frac{d}{dl}\left(l \int_0^l w_l^2\,dy \right). \tag{4.7}$$

Proof Let us denote $\phi_l = \mathscr{L}^{-1} w_l$. Then ϕ_l satisfies

$$\phi_{l,yy} - \phi_l + 3w_l^2 \phi_l = w_l, \quad \phi_{l,y}(0) = \phi_{l,y}(l) = 0.$$

Set

$$\phi_l = \frac{1}{2} w_l + \frac{1}{2} y w_{l,y}(y) + \Psi. \tag{4.8}$$

Then $\Psi(y)$ satisfies

$$\Psi_{yy} - \Psi + 3w_l^2 \Psi = 0, \quad \Psi_y(0) = 0, \quad \Psi_y(l) = -\frac{1}{2} l w_{l,yy}(l). \tag{4.9}$$

On the other hand, let $\Psi_0 = \frac{\partial w_l}{\partial M}$. Then Ψ_0 satisfies

$$\Psi_{0,yy} - \Psi_0 + 3w_l^2 \Psi_0 = 0, \quad \Psi_0(0) = 1, \quad \Psi_{0,y}(0) = 0. \tag{4.10}$$

Integrating (4.10), we have

$$\Psi_{0,y}(l) = \int_0^l \frac{\partial w_l}{\partial M}\, dy - 3 \int_0^l w_l^2 \frac{\partial w_l}{\partial M}\, dy = \frac{d}{dM} \left(\int_0^l (w_l - w_l^3)\, dy \right) - (w_l(l) - w_l^3(l)) \frac{dl}{dM}.$$

Using the equation for w_l, we have $\int_0^l (w_l - w_l^3)\, dy = 0$. Thus we obtain

$$\Psi_{0,y}(l) = -(w_l(l) - w_l^3(l)) \frac{dl}{dM}. \tag{4.11}$$

Comparing (4.9) and (4.11), we derive the following important relation:

$$\Psi(x) = \frac{1}{2} \frac{l}{\frac{dl}{dM}} \Psi_0(x). \tag{4.12}$$

Hence, we have

$$\begin{aligned} \int_0^l w_l \phi_l\, dy &= \int_0^l \left(\frac{1}{2} w_l + \frac{1}{2} y w_{l,y} + \Psi \right) w_l\, dy \\ &= \frac{1}{4} \int_0^l w_l^2\, dy + \frac{1}{4} l w_l^2(l) + \frac{l}{2} \left(\frac{dl}{dM} \right)^{-1} \int_0^l w_l \Psi_0\, dy. \end{aligned} \tag{4.13}$$

On the other hand,

$$\begin{aligned} \int_0^l w_l \Psi_0\, dy &= \int_0^l w_l \frac{\partial w_l}{\partial M}\, dy = \frac{1}{2} \frac{d}{dM} \int_0^l w_l^2\, dy - \frac{1}{2} w_l^2(l) \frac{dl}{dM} \\ &= \frac{1}{2} \left[\frac{d}{dl} \int_0^l w_l^2\, dy - \frac{1}{2} w_l^2(l) \right] \frac{dl}{dM}. \end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.13), we obtain that

$$\int_0^l w_l \phi_l\, dy = \frac{1}{4} \int_0^l w_l^2\, dy + \frac{1}{4} l \frac{d}{dl} \int_0^l w_l^2\, dy = \frac{1}{4} \frac{d}{dl} \left(l \int_0^l w_l^2\, dy \right). \tag{4.15}$$

This finishes the proof of the lemma. □

5 A nonlocal eigenvalue problem

Let (A, B) be the solution of (1.2) with L arbitrary. In Lemma 13 given in the appendix it was shown that the stability or instability of (A, B) is determined by the spectrum of the following self-adjoint, nonlocal eigenvalue problem

$$\begin{cases} \phi_{xx} - a\phi + 3bA^2\phi - 2\mu'\langle A\phi\rangle A = \lambda\phi, \\ \phi \in \mathscr{X}_L, \end{cases} \tag{5.1}$$

where

$$\mathscr{X}_L = \left\{ \phi \in H^1\left(0, \frac{L}{2}\right) \,\middle|\, \phi_x(0) = \phi_x\left(\frac{L}{2}\right) = 0 \right\}.$$

The derivation of (5.1) is crucial for the analysis. A similar argument, which was given for the case of sufficiently large L in Norbury *et al.* [15], is recalled in the appendix.

Recall the following relation:

$$A(x) = \frac{l}{L(\sqrt{\mu'} - 1)} w_l\left(\frac{2lx}{L}\right).$$

By scaling

$$y = \frac{2l}{L}x, \quad \Phi(y) = \phi(x) \tag{5.2}$$

it is easy to see that (5.1) is equivalent to the following

$$\begin{cases} \Phi_{yy} - \Phi + 3w_l^2\Phi - \gamma(\int_0^l w_l\Phi\, dy)w_l = \lambda\Phi, \quad 0 < y < l, \\ \Phi_y(0) = \Phi_y(l) = 0, \end{cases} \tag{5.3}$$

where

$$\gamma = \frac{2\mu'}{l(\mu' - 1)} = \frac{2t}{l} > 0 \tag{5.4}$$

and w_l is the unique solution of (2.6). (Note that (5.3) is self-adjoint.)

In this section, we give a complete study of (5.3). We remark that similar nonlocal eigenvalue problems have been studied elsewhere [6, 7, 10, 20, 21, 22, 23]. However in those papers the eigenvalue problems are in general not self-adjoint. Therefore they allow Hopf bifurcation which describe the onset of oscillatory phenomena. This does not happen here. Because of the oscillations it is harder to determine the bifurcation point analytically and this only possible in some special cases.

In Wei & Winter [23] the shadow system of the Gierer-Meinhardt system (i.e. infinite diffusion constant of the inhibitor) is studied on a bounded interval with finite diffusion constant of the activator. It is shown that there exists a unique Hopf bifurcation point which is transversal. The method of proof is similar to this paper in that Jacobi elliptic integrals are used as well. The reason for this similarity is because a finite diffusion constant of the activator is considered there (and not the singularly perturbed problem with sufficiently small diffusion constant). However, the destabilization mechanism is different in the two cases (Hopf bifurcation for a non-selfadjoint eigenvalue problem versus an eigenvalue crossing the imaginary axis for a self-adjoint eigenvalue problem).

We first have

Lemma 10 $\lambda = 0$ *is an eigenvalue of* (5.3) *if and only if*

$$\gamma \int_0^l w_l \mathscr{L}^{-1} w_l \, dy = 1. \tag{5.5}$$

Proof Suppose $\lambda = 0$. Then we have

$$0 = \mathscr{L}[\Phi] - \gamma \left(\int_0^l w_l \Phi \, dy \right) w_l$$

which implies that

$$\Phi = \gamma \left(\int_0^l w_l \Phi \, dy \right) \mathscr{L}^{-1} w_l. \tag{5.6}$$

Multiplying (5.6) by w_l and integrating, we obtain (5.5) since $\int_0^l w_l \Phi \, dy \neq 0$ (as otherwise $\mathscr{L}\Phi = 0$ and hence $\Phi = 0$). □

The following is the main result of this section:

Lemma 11 *All eigenvalues of* (5.3) *are real and*
(a) if $\gamma \int_0^l w_l \mathscr{L}^{-1} w_l \, dy > 1$, *then for all eigenvalues of* (5.3) *we have* $\lambda < 0$*;*
(b) if $\gamma \int_0^l w_l \mathscr{L}^{-1} w_l \, dy = 1$, *then for all eigenvalues of* (5.3) *we have* $\lambda \leqslant 0$ *and zero is an eigenvalue of* (5.3) *with eigenfunction* $\mathscr{L}^{-1} w_l$*;*
(c) if $\gamma \int_0^l w_l \mathscr{L}^{-1} w_l \, dy < 1$, *then there exists an eigenvalue* $\lambda_0 > 0$ *of* (5.3).

From Lemma 11, we see that $\gamma \int_0^l w_l \mathscr{L}^{-1} w_l \, dy = 1$ is the borderline case between stability and instability of (5.3).

Proof The nonlocal eigenvalue problem (5.3) is self-adjoint and hence all eigenvalues are real. Let $\lambda \geqslant 0$ be an eigenvalue of (5.3). We first claim that $\lambda \neq \lambda_1$, where λ_1 is the first eigenvalue of $\mathscr{L}$ given by Lemma 8. In fact, if $\lambda = \lambda_1$, then we have

$$\gamma \int_0^l w_l \Phi_1 \, dy = 0,$$

where Φ_1 is the eigenfunction to the eigenvalue λ_1 for the operator $\mathscr{L}$. This implies

$$\int_0^l w_l \Phi_1 \, dy = 0,$$

which is impossible since $\Phi_1 > 0$.

So $\lambda \neq \lambda_1$. By Lemma 8, $(\mathscr{L} - \lambda)^{-1}$ exists and hence $\lambda > 0$ is an eigenvalue of (5.3) if and only if it satisfies the following algebraic equation:

$$1 - \gamma \int_0^l [((\mathscr{L} - \lambda)^{-1} w_l) w_l] \, dy = 0. \tag{5.7}$$

Let

$$\rho(t) = 1 - \gamma \int_0^l [((\mathscr{L} - \lambda)^{-1} w_l) w_l]\, dy, \quad t \geqslant 0, \quad t \neq \lambda_1.$$

Then $\rho(0) = 1 - \gamma \int_0^l (w_l \mathscr{L}^{-1} w_l)\, dy$ and

$$\rho'(t) = -\gamma \int_0^L [((\mathscr{L} - t)^{-2} w_l) w_l]\, dy < 0.$$

On the other hand,

$$\rho(t) \to -\infty \ \text{ as } \ t \to \lambda_1, t < \lambda_1$$

$$\rho(t) \to +\infty \ \text{ as } \ t \to \lambda_1, t > \lambda_1$$

$$\rho(t) \to 1 \ \text{ as } \ t \to +\infty.$$

Thus $\rho(t) > 0$ for $t > \lambda_1$ and $\rho(t)$ has a (unique) zero in $(0, \lambda_1)$ if and only if $\rho(0) > 0$ which is equivalent to $1 - \gamma \int_0^l (w_l \mathscr{L}^{-1} w_l)\, dy > 0$. This proves the lemma. □

6 The proof of Theorem 3

Proof Now we can finish the proof of Theorem 3.

By Lemma 11, we have stability of (5.3) if

$$\gamma \int_0^l (w_l \mathscr{L}^{-1} w_l)\, dy > 1 \tag{6.1}$$

and instability if

$$\gamma \int_0^l (w_l \mathscr{L}^{-1} w_l)\, dy < 1.$$

By Lemma 9, we have

$$\gamma \int_0^l (w_l \mathscr{L}^{-1} w_l)\, dy = \frac{t \frac{d}{dl}(l \int_0^l w_l^2\, dy)}{2l}. \tag{6.2}$$

Thus (6.1) is equivalent to

$$2l - t \frac{d}{dl} \left(l \int_0^l w_l^2\, dy \right) < 0 \tag{6.3}$$

which by definition (2.8) is equivalent to

$$\frac{dg}{dl} < 0. \tag{6.4}$$

Thus, for $\mu_2 < \mu' < \mu_1$, by Lemma 7(c), the solution with small period l_1 is stable while the solution with large period l_2 is unstable. For $\mu' \leqslant \mu_2$, the only solution is unstable. When $\mu' = \mu_1$, we have $g'(l) = 0$, where g was defined in (2.8). This implies that the only solution is neutrally stable.

When $\mu_2 < \mu' < \mu_1$, let us compute the amplitude of A:

$$A(x) = \frac{2l}{L\sqrt{\mu' - 1}} w_l\left(\frac{2lx}{L}\right).$$

So the maximum of $A(x)$ is given by

$$\begin{aligned} A(0) = \max_{x \in I} A(x) &= \frac{2l}{L\sqrt{\mu' - 1}} w_l(0) = \frac{2l}{L\sqrt{\mu' - 1}} M \\ &= \frac{2}{L\sqrt{\mu' - 1}} \sqrt{2 - k^2} K(k) \frac{1}{\sqrt{1 - \frac{k^2}{2}}} = \frac{2\sqrt{2}}{L\sqrt{\mu' - 1}} K(k). \end{aligned}$$

So if $l_1 < l_2$, then by (3.12) $k_1 < k_2$ and the maximum of A for l_1 is smaller than the maximum of A for l_2.

Thus, for $\mu_2 < \mu' < \mu_1$, the solution with small amplitude is stable and the one with large amplitude is unstable.

This finishes the proof of Theorem 3. □

7 Instability of other solutions: The proof of Theorem 2

Proof In this section, we will show that all other solutions of (2.6) must be unstable. In fact, let w be any solution of the following ordinary equation:

$$w_{yy} - w + w^3 = 0, \quad w_y(0) = w_y(l) = 0 \tag{7.1}$$

and consider its associated eigenvalue problem

$$\begin{cases} \mathscr{L}\phi = \phi_{yy} - \phi + 3w^2\phi = \lambda\phi, & 0 < y < l, \\ \phi_y(0) = \phi_y(l) = 0. \end{cases} \tag{7.2}$$

The Morse index of a solution w of (7.1) is the number of positive eigenvalues of (7.2).

Since w_l is the unique positive solution of (7.1) it has least energy under all solutions of (7.1) and hence its Morse index is 1.

We claim that

Lemma 12 *All other solutions of* (7.1) *have Morse index at least* 2.

Proof Let $w \not\equiv w_l$ be a solution of (7.1). Let $\lambda_1 > 0$ be the principal eigenvalue of w. The associated eigenfunction Φ_1 can be made positive. By definition,

$$-\lambda_2 = \inf_{\phi \in H^1(0,l), \phi \not\equiv 0, \int_0^l \Phi_1 \phi \, dy = 0} \frac{\int_0^l [(\phi_y)^2 + \phi^2 - 3w^2\phi^2] \, dy}{\int_0^l \phi^2 \, dy}. \tag{7.3}$$

We now show that $\lambda_2 > 0$. There are two cases to be considered.

Case 1. w is a changing-sign solution. In this case, we suppose that $w(y) > 0, y \in (0, y_1)$ and $w(y) < 0, y \in (y_1, y_2)$ where $y_2 \leqslant l$. We may assume that $w(y_1) = 0$ and $w(y_2) = 0$ if $y_2 < l$ and $w_y(y_2) = 0$ if $y_2 = l$. Now let $\phi(y) = c_1 w(y), y \in (0, x_1)$ and $\phi(y) = c_2 w(y), y \in (x_1, x_2)$. We can choose the two constants c_1, c_2 such that

$$\int_0^l \Phi_1 \phi \, dy = 0.$$

Then, by simple computations, we have

$$\int_0^l \left[(\phi_y)^2 + \phi^2 - 3w^2\phi^2 \right] dy < 0$$

which implies that $-\lambda_2 < 0$ and hence $\lambda_2 > 0$.

Case 2. w is a positive solution. Since $w \neq w_l$, w_y must change sign. We may assume that $w_y < 0, y \in (0, y_1)$ and $w_y > 0, y \in (y_1, y_2)$. Furthermore, we may also assume that $w_y(y_1) = 0$ and $w_y(y_2) = 0$. Then similar to Case 1, we take $\phi = c_1 w_y(y), y \in (0, y_1)$ and $\phi(y) = c_2 w_y(y), y \in (y_1, y_2)$, where the constants c_1, c_2 are chosen such that $\int_0^l \Phi_1 \phi \, dy = 0$. Then, by simple computations, we have

$$\int_0^l ((\phi_y)^2 + \phi^2 - 3w^2\phi^2) \, dy \leqslant 0$$

and hence $-\lambda_2 \leqslant 0$ and $\lambda_2 \geqslant 0$. If $\lambda_2 = 0$, then $\phi(x)$ becomes an eigenfunction which satisfies (7.2) and hence is smooth. This is impossible since ϕ is not smooth at y_1 (as otherwise $w_y(y_1) = w_{yy}(y_1) = 0$ and hence $w \equiv 1$). □

From Lemma 12 it follows that the Morse index of all other solutions is at least 2. In other words, problem (7.2) has at least two positive eigenvalues. Let λ_1 be the principal eigenvalue and $0 < \lambda_2 \leqslant \lambda_1$ be the second eigenvalue. Let the corresponding eigenfunctions be Φ_1, Φ_2. Since λ_1 is the principal eigenvalue, we may assume that $\Phi_1 > 0$.

If $\int_0^l w\Phi_2 \, dy = 0$, we choose $\phi = \Phi_2$. If $\int_0^l w\Phi_2 \, dy \neq 0$, then we choose c such that

$$\int_0^l w\Phi_1 \, dy + c \int_0^l w\Phi_2 \, dy = 0$$

and

$$\phi = \Phi_1 + c\Phi_2.$$

In any case, we obtain that $\int_0^l w\phi \, dy = 0$.

Then we have

$$\int_0^l \left[(\phi_y)^2 + \phi^2 - 3w^2\phi^2 \right] dy - \gamma \left(\int_0^l w_l \phi \, dy \right)^2 = \int_0^l \left((\phi_y)^2 + \phi^2 - 3w_l^2\phi^2 \right) dy < 0, \tag{7.4}$$

which implies that for any constant γ there exists a positive eigenvalue to (5.3) and, by the scaling argument in Section 5, also to (5.1). Now, by the reduction lemma (Lemma 13), all solutions of (1.3) for which either A or A_x changes sign, must be unstable.

This finishes the proof of Theorem 2. □

8 Conclusion

In this paper we have given a rigorous analysis of the linearized stability of stationary periodic patterns on the real line with an arbitrary minimal period L for a Ginzburg-Landau equation with a mean field. This equation arises in many contexts in the sciences.

Periodic steady-states on the real axis are represented by steady-states on an interval of suitable length with Neumann boundary conditions. In this setting existence and multiplicity of solutions has been established and linear stability was proved rigorously. The proof is based on a self-adjoint, nonlocal eigenvalue problem, variational characterization of eigenvalues and Jacobi elliptic integrals. In particular the latter are well-suited for problems on finite domains.

In this paper we have rigorously shown results on a particular destabilization mechanism which acts on the space of periodic functions and is closely connected with the size of nonlocal terms. It has lead to an eigenvalue crossing the imaginary axis. We have not considered other possible instabilities acting on the whole real line such as absolute instabilities (see, for example, Sandstede & Scheel [19]).

In this paper stationary patterns are considered for the real Ginzburg-Landau equation. It would be very interesting to try and extend our methods to the complex Ginzburg-Landau equation with a mean field. For recent progress in this direction see Winterbottom *et al.* [24].

Another interesting but challenging direction would be the real or the complex case in several space dimensions.

Appendix: The linearized operator – a reduction lemma

In this appendix, we derive the linearized operator to system (1.3) and study some of its properties.

In particular, we show that its eigenvalues must all be real and that the system of eigenvalue equations reduces to a self-adjoint nonlocal eigenvalue problem. This is similar to our previous paper [15]. For the sake of completeness, we include all this material in this appendix.

To study the linear stability of (1.3), we perturb $(A(x), B(x))$ as follows:

$$A_\epsilon(x,t) = A(x) + \epsilon\phi(x)e^{\lambda_L t}, \quad B_\epsilon(x,t) = B(x) + \epsilon\psi(x)e^{\lambda_L t}, \tag{8.1}$$

where $\lambda_L \in \mathscr{C}$ – the set of complex numbers – and

$$\phi, \psi \in \mathscr{X}_L,$$

where $\mathscr{X}_L$ was defined at the beginning of Section 5.

Substituting (8.1) into (1.2) and considering the linear part, we obtain the following eigenvalue problem:

$$\begin{cases} \phi_{xx} + (1-B)\phi - 3A^2\phi - A\psi = \lambda_L\phi, & 0 < x < \frac{L}{2}, \\ \psi_{xx} + 2\mu'(A\phi)_{xx} = \tau\lambda_L\psi, & 0 < x < \frac{L}{2}, \\ \lambda_L \in \mathscr{C}, \quad \phi, \psi \in \mathscr{X}_L. \end{cases} \tag{8.2}$$

Now in a similar way as B was substituted in (1.3) we can substitute ψ in (8.2) and reduce it to a nonlocal, self-adjoint eigenvalue problem in ϕ only.

Let

$$\psi = -2\mu' A\phi + 2\mu'\langle A\phi\rangle + \tau\lambda_L\hat{\psi}, \tag{8.3}$$

where

$$\langle\hat{\psi}\rangle = 0.$$

Our goal in this appendix is to show that setting $\hat{\psi} = 0$ does not change the stability properties of the eigenvalue problem. This idea will enable us to reduce the eigenvalue problem (8.2) to the nonlocal, self-adjoint eigenvalue problem (8.6) below.

Equation (8.3) together with (8.2) implies

$$\hat{\psi}_{xx} - \tau\lambda_L\hat{\psi} = -2\mu' A\phi + 2\mu'\langle A\phi\rangle. \tag{8.4}$$

Substituting (2.1) and (8.3) into the first equation of (8.2), we obtain that

$$\phi_{xx} - a\phi + 3bA^2\phi - 2\mu'\langle A\phi\rangle A - \tau\lambda_L A\hat{\psi} = \lambda_L\phi, \quad 0 < x < \frac{L}{2}, \tag{8.5}$$

where a and b are given by (2.3).

If $\tau = 0$, then (8.5) becomes

$$\begin{cases} \phi_{xx} - a\phi + 3bA^2\phi - 2\mu'\langle A\phi\rangle A = \lambda_L\phi, & 0 < x < \frac{L}{2}, \\ \phi \in \mathscr{X}_L. \end{cases} \tag{8.6}$$

We now recall the following reduction lemma (Lemma 3 of Norbury *et al.* [15]):

Lemma 13 *(a) All eigenvalues of (8.2) are real.*

(b) If all eigenvalues of (8.6) are negative, then all eigenvalues of (8.2) are negative.

(c) If problem (8.6) has a positive eigenvalue, then problem (8.2) also has a positive eigenvalue.

Lemma 13 implies that the stability and instability properties of (8.2) and (8.6) are the same. Thus we have reduced our stability problem to the study of the self-adjoint nonlocal eigenvalue problem (8.6).

Proof We first prove part (a).

Multiplying (8.5) by $\overline{\phi}$ – the conjugate function of ϕ – and integrating over $I = (0, L/2)$, we obtain

$$\lambda_L \int_I |\phi|^2\,dx = -\int_I [|\phi_x|^2 + a|\phi|^2 - 3bA^2|\phi|^2]\,dx - \frac{2\mu'}{L}\left|\int_I (A\phi)\,dx\right|^2 - \tau\lambda_L \int_I A\hat{\psi}\overline{\phi}\,dx. \quad (8.7)$$

Multiplying the conjugate of (8.4) by $\hat{\psi}$ and integrating over R we get

$$\int_I A\overline{\phi}\hat{\psi}\,dx = \frac{1}{2\mu'}\int_I |\hat{\psi}_x|^2\,dx + \frac{\tau\bar{\lambda}_L}{2\mu'}\int_I |\hat{\psi}|^2\,dx. \quad (8.8)$$

Substituting (8.8) into (8.7) gives

$$\lambda_L \int_I |\phi|^2\,dx + \int_I [|\phi_x|^2 + a|\phi|^2 - 3bA^2|\phi|^2]\,dx + \frac{2\mu'}{L}\left|\int_I (A\phi)\,dx\right|^2 + \frac{\tau\lambda_L}{2\mu'}\int_I |\hat{\psi}_x|^2\,dx + \frac{\tau^2|\lambda_L|^2}{2\mu'}\int_I |\hat{\psi}_x|^2\,dx = 0. \quad (8.9)$$

Taking the imaginary part of (8.9), we obtain

$$\lambda_i \left(\int_I |\phi|^2\,dx + \frac{\tau}{2\mu'}\int_I |\hat{\psi}_x|^2\,dx\right) = 0, \quad (8.10)$$

where $\lambda_L = \lambda_r + \sqrt{-1}\lambda_i$.

Equation (8.10) implies

$$\lambda_i = 0 \quad (8.11)$$

and therefore λ_L is real. Since the spectrum of (8.5) coincides with that of (8.2) the proof of part (a) is complete.

Next we prove parts (b) and (c) of Lemma 13.

We use variational techniques. To this end, we need to introduce two quadratic forms: Let

$$L[\phi] = \int_I (|\phi_x|^2 + a|\phi|^2 - 3bA^2|\phi|^2)\,dx + \frac{2\mu'}{L}\left|\int_I A\phi\,dx\right|^2, \quad \phi \in \mathscr{X}_L \quad (8.12)$$

and

$$\mathscr{L}_\lambda[\phi] = L[\phi] + \frac{\tau\lambda}{2\mu'}\int_I (|\hat{\psi}_x|^2 + \tau\lambda|\hat{\psi}|^2)\,dx, \quad (8.13)$$

where $\hat{\psi}$ is the unique solution of the problem

$$\begin{cases} \hat{\psi}_{xx} - \tau\lambda\hat{\psi} = -2\mu' A\phi + 2\mu'\langle A\phi\rangle, \\ \hat{\psi} \in \mathscr{X}_L, \quad \langle\hat{\psi}\rangle = 0. \end{cases} \quad (8.14)$$

Observe that for $\tau \geqslant 0$ and $\lambda \geqslant 0$

$$\mathscr{L}_0[\phi] = L[\phi], \quad L[\phi] \leqslant \mathscr{L}_\lambda[\phi]. \quad (8.15)$$

To prove (b), we note that if all eigenvalues of (8.6) are negative, then the quadratic form $L[\phi]$ is positive definite, which by (8.15) yields that $\mathscr{L}_\lambda$ is positive definite if $\lambda \geqslant 0$.

Let $\lambda \geqslant 0$ be an eigenvalue of (8.2), then, by (8.9), we obtain that

$$\lambda \int_I |\phi|^2\, dx + \mathscr{L}_\lambda[\phi] = 0 \tag{8.16}$$

which is clearly impossible if $\lambda \geqslant 0$. Thus we have shown that all eigenvalues of (8.2) must be negative.

To prove (c), suppose that (8.6) has a positive eigenvalue. Then the eigenvalue problem

$$-\mu_L = \min_{\phi \in \mathscr{X}_L, \int_I \phi^2\, dx = 1} L[\phi] \tag{8.17}$$

has a positive value $\mu_L > 0$. We now claim that (8.2) admits a positive eigenvalue.

Fixing $\lambda \in [0, +\infty)$, let us consider another eigenvalue problem

$$-\mu(\lambda) = \min_{\phi \in \mathscr{X}_L, \int_I \phi^2\, dx = 1} \mathscr{L}_\lambda[\phi]. \tag{8.18}$$

A minimizer ϕ of (8.18) satisfies the equation

$$\phi_{xx} - a\phi + 3bA^2\phi - 2\mu'\langle A\phi\rangle A - A\hat{\psi} = \mu(\lambda)\phi, \quad \phi \in \mathscr{X}_L, \tag{8.19}$$

where $\hat{\psi}$ is given by (8.14).

By (8.15), $-\mu(\lambda) \geqslant -\mu_L$. Hence $\mu(\lambda) \leqslant \mu_L$. Moreover, since $\hat{\psi}$ is continuous with respect to λ in $[0, +\infty)$, we see that $\mu(\lambda)$ is also continuous in $[0, +\infty)$.

Let us consider the following algebraic equation

$$h(\lambda) := \mu(\lambda) - \lambda = 0, \quad \lambda \in [0, +\infty). \tag{8.20}$$

By our assumption, $h(0) = \mu(0) = \mu_L > 0$. On the other hand, for $\lambda > 2\mu_L$, $h(\lambda) \leqslant \mu_L - \lambda < -\mu_L < 0$. By the mean-value theorem, there exists a $\lambda_L \in (0, \mu_L)$ such that $h(\lambda_L) = 0$.

Substituting $\mu(\lambda_L) = \lambda_L$ into (8.19), we see that λ_L is an eigenvalue of problem (8.2).

Part (c) of Lemma 13 is thus proved. □

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