UNSTABLE PERIODIC ORBITS OF PERTURBED LORENZ EQUATIONS

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Abstract
The extended Malkus-Robbins dynamo [Moroz, 2003] reduces to the Lorenz equations when one of the key parameters, \( \beta \), vanishes. In a recent study [Moroz, 2004] investigated what happened to the lowest order unstable periodic orbits of the Lorenz limit as \( \beta \) was increased to the end of the chaotic regime, using the classic Lorenz parameter values of \( r = 28, \sigma = 10 \) and \( b = 8/3 \). In this paper we return to the parameter choices of [Moroz, 2003], reporting on two of the cases discussed therein.

Key words
Dynamos, unstable periodic orbits, Lorenz equations.

1 Introduction
Self-exciting dynamos can be nonlinear electro-mechanical devices or naturally occurring magnetohydrodynamic (MHD) fluid systems, such as the Earth’s geodynamo, operating within the liquid metallic outer core. These dynamos convert mechanical energy into magnetic energy through the action of motional induction. Since the governing equations are fully nonlinear partial differential equations in four independent variables, valuable insights into on-going physical processes can be achieved by investigating much lower dimensional models, which contain the key ingredients for producing dynamo action, yet render themselves accessible for systematic study.

One such model is the extended Malkus-Robbins dynamo (EMR) [Moroz, 2003], a four-mode system of nonlinear coupled ordinary differential equations, which reduce to the Malkus-Robbins dynamo [Robbins, 1977] when the parameter \( \beta \), measuring the inverse moment of inertia of the armature of the dynamo motor, vanishes. The EMR dynamo is of interest since it becomes the Lorenz system in this limit.

There has been considerable interest in recent years in characterising chaotic attractors of nonlinear dynamical systems, such as the Lorenz equations, in terms of the spectrum of unstable periodic orbits (upos), determined from time series. While a chaotic attractor possesses an infinite number of upos, it is believed that many properties may be determined from those orbits of lowest period.

In a recent study [Moroz, 2004] investigated what happened to the lowest order upos of the Lorenz limit of the EMR dynamo as \( \beta \) was increased, using the classic Lorenz parameter choices of \( r = 28, \sigma = 10 \) and \( b = 8/3 \), for two cases considered in [Moroz, 2003]. In one case it was possible to trace the evolution of some of the leading upos through to the onset of stable periodic oscillations. In the other case, the effects of non-zero \( \beta \) rapidly destroyed the simple character of the Poincaré return map.

In this paper, we extend the study of [Moroz, 2004] and present the results of investigations of the EMR dynamo in which the \( \beta = 0 \) limit does not yield chaotic behaviour, by choosing different values for \( r, \sigma \) and \( b \).

2 The Extended Malkus-Robbins Dynamo
The Malkus-Robbins dynamo comprises a conducting disk, driven to rotate with dimensionless angular speed \( \dot{W}(t) \) by a constant torque, with a magnetic field perpendicular to the disk inducing a radial e.m.f. A dimensionless current \( Z(t) \) flows in the disk, is removed by a ring of brushes and is fed to a coil and external load. The coil is aligned in such a way that its dimensionless current \( Y(t) \) produces a magnetic field to reinforce the original field. If \( U(t) \) is the angular velocity of the motor, then the EMR dynamo takes the form [Moroz, 2003; Moroz, 2004].

\[
\begin{align*}
Y' &= \sigma(Z - Y) - \beta U, \\
Z' &= Y[(1 + \frac{\beta}{2\sigma}) - \dot{W}] - Z, \\
W' &= ZY - \nu(W + \dot{W}), \\
U' &= Y - \dot{U},
\end{align*}
\]
and resistances of the coil and brushes; the mechanical friction of the motor and mechanical friction;

2.1 The Lorenz Limit
When \( \beta = 0 \), the evolution of \( U \) decouples from the first three equations of (1), which reduce to the Lorenz equations under the identification

\[
\begin{aligned}
Y, Z, W + W_e, U &\mapsto (x, y, z, 0), \\
(\sigma, R/\nu, \nu, \beta) &\mapsto (\sigma, r, b, 0).
\end{aligned}
\]  

Our choice for the Poincaré section is motivated by Koga, 1986, who, for the classic parameter choice of \( r = 28, b = 8/3 \) and \( \sigma = 10 \) defined his Poincaré section as

\[
S = [(x, y) : \dot{z} = 0, \dot{z} > 0, x > 0],
\]

where \( \dot{z} = z - (r - 1) \). He was able to construct two one-dimensional return maps

\[
X_{n+1} = f_1(X_n), \quad Y_{n+1} = f_2(Y_n),
\]

whose fixed points correspond to the upos of the series \( L_{n1} \) for \( \nu = 1, 2, ..., \). \( L_{nm} \) refers to \( n \) rotations around the left focus and \( m \) around the right focus, where the foci are the nontrivial equilibrium solutions of the Lorenz equations. Upos with \( L_{nm} \) for \( m > 1 \), are obtained by iterating the return maps \( m \) times and finding their fixed points.

In anticipation of what is to come, we remind the reader that the subcritical Hopf bifurcation for the nontrivial equilibria in the Lorenz system occurs when

\[
r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - 1 - b}
\]

so that \( \sigma > 1 + b \) must be satisfied for its existence.

2.2 The Poincaré section for the EMR dynamo
By analogy with the Lorenz equations, we have chosen the Poincaré section for the EMR dynamo to be:

\[
\hat{S} = [(Y, Z, U) : W = 0, \dot{W} > 0, Y > 0].
\]

The nontrivial equilibrium states are then \( W_e = (Y_e, Z_e, 0, U_e) \), where

\[
\begin{aligned}
Y_e = \pm \sqrt[3]{\frac{\nu W_e}{1 + \frac{\beta}{3 \nu}}}, \\
Z_e = Y_e(1 + \frac{\beta}{3 \nu}), \\
U_e = Y_e/\Lambda,
\end{aligned}
\]

while the trivial equilibrium state is

\[
(Y_0, Z_0, W_0, U_0) = (0, 0, -W_e, 0).
\]

We remark that a double-zero Takens-Bogdanov bifurcation occurs when

\[
(R_d, \beta_d) = \left( \frac{\nu(\Lambda + \sigma)}{\sigma(1 - \Lambda)}, \frac{\Lambda^2(1 + \sigma)}{1 - \Lambda} \right),
\]

provided \( \Lambda < 1 \), while the subcritical Hopf bifurcation condition in the Lorenz limit translates to \( \sigma > 1 + \nu \).

2.3 Parameter Regimes
As discussed in detail in [Moroz, 2003], there are four different parameter regimes of interest: \( \Lambda \gtrless 1 \) and \( \sigma \gtrsim 1 + \nu \) [Moroz, 2003]. The conditions on \( \Lambda \) correspond to the existence or otherwise of a codimension-two double-zero bifurcation, associated with each of the equilibrium solutions (see [Moroz, 2003] for details). The conditions on \( \sigma \) relate to the termination or otherwise of the limit cycle, which emerges from the multiple bifurcation point, on the subcritical Hopf bifurcation point, associated with the Lorenz limit of \( \beta = 0 \) [Moroz, 2004] chose to work with the classic Lorenz parameter values, which correspond to \( \sigma > 1 \). In this contribution, we consider the alternative choice of \( \sigma < 1 \), with \( \Lambda \gtrsim 1 \). This necessitates a different selection for \( r, \sigma \) and \( b \), and the ones we chose were those in [Moroz, 2003] of \( \nu = b = 1, \sigma = 1.9 \) and either \( \Lambda = 0.5 \) or \( \Lambda = 1.2 \). These are cases (iii) and (iv) of [Moroz, 2003].

2.4 \( \Lambda < 1 \) and \( 0 < \sigma < 1 + \nu \)
When \( \Lambda = 0.5, \nu = 1 \) and \( \sigma = 1.9 \), a linear stability analysis shows the curves of Figure 3 of [Moroz, 2003]. The are curves of steady state solutions and curves of Hopf bifurcation, emanating from the single double-zero bifurcation point. Because of the violation of the Hopf bifurcation criterion in the \( \beta = 0 \) limit, the curve of subcritical limit cycles, asymptotes for the \( \beta = 0 \) axis, but does not touch it.

Numerical integrations reveal a small region of parameter space to be occupied by chaotic solutions, with most of the rest dominated by simple periodic behaviour [Moroz, 2003].
2.5 $\Lambda > 1$ and $0 < \sigma < 1 + \nu$

When $\Lambda = 1.2$, $\nu = 1$ and $\sigma = 1.9$, neither the double-zero nor the subcritical Hopf bifurcation exist. A linear stability analysis yields one curve of steady state bifurcations and a curve of (subcritical) Hopf bifurcations (associated with the nontrivial equilibria) which no longer intersect (see Figure 5 of [Moroz, 2003]).

Numerical integrations there show two regions of chaotic behaviour, separated by a relatively small region of periodic behaviour, which increases as $R$ and $\beta$ increase. In this and in the other case considered here, chaos is found near the left hand Hopf stability boundary, marking the transition to steady state solutions.

3 Numerical integrations

In the numerical investigations, we integrated the EMR dynamo equations for 60,000 seconds using time steps of either 0.01, 0.005 or 0.001 seconds, discarding the first 100 seconds as representing transients. The selection of a close return on the Poincaré section, regardless of the size of the integration step, by adopting the approach described in [Henon, 1982]. This is achieved by rewriting the EMR equations with $W$ as the independent variable instead of $t$, and integrating the resulting equations for 1 step from $W_n$ to $W = 0$. Here $t_n$ is the time just before $W$ changes sign.

Histograms of periods of the upos were constructed based upon close returns to within $\epsilon$ of the fixed points of the $m^{th}$ iterates of the return map for $m = 1, \ldots, 10$. In our numerical integrations we used the same initial conditions of $(Y, Z; W, U) = (-2.159025, -6.5509769, -8.7136071, -4.81492)$.

3.1 $\Lambda < 1$ and $0 < \sigma < 1 + \nu$

Figure 3 of [Moroz, 2003] shows the linear and nonlinear stability regions for $\Lambda = 0.5$ and $\sigma = 1.9$ between the ranges of $0 \leq R \leq 20$ and $0 \leq \beta \leq 10$. For our current investigations, we chose $R = 20$, which, for $\beta$ increasing, yields the onset of chaotic behaviour at $\beta \approx 0.38$ as a sudden jump up from the zero equilibrium state of no dynamo action. For $\beta$ decreasing, the loss occurs at $\beta \approx 0.31$, again as a sudden jump, demonstrating the presence of hysteresis effects. Such behaviour is consistent with the subcritical nature of the Hopf bifurcation curve which no longer terminates on $\beta = 0$. The loss of chaotic behaviour is observed at $\beta \approx 2.34$, via an intermittent destabilisation of a periodic orbit.

[Moroz, 2004] found that for very small perturbations about the Lorenz limit of $\beta = 0$ and for the Lorenz parameter choices for $\sigma$ and $\nu$, the $\Lambda < 1$ case appears to be far more sensitive to the number of close returns than does the case of $\Lambda > 1$. We have also found the same to hold for the current study. When $\beta = 0.4$, we found only 65 close returns with 60 million time steps, the lowest order upo having a period of 3.96 sec. Figure 1 shows the time series of $Y(t)$ for $\beta = 0.4$, while Figure 2 shows the first return map in the $(Y_n, Y_{n+1})$-plane. As $\beta$ increases into the chaotic regime, the number of close returns increases, while the period of the dominant upo decreases. For example when $\beta = 2.0$, we counted 2800 close returns with the only observed lowest order upo having a period of 3.33 sec. Figure 1 shows a section of the time series for $\beta$ and Figure 2 the lowest order upo in the $(Y, Z)$-plane for $\beta = 2$. The time series shows evidence of an intermittent destabilisation of an underlying periodic orbit, corroborated in the corresponding phase portrait (not shown here). As the chaotic regime gives way to stable...
periodic limit cycle behaviour, we found, at $\beta = 2.35$, the limit cycle’s period to be $3.259\text{ sec}$.

3.2 $\Lambda > 1$ and $0 < \sigma < 1 + \nu$

Figure 4 of [Moroz, 2003] shows the linear and nonlinear stability curves for $\sigma = 1.9$ with $\Lambda = 1.2$ between the ranges $0 \leq \beta \leq 100$ and $0 \leq R \leq 50$. Numerical integrations showed the existence of two regions of chaotic behaviour, contained within the domain bounded by the subcritical Hopf bifurcation curve. The larger region occurred for small values of $\beta$, loss of stability being a discontinuous jump down to steady (non-trivial) equilibrium states. The smaller region occurred in an expanding band, for larger values of $\beta$, near the righthand boundary of the subcritical Hopf curve. Figure 13 of [Moroz, 2003] gave a typical example of a bifurcation transition curve for $R = 50$. For the current study, we chose $R = 20$, as in the previous subsection. In this case, the initial onset of the larger chaotic band occurs at $\beta \approx 0.93$, terminating in a stable periodic limit cycle when $\beta = 4.6$. This periodic limit cycle maintains its stability, growing in amplitude as $\beta$ increases, until the onset of the second band of chaotic solutions at $\beta \approx 23.8$, with its subsequent loss at $\beta \approx 24.1$.

Following [Koga, 1986; Moroz, 2004], we adopt the terminology $L_{nm}$, which refers to the trajectory rotating $n$ times around the left focus and $m$ times the right focus, where the foci are the nontrivial equilibrium solutions of equation (8). The basic upo of the previous subsection would therefore be labelled $L_{11}$.

When $\beta = 2$ we observed 857 close returns. The basic upo with period $\approx 3.79\text{ sec}$ resembles that shown in Figure 2 above. Figures 3 and 4 show examples of $L_{21}$ and $L_{22}$ upos with periods $5.8\text{ sec}$ and $7.58\text{ sec}$ respectively. When $\beta = 4$ there were 6031 close returns with the $L_{11}$ upo of period $3.454\text{ sec}$ accounting for nearly all of the $L_{m1}$ series. Examples of $L_{22}$ and $L_{33}$ are shown in Figures 5 and 6 respectively. The examples shown had periods of $8.098\text{ sec}$ and $11.74\text{ sec}$. When $\beta = 24$, we observed 7520 close returns with $\approx 1800$ of those centred about a basic upo of period $9.1\text{ sec}$. Figure 6 shows a typical $L_{11}$ upo. As in the $\Lambda > 1$ case in [Moroz, 2004], the Poincaré return maps show a cleaner banded structure than in the $\Lambda < 1$ case (see for example Figure 7 for $\beta = 1.0$). As $\beta$ increases,
more of the bands disappear until very few remain (see Figure 8) prior to the loss of chaotic dynamo action to a steady state.

4 Discussion

As noted in [Moroz, 2004], the purpose of studying such low-dimensional dynamos lies in their incorporation of some of the key features of much larger-scale naturally-occurring MHD dynamos. They exhibit steady dynamo action, irregular reversals as well as sustained periodic/multiply periodic oscillations. However with so many different low order dynamos of this family in existence [Moroz, 2001], the issue naturally arises as to how to distinguish between them. The identification of the spectrum of unstable periodic orbits, as well as their behaviour when key parameters vary provides one means of comparison, especially if similar approaches are applied to identify upos of the large-scale dynamo models. The present study is one more step in that direction.

References

Figure 11. First return map for $\beta = 24.0$ with $\Lambda = 1.2$