A ‘twistor transform’ for complex manifolds with connection
by Dominic Joyce.


In this note we will briefly describe the geometry of a class of complex manifolds, to be
called complex-flat manifolds, that have a connection \( \nabla \) satisfying a curvature condition
given in §1, which is the curvature condition satisfied by the Levi-Civita connection of a
Kähler manifold. The structure has a sort of twistor transform: in §2, \( \nabla \) will be used to
define an almost complex structure \( J \) on the tangent bundle of \( X \), and it will be shown
that \( J \) is integrable exactly when the curvature condition holds.

It therefore gives a miniature picture of the Penrose transform for conformal 4-
manifolds, where the Cartan conformal connection is used to define a complex structure
on a bundle, and the integrability condition is a condition on the conformal curvature. In
§3 we give some examples of complex-flat manifolds.

1. Connections, curvature and complex structures

We begin by recalling how to decompose tensors relative to a complex structure \( I \).
Let \( X \) be a complex manifold, with complex structure \( I \), which will be written with indices
as \( I^a \) with respect to some real coordinate system \( (x^1, \ldots, x^{2n}) \). Let \( K = K^a_{\cdot\cdot\cdot} \) be a tensor
on \( X \), taking values in \( \mathbb{C} \). Here \( a \) is a contravariant index of \( K \), and any other indices of
\( K \) are represented by dots. The Greek characters \( \alpha, \beta, \gamma, \delta, \epsilon \), and the starred characters
\( \alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^* \), will be used in place of the Roman indices \( a, b, c, d, e \) respectively. They
are tensor indices with respect to \( (x^1, \ldots, x^{2n}) \) in the normal sense, and their use is actually
a shorthand indicating a modification to the tensor itself.

Define \( K^{a\cdot\cdot\cdot} = (K^{a\cdot\cdot\cdot} + i I^a_j K^{j\cdot\cdot\cdot})/2 \) and \( K^{\cdot\cdot\cdot a*} = (K^{\cdot\cdot\cdot a*} - i I^a_j K^{j\cdot\cdot\cdot})/2 \). In the same way,
if \( b \) is a covariant index on a complex-valued tensor \( L_{\cdot\cdot\cdot b} \), define \( L_{\cdot\cdot\cdot b} = (L_{\cdot\cdot\cdot b} - i I^b_j L^{j\cdot\cdot\cdot})/2 \)
and \( L_{\cdot\cdot\cdot b*} = (L_{\cdot\cdot\cdot b*} + i I^b_j L^{j\cdot\cdot\cdot})/2 \). Then \( K^{a\cdot\cdot\cdot} \) and \( L_{\cdot\cdot\cdot b} \) are the components of \( K \) and \( L \)
that are complex linear w.r.t. \( I \), and the starred versions are the components that are complex
antilinear w.r.t. \( I \). These operations are projections, and satisfy \( K^{a\cdot\cdot\cdot} = K^{\cdot\cdot\cdot a*} \) and
\( L_{\cdot\cdot\cdot b} = L_{\cdot\cdot\cdot b*} + L_{\cdot\cdot\cdot b*} \). The complex decomposition of a real-valued tensor is self-adjoint.
This means that changing round starred and unstarr\( h \)d indices has the same effect as complex conjugation. All the tensors we deal with will be self-adjoint.

Let \( \nabla \) be a torsion-free connection on \( X \) satisfying \( \nabla I = 0 \). The connection will be
written in the usual way as \( \Gamma^a_{\cdot\cdot\cdot} \), relative to the coordinate system \( (x^1, \ldots, x^{2n}) \). In this
fixed coordinate system, \( \Gamma \) may be decomposed into components relative to \( I \) as above, but
as \( \Gamma \) is not a tensor this decomposition does depend on the coordinate system. Therefore,
we shall consider only coordinate systems \( (x^1, \ldots, x^{2n}) \) with the property that \( I \) is constant
in coordinates, i.e. \( \partial I^a_j/\partial x^c = 0 \) for all \( a, b, c \).

As \( \nabla I = 0 \) we have \( \Gamma^a_{bc} = \Gamma^a_{\cdot\cdot\cdot} + \Gamma^a_{\cdot\cdot\cdot} + \Gamma^a_{\cdot\cdot\cdot} \), and as \( \nabla \) is torsion-free \( \Gamma^a_{bc} = \Gamma^a_{cb} \). Together
these imply that \( \Gamma^a_{bc} = \Gamma^a_{\cdot\cdot\cdot} + \Gamma^a_{\cdot\cdot\cdot} \). Now the curvature \( R^a_{bcd} \) of \( \nabla \) is given by
\( R^a_{bcd} = \partial \Gamma^a_{bd}/\partial x^c - \partial \Gamma^a_{bc}/\partial x^d + \Gamma^a_{jcd} \Gamma^j_{bd} - \Gamma^a_{jcd} \Gamma^j_{bd} \). Substituting in for \( \Gamma \) gives
\( R^a_{bcd} = R^a_{\cdot\cdot\cdot} + R^a_{\cdot\cdot\cdot} \). Because \( \nabla \) is torsion-free, \( R \) satisfies the Bianchi identity
\( R^a_{bcd} + R^a_{cde} + R^a_{dyz} = 0 \), and thus \( R^a_{\cdot\cdot\cdot} + R^a_{\cdot\cdot\cdot} \). But from above the last two terms are zero,
and so \( R^a_{\cdot\cdot\cdot} = 0 \). Therefore
defines an almost complex structure \( J \). Then let \( (x, y) \) be a vector in \( J \). Let \( v \) be a vector in \( J \). Then \( Jv = (v_1, v_2) \) for all vectors \( v \), and for all \( x \in X, y \in T_x X \). This defines an almost complex structure \( J \) on the total space of \( TX \), commuting with \( I \) and projecting down to \( I \) on \( X \).

We will write \( J \) out explicitly in terms of the connection components \( \Gamma \), and calculate the Nijenhuis tensor \( N_J \) of \( J \), which will give the condition for \( J \) to be integrable. Let \((x^1, \ldots, x^{2n})\) be a coordinate system as in §1, for some open set \( U \subset X \). Let \((y^1, \ldots, y^{2n})\) be coordinates w.r.t. the basis \((\partial/\partial x^1, \ldots, \partial/\partial x^{2n})\) for the fibres of \( TU \). Then \((x^1, \ldots, x^{2n}, y^1, \ldots, y^{2n})\) are coordinates for \( TU \). In these coordinates, \( J \) is

\[
J \left( p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a} \right) = I_b^a p^b \frac{\partial}{\partial x^a} - I_b^a q^b \frac{\partial}{\partial y^a} - 2I_d^c \Gamma_{bc}^a y^b \frac{\partial}{\partial y^d}.
\]

Decomposing this expression w.r.t. \( I \) leads to some simplifications, as we may use the facts that \( \Gamma_{bc}^a = \Gamma^a_{\beta\gamma} + \Gamma^*_{\beta\gamma} \) and \( I_b^a = i\delta_{\beta}^a - i\delta_{\beta}^* \). So we have

\[
J \left( p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a} \right) = i p^a \frac{\partial}{\partial x^a} - i q^a \frac{\partial}{\partial y^a} + i q^* \frac{\partial}{\partial y^a} - 2i \Gamma_{\beta\gamma}^a y^\beta \frac{\partial}{\partial y^\gamma} + 2i \Gamma_{\beta\gamma}^* y^\beta \frac{\partial}{\partial y^\gamma}.
\]

Theorem. The almost complex structure \( J \) is integrable if and only if \( R^\alpha_{\beta\gamma\delta} = 0 \).
**Proof.** By the Newlander-Nirenberg theorem, a necessary and sufficient condition for the integrability of $J$ is the vanishing of the Nijenhuis tensor $N_J$ of $J$, which is given by $N_J(v, w) = [v, w] + J([J, v, w] + [v, J, w]) - [J, v, J, w]$. We shall evaluate $N_J$ with $v = p^a \partial / \partial x^a + q^a \partial / \partial y^a$ and $w = r^a \partial / \partial x^a + s^a \partial / \partial y^a$, where $p^a, q^a, r^a$ and $s^a$ are constants independent of $x^a, y^a$. It is easy to see that $[v, w] = 0$. Using the fact that $J$ acts as $-I$ on $V$, one calculates that

$$J([J, v, w]) = 2r^a \frac{\partial \Gamma_{\gamma}^a}{\partial x^d} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2r^a \frac{\partial \Gamma_{\gamma}^a}{\partial x^d} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2\Gamma_{bc}^a b^c \frac{\partial}{\partial y^\alpha},$$

$$J([v, J, w]) = -2p^a \frac{\partial \Gamma_{\gamma}^a}{\partial x^d} y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} - 2p^a \frac{\partial \Gamma_{\gamma}^a}{\partial x^d} y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} - 2\Gamma_{bc}^a q^b r^c \frac{\partial}{\partial y^\alpha},$$

and

$$[J, v, J, w] =$$

$$\left( i p^\delta \frac{\partial}{\partial x^\delta} - i p^\delta \frac{\partial}{\partial x^\delta} \right) \left( -2i \Gamma_{\delta}^\alpha y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} + 2i \Gamma_{\delta}^\alpha y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} \right)$$

$$- \left( i r^\delta \frac{\partial}{\partial x^\delta} - i r^\delta \frac{\partial}{\partial x^\delta} \right) \left( -2i \Gamma_{\delta}^\alpha y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2i \Gamma_{\delta}^\alpha y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} \right)$$

$$- 4\Gamma_{\beta}^\gamma y^\beta p^\gamma \Gamma_{\delta}^\epsilon r^\epsilon \frac{\partial}{\partial y^\alpha} - 4\Gamma_{\beta}^\gamma y^\beta p^\gamma \Gamma_{\delta}^\epsilon r^\epsilon \frac{\partial}{\partial y^\alpha}$$

$$+ 4\Gamma_{\beta}^\gamma y^\beta r^\gamma \Gamma_{\delta}^\epsilon p^\epsilon \frac{\partial}{\partial y^\alpha} + 4\Gamma_{\beta}^\gamma y^\beta r^\gamma \Gamma_{\delta}^\epsilon p^\epsilon \frac{\partial}{\partial y^\alpha}$$

$$- 2\Gamma_{\beta}^\gamma q^\beta r^\gamma \frac{\partial}{\partial y^\alpha} - 2\Gamma_{\beta}^\gamma q^\beta r^\gamma \frac{\partial}{\partial y^\alpha} + 2\Gamma_{\beta}^\gamma s^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2\Gamma_{\beta}^\gamma s^\beta p^\gamma \frac{\partial}{\partial y^\alpha}.$$

Combining the above gives

$$N_J(v, w) = 4R_{\beta}^\alpha y^\beta r^\gamma p^\delta \frac{\partial}{\partial y^\alpha} + 4R_{\beta}^\alpha y^\beta r^\gamma p^\delta \frac{\partial}{\partial y^\alpha},$$

using the expression for $R$ in §1. As this holds for all $v, w$ and $y^a$ for each fixed $x$, $N_J = 0$ identically if and only if $R_{\beta}^\alpha y^\beta = 0$. 

\[\square\]

3. Examples

The simplest examples of complex-flat manifolds are Kähler manifolds, taking $\nabla$ to be the Levi-Civita connection of the Kähler metric. However, there are many other examples of complex-flat manifolds with no compatible Kähler metric. We shall comment briefly on three such families. Firstly, using the work of [J] for hypercomplex manifolds it is possible to define a quotient construction for complex-flat manifolds analogous to the Kähler quotient. Starting with a flat complex-flat structure one may produce non-Kähler complex-flat structures by choosing a moment map not compatible with any Kähler metric.

Another way of constructing examples is to consider complex submanifolds of complex-flat manifolds. To induce a connection on the tangent bundle of a submanifold $M$ of $X$ we need a splitting $TX|_M = TM \oplus V$ for some vector bundle $V$; for the induced connection to be complex-flat, it turns out that $V$ must be a holomorphic subbundle w.r.t. $J$. In the
case, say, of projective varieties in $X = \mathbb{CP}^n$, there may be many different choices of $V$ satisfying this condition, and each will give a distinct complex-flat connection on $M$.

Our final family of examples are hypercomplex manifolds. A hypercomplex manifold is a manifold $M^{4n}$ with complex structures $I_1, I_2$ and $I_3$ satisfying $I_1 I_2 = I_3$. By [S], §6, there is a unique connection $\nabla$ on $M$ called the Obata connection, that is torsion-free and satisfies $\nabla I_j = 0$. We shall show that $\nabla$ is a complex-flat connection for each of the complex structures $I_1, I_2, I_3$. Thus hypercomplex manifolds are examples of complex-flat structures that in general do not come from Kähler structures.

**Proposition.** Let $M$ and $\nabla$ be as above. Then the curvature $R^a_{\ bcd}$ of $\nabla$ satisfies $R^\alpha_{\ \beta\gamma\delta} = 0$ in the complex decomposition with respect to each complex structure $I_j$. Thus $(M, I_j, \nabla)$ is a complex-flat manifold.

**Proof.** We shall prove the result for $I_1$, for by symmetry it then holds for $I_2, I_3$. As $\nabla$ is torsion-free and $\nabla I_2 = 0$, from §1 the curvature $R$ satisfies $R^a_{\ bcd} = R^\alpha_{\ \beta cd} + R^\alpha_{\ \beta cd}$ in the complex decomposition w.r.t. $I_2$, and so $R^a_{\ bcd} = -(I_2)^a_j(2)_b^k R^j_{\ kcd}$. Also, from §1 the component $R^\alpha_{\ \beta\gamma\delta}$ is zero in the complex decomposition w.r.t. $I_1$. Therefore

$$0 = (1 - iI_1)\alpha_p (1 + iI_1)\beta_b (1 - iI_1)\gamma_c (1 - iI_1)\delta_d R^p_{\ qrs}$$

$$= (1 - iI_1)\alpha_p (1 + iI_1)\beta_b (1 - iI_1)\gamma_c (1 - iI_1)\delta_d (I_2)^p_j (I_2)^k_q R^j_{\ krs}$$

$$= (I_2)^a_j (I_2)^k_b (1 + iI_1)\beta_p (1 - iI_1)\gamma_q (1 - iI_1)\delta_r R^p_{\ qrs},$$

where $I_1 I_2 = -I_2 I_1$ is used in the last line. So

$$\frac{1}{16} (1 + iI_1)\alpha_p (1 - iI_1)\beta_b (1 - iI_1)\gamma_c (1 - iI_1)\delta_d R^p_{\ qrs} = R^\alpha_{\ \beta\gamma\delta} = 0$$

(3)

in the complex decomposition w.r.t. $I_1$, which is the condition for $(I_1, \nabla)$ to be a complex-flat structure on $M$. $\square$

Thus the results of §2 apply to hypercomplex manifolds, and lead to some new ideas about the Obata connection and complex structures on the tangent and cotangent bundles of a hypercomplex manifold.

**References**

