The 80% Rule in Mean-Variance Portfolio Selection with Random Interest Rate

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A thesis submitted in partial fulfilment of the MSc in
Mathematical and Computational Finance
Trinity Term 2014
Acknowledgements

I would like to express my deep gratitude to Professor Xun Yu Zhou, my supervisor, for offering me the opportunity of working on this topic under his supervision, and for his enlightening guidance and advice. I also wish to thank my parents and my girlfriend for their infinite and continuous support all along the work.
Abstract

In this thesis, we study the continuous-time mean-variance portfolio selection problem, which aims at minimising the variance of the return of a portfolio at the end of a certain investment horizon with a targeted expected return at the terminal date specified. It has been shown in the literature that, when the market parameters in the problem are all deterministic functions in time, the wealth process of the optimal portfolio can reach the discounted targeted return before or at the terminal time with a probability of at least 80%. This thesis extends the above result to some more general cases by showing that the 80% rule is still valid when the interest rate becomes random while all the other parameters are kept deterministic. This is achieved through application of backward stochastic differential equations (BSDEs) and by using some results in fixed income markets modelling.
Contents

1 Introduction 1

2 Mean-Variance Model and Solution 3
   2.1 Problem Formulation 3
   2.2 Some Theory in Interest Rates and Bonds 5
   2.3 Solution of the Mean-Variance Problem 7
      2.3.1 Problem Solution 7
      2.3.2 Solution of Equation (2.20) 8
      2.3.3 Solution of Equation (2.21) 10
      2.3.4 Relation between $v$ and $w$ 11
      2.3.5 Optimal Portfolio Rewritten 12

3 Probability of Target Reaching: The 80% Rule 13
   3.1 Proof of the Rule 13
   3.2 Numerical Application 15
      3.2.1 Numerical Demonstration of the 80% Rule 16
      3.2.2 Application to Pension Fund Investment 19

4 Discussions and Conclusion 21

A Matlab Code for Section 3.2.1 24

B Matlab Code for Section 3.2.2 27
Chapter 1

Introduction

A mean-variance portfolio selection problem was first introduced and studied by Harry Markowitz in the 1950s. It concerns finding optimal investment strategies over a single period under the assumption that risk is undesirable for investors. If the final expected return is targeted, then the optimal portfolio is achieved when the variance of the return of the portfolio, as a measure of risk, is minimised. In recent years, a considerable amount of works have been contributed to this optimisation problem, notably in continuous time cases. Zhou and Li [6] solved the problem with deterministic parameters and provided closed analytical forms of the optimal portfolio process and the associated minimum variance of the final return. Two years later, Lim and Zhou [3] extended the study to a random parameters set-up. Theory in Stochastic Control for optimisation purpose and Backward Stochastic Differential Equations (BSDE), and related techniques developed have been critical to these works.

The mean-variance portfolio selection model is not only mathematically aesthetic but presents considerable potential use in real investment activity as well. Nevertheless, the relevance of the model with the real world has always been challenged. The discussions have been focused on: first, whether the return variance is an appropriate measure for risk; second, whether a geometric Brownian motion is a pertinent model for asset pricing; and third, whether an investor in the real world really cares about the optimality implied by this model, the validity of which is based on an infinity of possible scenarios while the investor will only experience one.

Li and Zhou dealt with the portfolio selection problem by a different approach in their paper (see Li and Zhou [2]). They showed that in a setting with continuous time, deterministic market parameters and complete market assumptions, the discounted targeted return could be reached before or at the terminal time with a probability of at least 80% by applying the mean-variance optimal selection policy. Yan and Zhou
[4] further studied the approach and revised the strategies so that the target-reaching probability could be even higher.

As stated in the paper of Li and Zhou [2], the main assumption for the 80% rule to hold is that market parameters are deterministic functions in time, and how the rule changes with stochastic parameters remains an open question. Motivated by the unsolved problem, this paper makes a first attempt to answer the question by proving that the 80% rule still holds more generally where the interest rate is a uniformly bounded stochastic process whose dynamics are of a specific case and other parameters remain deterministic. Whether stochastic volatilities and stochastic relations between appreciation rates and the interest rate may change the rule, or if they do, to what extent, remains an open question.

The outline of the thesis is as follows. In Chapter 2, we formulate the mean-variance portfolio selection problem, introduce some preliminary knowledge and theory in term-structure modelling, and present the solution to the problem. Chapter 3 shows how the main result of the thesis is obtained and numerical simulations are carried out to support and demonstrate the 80% rule. Discussions and the conclusion are made in Chapter 4.
Chapter 2

Mean-Variance Model and Solution

2.1 Problem Formulation

Throughout this thesis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ denotes a fixed filtered complete probability space where $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by a standard $m$-dimensional Brownian motion $(W_t)_{t \geq 0}$ with $W_t = (W^1_t, W^2_t, \ldots, W^m_t)^\top$. Let $T > 0$ denote the investment horizon. For any matrix $M$ let $M^\top$ denote its transpose. For any $p \in \mathbb{N}^\ast$ let $L^p_\mathcal{F}(0, T; \mathbb{R}^m)$ denote the space of all $\mathbb{R}^m$-valued, $\mathcal{F}_t$-adapted process $f$ on $[0, T]$ such that $\mathbb{E}\int_0^T |f(t)|^p dt < \infty$ where $|.|$ denotes the usual Euclidean norm for a vector of any finite dimension, let $L^\infty_\mathcal{F}(0, T; \mathbb{R}^m)$ denote the set of $\mathcal{F}_t$-adapted uniformly bounded processes, let $L^{\infty}_\mathcal{F}(\Omega; C(0, T; \mathbb{R}^m))$ denote the set of $\mathcal{F}_t$-adapted uniformly bounded continuous processes, and let $L^{\infty}_\mathcal{T}\mathcal{F}_T(\Omega, \mathbb{R}^m)$ denote the set of all bounded $\mathcal{F}_T$-measurable random variables. Finally, for any random process $X$ we use $X_t$ to denote the random variable at time $t$ while for any deterministic function $Y$ of time we use $Y(t)$ to denote its value at time $t$.

Consider a market with $m + 2$ continuously traded securities, one of which is a bank account whose value process $S^0_t$ satisfies the following differential equation:

$$
\begin{cases}
    dS^0_t &= r_t S^0_t dt, \quad \text{for } t \geq 0 \\
    S^0_0 &= s_0 > 0
\end{cases}
$$

(2.1)

where $(r_t)_{t \geq 0} \in L^\infty_\mathcal{F}(\Omega; C(0, T; \mathbb{R}^m))$ is the interest rate process whose stochastic dynamics are specified at a later time. The second is a zero-coupon bond maturing at $T$ whose value process $P(t, T)$ satisfies:

$$
P(t, T) = \mathbb{E}^Q [e^{-\int_t^T r_s \, ds} | \mathcal{F}_t]
$$

(2.2)

where $\mathbb{E}^Q$ denotes the expectation associated with the risk neutral probability measure $Q$ which is defined by a Girsanov transformation from the physical measure $\mathbb{P}$ and
will be specified at a later time in the thesis. The other \( m \) securities are risky assets whose value processes \( S^1_t, ..., S^m_t \) satisfy the following dynamics:

\[
\begin{align*}
    dS^k_t &= S^k_t \left( \mu^k_t dt + \sum_{j=1}^{m} \sigma_{kj}(t) dW^j_t \right), \quad \text{for } t \geq 0 \\
    S^k_0 &= s_k > 0, \quad \text{for } k = 1, 2, ..., m.
\end{align*}
\]

(2.3)

where \( \mu^k_t \) is the appreciation rate process for the \( k \)-th risky asset, and \( \sigma_{kj}(t) \) is the volatility process of the asset with respect to the \( j \)-th component of the Brownian motion. We assume that \( \mu^k_t \) is a stochastic process in time while \( \sigma_{kj}(t) \) is a deterministic function in time.

Now consider an agent with an initial capital endowment \( x_0 \) and an investment horizon \([0, T]\). Let \( x_t \) denote the total wealth of the agent at time \( t \in [0, T] \) (\( x_0 \) is used as a constant value and as the total wealth at time zero indifferently). If we assume that the agent only invests in the bank account and the \( m \) risky assets, the trading takes place continuously, the portfolio is self-financing, and transaction costs and consumptions are not considered, then \( x_t \) satisfies:

\[
\begin{align*}
    dx_t &= \left( r_t x_t + \sum_{i=1}^{m} (\mu^i_t - r_t) u^i_t \right) dt \\
    &\quad + \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij}(t) u^i_t dW^j_t, \quad \text{for } 0 \leq t \leq T \\
    x_0 &= x_0
\end{align*}
\]

(2.4)

where \( u^i_t \) for \( i = 1, 2, ..., d \) denotes the total investment in the \( i \)-th risky asset so that the sum of \( u^i_t \) is \( x_t \) for \( 0 \leq t \leq T \). We define the vector process \( u_t = (u^1_t, ..., u^m_t)^\top \) to be the portfolio of the agent at time \( t \).

Let vector \( B(t) \) denote vector \((\mu^1_t - r_t, ..., \mu^m_t - r_t)^\top\). Then Equation (2.4) can be rewritten in a matrix form as follows:

\[
\begin{align*}
    dx_t &= \left( r_t x_t + B(t)^\top u_t \right) dt + u_t^\top \sigma(t) dW_t, \quad \text{for } 0 \leq t \leq T \\
    x_0 &= x_0
\end{align*}
\]

(2.5)

where \( \sigma(t) \) is the volatility matrix. We assume that \( B(t) \) is a \( \mathbb{R}^m \)-valued deterministic function in time.

**Definition 2.1.** A portfolio \( u \) is said to be admissible if \( u \in L^2_F(0, T; \mathbb{R}^m) \) and the SDE (2.5) has a unique solution \( x \) corresponding to \( u \).

Given a targeted expected return \( d \). The agent’s objective is to find an admissible portfolio process \( u \) such that \( \mathbb{E}[x_T] = d \) and \( \text{Var}(x_T) \) is minimised. The requirement \( d \geq x_0/P(0, T) \) is imposed (suppose that the dynamics of \( r_t \) is known so that the value of the bond is known), or otherwise the agent could simply put all his capital in the zero-coupon bonds to reach the target. The problem of finding such a portfolio is referred to as the *mean-variance portfolio selection problem*, and more rigorously, it is formulated as follows:
Definition 2.2. The mean-variance portfolio selection problem is a constrained stochastic optimisation problem with \( d \geq x_0/P(0, T) \), described as:

\[
\begin{array}{ll}
\text{minimise } & \mathbb{E}[(x_T - d)^2] \\
\text{subject to: } & \mathbb{E}[x_T] = d, \\
& u \text{ is admissible.}
\end{array}
\] (2.6)

Before proceeding to the solution, we present some preliminary theory in fixed-income markets in the next section, which will be used in the solution of the above problem.

2.2 Some Theory in Interest Rates and Bonds

A zero-coupon bond (ZCB) is a bond which does not pay any dividend and is redeemed at face value at maturity. Mathematically, if we denote \( P(t, T) \) the value of a ZCB at \( t \) maturing at \( T \) (named \( T-Bond \) hereafter), then \( P(t, T) \) is a stochastic process in \( t \) satisfying: (i) \( P(T, T) = 1 \) and (ii) \( t_0 \) being fixed, \( P(t_0, T) \) is a differentiable function in \( T \). We assume that the market is arbitrage free and complete.

For \( t \in [0, T] \) we define the instantaneous forward rate \( f(t, T) \) at time \( t \) with maturity \( T \) as:

\[
f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}
\] (2.7)

or equivalently:

\[
P(t, T) = e^{-\int_t^T f(t, s)ds}.
\]

The instantaneous short rate (interest rate) at time \( t \) is defined by:

\[
r_t = f(t, t)
\] (2.8)

As shown in Section 4 - Arbitrage Theory in Filipovic [1], we have the following result using arbitrage free arguments:

Proposition 2.3. In a complete and arbitrage free market, assume there exists a unique equivalent martingale measure \( Q \). Then the value process of the \( T-Bond \) \( P(t, T) \) satisfies

\[
P(t, T) = \mathbb{E}^Q[e^{-\int_t^T r_s ds} | \mathcal{F}_t] \] (2.9)

where \( r_t \) is the short rate at \( t \).
Now we assume that the forward rate \( f(t, T) \) follows the Ito dynamics

\[
f(t, T) = f(0, T) + \int_{0}^{t} \alpha(s, T)ds + \int_{0}^{t} \sigma(s, T)^{\top}dW_s
\]

(2.10)

where \( f(0, T) \) is the initial forward curve which is given, \( W_t \) is a standard \( m \)-dimensional Brownian motion as described in Section 2.1, \( \alpha(t, T) \) is a \( \mathcal{F}_t \)-adapted scalar process such that \( \int_{0}^{T} \int_{0}^{T} |\alpha(s, t)| ds dt < \infty \) for all \( T \), and \( \sigma(t, T) \) is a \( \mathcal{F}_t \)-adapted, \( \mathbb{R}^m \)-valued process such that \( \sup_{s,t<T} |\sigma(s, t)| < \infty \) for all \( T \). Then the short rate \( r_t \) has the following dynamics

\[
r_t = f(0, t) + \int_{0}^{t} \alpha(s, t)ds + \int_{0}^{t} \sigma(s, t)^{\top}dW_s.
\]

(2.11)

On p.94-95 of his book (Filipovic [1]), Filipovic proposes the following lemma and theorem to determine the dynamics of the value process \( P(t, T) \) of the T-Bond, and the constraint on the drift term \( \alpha(t, T) \) of the forward rate \( f(t, T) \) in order to comply with the arbitrage-free principle.

**Lemma 2.4.** For every maturity \( T \), the T-Bond price follows an Ito process of the form

\[
dP(t, T) = \left( r_t + A(t, T) + \frac{1}{2} |v(t, T)|^2 \right) P(t, T)dt + P(t, T)v(t, T)^{\top}dW_t
\]

(2.12)

where

\[
A(t, T) = -\int_{t}^{T} \alpha(t, s)ds
\]

and

\[
v(t, T) = -\int_{t}^{T} \sigma(t, s)ds
\]

(2.13)

is the volatility vector of the bond.

**Proof** The fact that \( \log P(t, T) = -\int_{t}^{T} f(t, s)ds \) and the use of Fubini Theorem twice on stochastic integrals give the dynamics of \( \log P(t, T) \). Applying Ito’s formula to \( \exp(\log P(t, T)) \) gives the above dynamics of \( P(t, T) \). For more details of the proof, see Lemma 6.1 on p.94 of Filipovic [1].
Theorem 2.5. (HJM Drift Condition) In an arbitrage free market, let $\mathbb{Q}$ be an equivalent martingale measure and let $W^\mathbb{Q}$ be a $m$-dimensional Brownian motion under this measure. Then the drift term $\alpha(t,T)$ satisfies

$$
\alpha(t,T) = \sigma(t,T)^\top \int_t^T \sigma(s,s)ds + \lambda(t)^\top \sigma(t,T) \tag{2.14}
$$

where $\lambda(t)$ is the vector related to the Girsanov translation between $\mathbb{P}$ and $\mathbb{Q}$ such that $dW_t^\mathbb{Q} = \lambda(t)dt + dW_t$.

Proof Using the fact that $\exp\left(-\int_0^t r_s ds\right) P(t,T)$ is a $\mathbb{Q}$-martingale and applying Itô’s formula to it gives the above HJM drift condition. For mere details of the proof, see Theorem 6.1 on p.95 of Filipovic [1].

In view of the above theorem, Equation (2.11) becomes

$$
r_t = f(0,t) + \int_0^t \left( \sigma(s,t)^\top \int_s^t \sigma(s,u)du + \lambda(s)^\top \sigma(s,t) \right) ds + \int_0^t \sigma(s,t)^\top dW_s. \tag{2.15}
$$

and the dynamics of the value process of the T-Bond becomes

$$
dP(t,T) = (r_t + \lambda(t)^\top v(t,T)) P(t,T)dt + P(t,T)v(t,T)^\top dW_t. \tag{2.16}
$$

2.3 Solution of the Mean-Variance Problem

2.3.1 Problem Solution

Now we come back to our original problem. We keep all the notation in Section 2.1, and in addition to the assumptions already made, we further assume that:

1. There exists some $\delta > 0$ such that for all $t \in [0,T]$, $\sigma(t)^\top \sigma(t) \geq \delta I$.

2. The above assumption implies that matrix $\sigma(t)$ is invertible for all $t \in [0,T]$. Hereafter, we write $\theta(t) = \sigma(t)^{-1} B(t)$. Then $0 < \int_0^T |\theta(t)|^2 ds < \infty$.

3. The dynamics of $r$ are determined by the modelling of the corresponding forward rate, whose coefficients in the stochastic differential form are deterministic functions in $t$ and $T$. See Equation (2.27) and (2.34).
4. Let \((\psi, \xi) \in L^2_F(\Omega; C(0,T;\mathbb{R})) \times L^2_F(0,T;\mathbb{R}^m)\) be the unique solution to the BSDE

\[
\begin{aligned}
d\psi_t &= -r_t \psi_t dt + \xi_t^\top dW_t \\
\psi_T &= 1.
\end{aligned}
\] (2.17)

We assume \(\mathbb{E} \int_0^T |\psi_s B(s) + \sigma(t)\xi_t|^2 ds > 0\).

Lim and Zhou [3] provides an analytical solution to the mean-variance portfolio selection problem with random market parameters. It is stated as follows.

**Theorem 2.6.** Let \(u^*\) and \(x^*\) denote the optimal portfolio and the associated optimal wealth process of the stochastic control problem (2.6). Then the optimal portfolio is

\[
u_t^* = -\sigma(t)\sigma(t)^\top \Re^{-1} [(B(t) + \sigma(t)\frac{\Delta}{p_t})(x_t^* - d \cdot g_t) - d \cdot \sigma(t)\zeta_t \\
+ \lambda^* ((B(t) + \sigma(t)\frac{\Delta}{p_t})g_t + \sigma(t)\zeta_t)]
\] (2.18)

where

\[
\lambda^* = -\frac{p_0 g_0^2}{1 - p_0 g_0} \left( d - \frac{x_0}{g_0} \right),
\] (2.19)

\((p, \Lambda) \in L^\infty_F(\Omega; C(0,T;\mathbb{R})) \times L^2_F(0,T;\mathbb{R}^m)\) is the unique solution to

\[
\begin{aligned}
dp &= -[(2r - \theta^\top \theta)p - 2\theta^\top \Lambda - \frac{1}{p} \Lambda^\top \Lambda]dt + \Lambda^\top dW \\
p_T &= 1,
\end{aligned}
\] (2.20)

and \((g, \zeta) \in L^\infty_F(\Omega; C(0,T;\mathbb{R})) \times L^2_F(0,T;\mathbb{R}^m)\) is the unique solution to

\[
\begin{aligned}
dg &= (rg + \theta^\top \zeta)dt + \zeta^\top dW \\
g_T &= 1.
\end{aligned}
\] (2.21)

For details of the proof of the theorem, refer to Lim and Zhou [3]. In order to comply with the constraints imposed on the coefficients of the SDEs by that paper, the interest rate \(r_t\) should be a uniformly bounded process. Although this is not a necessary condition, we assume henceforth that the coefficients in the SDE of \(f(t,T)\) in our model are arranged in such a way that \(r_t\) is a uniformly bounded process.

### 2.3.2 Solution of Equation (2.20)

To solve (2.20), we consider BSDE

\[
\begin{aligned}
dY &= [(2r - \theta^\top \theta)Y + 2\theta^\top Z]dt + Z^\top dW \\
Y_T &= 1.
\end{aligned}
\] (2.22)
Both Equation (2.20) and (2.22) admit unique solutions, see Theorem 2.2, p.349 of Yong and Zhou [5]. If \((Y, Z)\) is the solution to (2.22), then \((p, \Lambda) = \left(\frac{1}{Y}, -\frac{Z}{Y^2}\right)\) will be the solution to (2.20). Using the Martingale Representation Theorem, \(Y\) is found to be

\[
Y_t = e^{\int_0^t |\theta(s)|^2 ds} \mathbb{E}[e^{-2rT}dW_t]\]  \(2.23\)

where \(\tilde{Q}\) is a new measure defined by the Girsanov change of measure

\[
\frac{d\tilde{Q}}{dP}\bigg|_{\mathcal{F}_t} = \exp \left( -\int_0^t 2\theta(s)^\top dW_s - 2\int_0^t |\theta(s)|^2 ds \right). \]  \(2.24\)

Note that \(\theta(t)\) is a deterministic vector function in \(t\). Hereafter we denote \(\int_0^t |\theta(s)|^2 ds\) by \(\beta(t)\) for all \(t \geq 0\).

Actually, \(\tilde{Q}\) would be the unique equivalent martingale measure in a fictitious world where the appreciation rates and interest rate are all doubled. In this world, the T-Bond price would be \(R(t, T) = \mathbb{E}[e^{-\beta(T)}|F_t]\) and

\[
Y_t = \exp(\beta(T) - \beta(t))R(t, T). \]  \(2.25\)

Applying Ito’s formula to \(R(t, T) = Y_t \exp(\beta(t) - \beta(T))\), we obtain the dynamics of \(R(t, T)\) as

\[
dR(t, T) = [2r_t R(t, T) + 2\theta^\top(t)e^{\beta(t) - \beta(T)}Z_t]dt + e^{\beta(t) - \beta(T)}Z^\top_t dW_t. \]  \(2.26\)

We assume that in this world the forward rate \(f_2(t, T)\) follows the dynamics

\[
df_2(t, T) = \alpha_2(t, T)dt + \sigma_2(t, T)^\top dW_t \]  \(2.27\)

with \(\alpha_2(t, T)\) and \(\sigma_2(t, T)\) being both deterministic functions in \(t\). Let \(w(t, T) = -\int_t^T \sigma_2(t, s)ds\). Then Lemma 2.4 and Theorem 2.5 impose that

\[
\alpha_2(t, T) = \sigma_2(t, T)^\top \int_t^T \sigma_2(t, s)ds + 2\theta(t)^\top \sigma_2(t, T) \]  \(2.28\)

and

\[
dR(t, T) = \left(2r_t + 2\theta(t)^\top w(t, T)\right) R(t, T)dt + R(t, T)w(t, T)^\top dW_t. \]  \(2.29\)

Comparing (2.26) with (2.29) we obtain

\[
Z_t = e^{\beta(T) - \beta(t)}R(t, T)w(t, T) \]  \(2.30\)
Finally, taking (2.25), (2.30), and the relation between \((p, \Lambda)\) and \((Y, Z)\), we obtain the solution to BSDE (2.20)

\[
p_t = \frac{\exp(\beta(t) - \beta(T))}{R(t, T)},
\]

(2.31)

\[
\Lambda_t = -\frac{\exp(\beta(t) - \beta(T))w(t, T)}{R(t, T)},
\]

(2.32)

where \(R(t, T)\) is the value of the T-Bond in a world described by \(\hat{Q}\) and \(w(t, T)\) is its volatility vector. Note that \(w(t, T)\) is a deterministic vector function in \(t\).

### 2.3.3 Solution of Equation (2.21)

BSDE (2.21) is quite straightforward to solve. It is not hard to observe that \(g_t\) is in fact the value of the T-Bond (this time, in the real world) at time \(t\). As a convention, we write \(g_t = P(t, T)\). Note that this time the equivalent martingale measure \(Q\) is defined by

\[
\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \exp \left( -\int_0^t \theta(s)^\top dW_s - \frac{1}{2} \int_0^t |\theta(s)|^2 ds \right)
\]

(2.33)

which is the true risk-neutral probability measure. We assume that the real world forward rate \(f_1(t, T)\) follows the dynamics

\[
df_1(t, T) = \alpha_1(t, T)dt + \sigma_1(t, T)^\top dW_t
\]

(2.34)

with \(\alpha_1(t, T)\) and \(\sigma_1(t, T)\) being both deterministic functions in \(t\). Let \(v(t, T) = -\int_t^T \sigma_1(t, s)ds\). Using once again Lemma (2.4) and Theorem (2.5) we find

\[
\alpha_1(t, T) = \sigma_1(t, T)^\top \int_t^T \sigma_1(t, s)ds + 2\theta(t)^\top \sigma_1(t, T)
\]

(2.35)

and

\[
dP(t, T) = (r_t + \theta(t)^\top v(t, T)) P(t, T)dt + P(t, T)v(t, T)^\top dW_t.
\]

(2.36)

Comparing (2.21) with (2.36) we obtain the solution to BSDE (2.21)

\[
g_t = P(t, T)
\]

(2.37)

\[
\zeta_t = P(t, T)v(t, T),
\]

(2.38)

where \(P(t, T)\) is the value of the T-Bond in the real world described by \(Q\) and \(v(t, T)\) is its volatility vector. Note that \(v(t, T)\) is a deterministic vector function in \(t\).
2.3.4 Relation between $v$ and $w$

From the expressions $P(t,T) = \mathbb{E}^Q[e^{-\int_t^T r_s ds}|\mathcal{F}_t]$ and $R(t,T) = \mathbb{E}^{\tilde{Q}}[e^{-\int_t^T 2r_s ds}|\mathcal{F}_t]$ where $Q$ and $\tilde{Q}$ are probability measures entirely defined by $P$ and $\theta$, we learn that $P(t,T)$ and $R(t,T)$ are related, which means that the dynamics of one of them entirely determine those of the other. As the dynamics of the T-Bond price is determined by those of the related forward rate process, whose drift term in the SDE is entirely determined by its volatility term (see Theorem 2.5 HJM Drift Condition), we deduce that the volatility vectors $v(t,T)$ of $P(t,T)$ and $w(t,T)$ of $R(t,T)$ are related.

**Proposition 2.7.** Let $f_1(t,T)$ be the forward rate for $r_t$ and $f_2(t,T)$ be the forward rate for $2r_t$. Let $\sigma_1(t,T)$ and $\sigma_2(t,T)$ be the volatility vectors of $f_1(t,T)$ and $f_2(t,T)$ respectively. If $\sigma_1(t,T)$ and $\sigma_2(t,T)$ both are deterministic functions in $t$, then $\sigma_2(t,T) = 2\sigma_1(t,T)$ for all $t \in [0,T]$. Moreover, if $P(t,T)$ is the T-Bond price with volatility vector $v(t,T)$ and $R(t,T)$ is the T-Bond price with volatility vector $w(t,T)$ in a fictitious world with doubled interest rate and doubled appreciation rates, then $w(t,T) = 2v(t,T)$.

**Proof** According to Equation (2.15) which is a direct result of Theorem 2.5:

$$r_t = f_1(0,t) + \int_0^t \left(\sigma_1(s,t)^\top \int_s^t \sigma_1(s,u) du + \theta(s)^\top \sigma_1(s,t)\right) ds + \int_0^t \sigma_1(s,t)^\top dW_s$$

and

$$2r_t = f_2(0,t) + \int_0^t \left(\sigma_2(s,t)^\top \int_s^t \sigma_2(s,u) du + 2\theta(s)^\top \sigma_2(s,t)\right) ds + \int_0^t \sigma_2(s,t)^\top dW_s.$$ 

Since $\sigma_1(t,T)$, $\sigma_2(t,T)$, $f_1(0,t)$, $f_2(0,t)$ and $\theta(t)$ are all deterministic functions in $t$, the last term in each equation is the only source of randomness. It follows that $\sigma_2(t,T) = 2\sigma_1(t,T)$.

For the second statement, Lemma 2.4 gives

$$v(t,T) = -\int_t^T \sigma_1(t,s) ds \text{ and } w(t,T) = -\int_t^T \sigma_2(t,s) ds.$$ 

It follows immediately that $w(t,T) = 2v(t,T)$.

\[\square\]
2.3.5 Optimal Portfolio Rewritten

Substituting $p_t, A_t, g_t$ and $\zeta_t$ in the optimal portfolio formula (2.18) with their expressions provided in (2.31), (2.32), (2.37) and (2.38), and replacing $w$ with $2v$, we obtain

$$u_t^* = -(\sigma(t)\sigma(t)^\top)^{-1}[(B(t) - 2\sigma(t)v(t, T))(x_t^* + (\lambda^* - d)P(t, T)) + (\lambda^* - d)P(t, T)\sigma(t)v(t, T)]$$

(2.39)

where

$$\lambda^* = \frac{e^{-\beta(T)}P(0, T)^2R(0, T)^{-1}}{1 - e^{-\beta(T)}P(0, T)^2R(0, T)^{-1}} \left( \frac{x_0}{P(0, T)} - d \right).$$

The expression of $\lambda^*$ can be further simplified by solving the SDEs of $P(t, T)$ and $R(t, T)$ directly. In fact, taking the expressions of (2.29) and (2.36), applying Ito’s lemma to $P(t, T)^2$, and substituting $w$ with $2v$, we deduce that

$$R(t, T) = R(0, T) \exp \left( \int_0^t (2r + 4\theta^\top v - 2|v|^2)ds + \int_0^t 2v^\top dW_s \right)$$

(2.40)

and

$$P(t, T)^2 = P(0, T)^2 \exp \left( \int_0^t (2r + 2\theta^\top v - |v|^2)ds + \int_0^t 2v^\top dW_s \right).$$

(2.41)

Then take $t = T$. As $P(T, T)^2 = R(T, T) = 1$, comparing these two equations gives

$$\frac{R(0, T)}{P(0, T)^2} \exp \left( \int_0^T (2\theta^\top v - |v|^2)ds \right) = 1$$

which is equivalent to

$$\frac{R(0, T)}{P(0, T)^2} \exp \left( \int_0^T |\theta|^2ds \right) = \exp \left( \int_0^T |\theta - v|^2ds \right).$$

Hereafter, we set $\theta'(t) = \theta(t) - v(t, T)$ and $\gamma(t) = \int_0^t |\theta'(s)|^2ds$ for $t \in [0, T]$. The above relation is rewritten as

$$\frac{R(0, T)}{P(0, T)^2} \exp(\beta(T)) = \exp(\gamma(T)).$$

Hence,

$$\lambda^* = \frac{e^{\gamma(T)}}{1 - e^{\gamma(T)}} \left( \frac{x_0}{P(0, T)} - d \right).$$

(2.42)
Chapter 3

Probability of Target Reaching:
The 80% Rule

3.1 Proof of the Rule

In this section, we follow steps similar to those in Section 3 of Li and Zhou [2] to show that in a market as described in Chapter 2, if an agent follows the investment strategy as specified in Theorem 2.6, then the probability that the wealth reaches the discounted targeted return before or at the terminal time is still greater than 80% percent even though the interest rate is random. All notation, assumptions and results from previous sections are preserved in this section.

First, we define the hitting time of the wealth process on the discounted value of the targeted return \( d \) (as specified in Chapter 2, \( d > x_0/P(0,T) \) is a prerequisite):

\[
\tau = \inf \{ t \in [0, T] : x_t^* = d \cdot P(t, T) \} \tag{3.1}
\]

where \( \inf \emptyset = \infty \).

Lemma 3.1.

\[
\tau = \inf \left\{ t \in [0, T] : \frac{3}{2} \int_0^t |\theta'(s)|^2 ds + \int_0^t \theta'(s)^\top dW_s = \int_0^T |\theta'(s)|^2 ds \right\} \tag{3.2}
\]

Proof This proof is inspired by that of Theorem 3.1 of Li and Zhou [2].

Set \( y_t = x_t^* + (\lambda^* - d)P(t, T) \). Using the wealth equation (2.5), the bond dynamics (2.36) and the optimal portfolio equation (2.39), we deduce that \( y \) follows

\[
\begin{cases}
  dy_t &= \left[ r_t - |\theta(t)|^2 + 2\theta(t)^\top v(t, T) \right] y_t dt - [\theta(t) - 2v(t, T)]^\top y_t dW_t \\
  y_0 &= x_0 + (\lambda^* - d)P(0, T).
\end{cases}
\]
Using (2.42) to eliminate $\lambda^*$ in $y_0$, we obtain

$$y_0 = \frac{x_0 - dP(0, T)}{1 - e^{-\gamma(T)}}.$$

The above equation has a unique solution

$$y_t = y_0 \exp \left( \int_0^t \left[ r_s - \frac{3}{2} |\theta(s)|^2 - 2|v(s, T)|^2 + 4\theta(s)^\top v(s, T) \right] ds 
- \int_0^t \left[ \theta(s) - 2v(s, T) \right]^\top dW_s \right).$$

Therefore, the wealth hits on the discounted targeted return

$$x_t^* = dP(t, T)$$
$$\iff y_t = \lambda^* P(t, T)$$
$$\iff \frac{x_0 - dP(0, T)}{1 - e^{-\gamma(T)}} \exp \left( \int_0^t (r_s - \frac{3}{2} |\theta|^2 - 2|v|^2 + 4\theta^\top v) ds - \int_0^t (\theta - 2v)^\top dW_s \right)$$
$$= \frac{e^{-\gamma(T)}}{1 - e^{-\gamma(T)}} (x_0 - dP(0, T)) \exp \left( \int_0^t (r - \frac{1}{2}|v|^2 + \theta^\top v) ds + \int_0^t v^\top dW_s \right)$$
$$\iff \int_0^t \frac{3}{2} |\theta - v|^2 ds + \int_0^t (\theta - v)^\top dW_s = \gamma(T)$$
$$\iff \frac{3}{2} \int_0^t |\theta'(s)|^2 ds + \int_0^t \theta'(s)^\top dW_s = \int_0^T |\theta'(s)|^2 ds.$$

which completes the proof.

Lemma 3.1 reveals a curious fact that the target hitting time depends on $T$, which is the investment horizon, on $\theta(\cdot)$, which is the market risk premium process reflecting the overall performance of the market, and on $v(\cdot, T)$, which reflects the volatility level of the zero coupon bond maturing at $T$, but does not depend on the targeted return $d$. Based on this lemma, the following theorem gives a more precise description of the stopping time $\tau$.

**Theorem 3.2.** The probability that the optimal wealth process $x^*$ reaches the discounted value of the targeted return $d$ before or at the terminal time $T$ is given by

$$\mathbb{P}(\tau \leq T) = N \left( \frac{1}{2} \sqrt{\gamma(T)} \right) + e^{\gamma(T)} N \left( -\frac{5}{2} \sqrt{\gamma(T)} \right) \tag{3.3}$$

where

$$\gamma(T) = \int_0^T |\theta'(s)|^2 ds = \int_0^T |\theta(s) - v(s, T)|^2 ds.$$
and $N$ is the cumulative distribution function of the standard normal distribution defined as

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du, \quad x \in \mathbb{R}.$$ 

Proof The proof is given by Li and Zhou [2]. Basically, it uses a time-change technique by finding a one-dimensional Brownian motion $\hat{W}_t$ on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ such that

$$\int_{0}^{t} (\theta(s)')^{\top} dW_s = \hat{W}_{\gamma(t)} \quad \text{for} \quad 0 \leq t \leq T$$

where

$$\gamma(t) = \int_{0}^{t} |\theta'(s)|^2 ds = \int_{0}^{t} |\theta(s) - v(s, T)|^2 ds.$$ 

Combined with Lemma 3.1, the problem is then converted to finding the distribution of the running maximum of a Brownian motion with drift, which requires no special techniques. For the detailed proof, see Li and Zhou [2].

□

Once we reach the above theorem, Li and Zhou [2] shows that the right hand side of Equation (3.3) is actually bounded from below by 0.8072 by proving the following theorem:

Theorem 3.3. Define function $f$ as

$$f(x) = N\left(\frac{1}{\sqrt{5}}x\right) + e^{3x^2} N\left(-\frac{5}{2}x\right).$$

Then

$$f(x) \geq N\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{12} \sqrt{\frac{10}{\pi}} e^{-\frac{1}{10}} \approx 0.8072, \quad \forall x \geq 0 \tag{3.4}$$

For details of the proof, refer to Section 3 of Li and Zhou [2].

Theorem 3.3 provides a lower bound for the accumulative distribution function shown in Equation (3.3), which completes the proof of the main result of this thesis.

### 3.2 Numerical Application

In this section, we carry out numerical simulations to demonstrate the 80% rule and apply the rule to a virtual case of pension fund investment.

In the first part of this section, we arbitrarily choose a set of values for $\gamma(T)$ by varying the market parameters, then assign a quasi-uniformly bounded stochastic
model to the interest rate process, and finally simulate the optimal wealth process, the interest rate process, and the bond process with different values of $\gamma(T)$ many times to obtain real target-reaching probabilities so as to compare with theoretical values as shown in Theorem 3.2.

In the second part, we try to answer the question how the mean-variance portfolio selection strategy and the 80% rule may be used in practice. We consider the planning of the investment of a pension fund. We assume that a certain amount of money is allocated in the fund every year and the fund manager strictly follows the mean-variance investment policy over a considerable length of time. Simulations show how this strategy is effective and by how much the wealth thus generated will exceed simple bank deposit.

### 3.2.1 Numerical Demonstration of the 80% Rule

In this part, we demonstrate the 80% rule through numerical simulation in Matlab. First of all, an appropriate model for interest rate is needed. As assumed in Chapter 2, the interest rate is a continuous uniformly bounded process. Since a uniformly bounded interest rate processed rarely has a stochastic differential form sufficiently simple for the corresponding bond price processes to be derived from while an interest rate process of simple form from which the bond price could be relatively easy to be derived is usually far from being uniformly bounded, a compromise between simplicity and uniform boundedness need to be made here in our simulation. For this reason, we consider the Hull-White-Vasicel (HMV) model as the most appropriate for the interest rate, which is of the form

$$dr_t = (b(t) + \beta r_t) dt + \hat{\sigma}^\top dW^Q$$

(3.5)

where $b(t)$ is a deterministic function in $t$ determined by the initial forward rate curve $f(0, t)$ for $t \in [0, T]$, $\beta$ is a negative number indicating the rapidity of the interest rate returning to its mean level, $\hat{\sigma}$ is a $m$-dimensional constant vector indicating the volatilities, and $W^Q$ is a $m$-dimensional standard Brownian motion under the risk-neutral measure $Q$. It is a mean-reverting process, and if $|\hat{\sigma}|^2 \beta^{-1}$ is chosen to be sufficiently small and $t$ sufficiently large (so the process is stabilised), the process will be in more than 99% of all cases bounded and positive, and situated between [mean + 3*standard deviation, mean - 3*standard deviation] which is a result given by the solution of the SDE.
As to \( b(t) \), Section 5.4.5 on p.90 in Filipovic [1] shows that

\[
b(t) = \frac{\partial}{\partial t} (f_0(t) + g(t)) - \beta (f_0(t) + g(t))
\]

where \( f_0(t) = f(0,t) \) is the initial forward rate curve, and \( g(t) \) is defined as

\[
g(t) = \frac{|\hat{\sigma}|^2}{2\beta^2} (e^{\beta t} - 1)^2.
\]

At the end of the page, Filipovic [1] also shows that the corresponding forward rate satisfies the relation with the interest rate

\[
f(t, T) = f_0(T) - e^{\beta(T-t)} f_0(t) - \frac{|\hat{\sigma}|^2}{2\beta^2} (e^{\beta(T-t)} - 1)(e^{\beta(T-t)} - e^{\beta(T+t)}) + e^{\beta(T-t)} r_t.
\]

As to \( f_0(t) \), for simplicity we assume that the initial bond price curve \( P(0,t) \) is an exponential function such that \( P(0,t) = \exp(-\rho_0 t/T) \) where \( \rho_0 \) is a constant. Hence, \( f_0(t) = \rho_0 \) for all \( t \in [0,T] \).

It remains to determine the form of the volatility vector \( v(t, T) \) for the T-bond \( P(t, T) \) to complete the model. In fact, differentiating Equation (3.6) gives the volatility vector of \( f(t, T) \) and then \( v(t, T) \) can be deduced through Lemma 2.4. Note that it can be verified that the drift term that appears in the SDE of \( f(t, T) \) of the above form does satisfy the HJM condition as stated in Theorem 2.5. After performing some calculations, we find

\[
v(t, T) = -\frac{1}{\beta} (e^{\beta(T-t)} - 1) \hat{\sigma}.
\]

In Matlab file we set \( \rho_0 = 3\% \), \( \beta = -2 \) and \( \hat{\sigma} = 0.01 \cdot 1_m \) where \( 1_m \) is a \( m \)-dimensional one-vector.

On the optimal portfolio side, we set maturity \( T = 1 \) as one year, \( d = 1.1 \) as 10\% annual return, and \( m = 10 \) which is the number of sources of randomness in the market. We set \( \sigma(t) \) to be a 10-by-10 constant matrix such that the elements on the diagonal are all 0.2, which are the annual volatilities of the stocks, and the off-diagonal elements are all 0.04, which are the correlations between stocks. Finally, we set the \( B \) vector which is the difference between the appreciation rates and the interest rate to be

\[
(0.01 + 0.001k) \cdot (7, 7, 8, 8, 9, 9, 10, 10, 11, 11)^T
\]

for \( k \) varying from 1 to 20 to obtain 20 different values of \( \gamma(T) \).
As to the discretisation scheme for the SDEs, we choose the Euler-Maruyama Scheme in which the $dt$ term is replaced by a small time step $\Delta t$ and the $dW_t$ term is replaced by a set of small Brownian increments $\sqrt{\Delta t}Z_t$ where $Z$ is a $m$-dimensional discontinuous independent process such that all components of $Z_t$ are independent and identically distributed $\mathcal{N}(0, 1)$ random variables. The discretisation precision is set at 250 as there are approximately 250 trading days in a calendar year.

For the simulations of each scenario (each $\gamma(T)$) we set a counter, which is added by 1 every time the discounted target is reached during one simulation. The value of the counter divided by the total number of simulations will give a rather precise probability of target reaching if sufficiently many simulations have been carried out.

Finally after 10 000 times of simulation for each scenario (i.e for each value of $\gamma(T)$), the following figure is obtained. The theoretical curve represents the cumulative distribution function provided in Theorem 3.2 with $\gamma(T)$ being the variable.

![Probability of Target Reaching](image)

Figure 3.1: *Real probabilities of target reaching compared with the theoretical curve.*

We observe that although the values of the real target-reaching probabilities stick quite close to the theoretical curve, they are in general below the curve by about 1%. This is due to the discretisation errors which are caused by the relatively large length of time steps.
The result of the simulations, from one perspective, corroborates the 80% rule claimed by this thesis.

### 3.2.2 Application to Pension Fund Investment

In this part, we apply the mean-variance optimal portfolio selection policy and the corresponding 80% rule to a real investment case. We are placed in a situation where an agent allocates regularly a portion of his income in a pension fund and manages the fund in such a way that the optimal portfolio strategy is strictly followed in order to maximise the value of the fund at his retirement. The result of the simulations shows how significantly the agent will benefit from this investment strategy compared to the situation where he simply deposits his money in a bank account.

The following assumptions and parameters are adopted in the simulation. The agent invests in the pension fund over a period of 40 years. Since the 80% rule holds in both deterministic interest rate case and stochastic interest rate case, the switch between the two kinds of behaviour of the interest rate would not make big difference, so we choose a constant interest rate $r = 3\%$ for all the period for computing time saving purpose. The inflation rate $\rho = 3\%$ is the same as the interest rate, which implies that the agent’s annual salary increases by 3% year by year. The agent allocates 20% of his annual salary to be injected into the fund every year. Finally, we set the annual target return at 10%.

Every year, the agent manages the portfolio in order to attain the annual target, and once the discounted target is reached, the agent will flat all his positions in the stocks, deposit all the cash in a bank account until the end of the year, and restart managing the portfolio at the beginning of the next year after the annual allocation is added.

We suppose in Year 0, the annual salary of the agent is 1, thus 0.2 is invested in the pension fund. As to the covariance matrix and the B vector of the risky assets, we keep the same ones as in the previous part.

The following figure shows 100 simulations of the pension fund’s wealth process starting at 0.2 following the strategy as described above, and the red curve at the bottom, starting at 0.2 and ending at approximately 27, is the wealth curve if the agent places the capital in a bank account instead of the pension fund.

The trajectories marked in green, which represent a good proportion of all trajectories, are those ending at over 54, which is more than the double of the final bank deposit value. In fact, with the number of simulations increased to 10 000 times, it is
shown that 76% of the realisations end with the pension fund wealth double the bank deposit value and 98% of them end at over the bank deposit level. The mean final wealth is 72, which means if the salary level will be at $1.03 \cdot 10^{10} = 3.26$ in 40 years, then this will represent approximately 24 years of the agent’s salary just before retirement, which is far greater than $27/3.26 \approx 8$ year-salary in the case of bank deposit.

For details in Matlab code for the simulations in Section 3.2.1 and 3.2.2, see Appendix A and B.
Chapter 4

Discussions and Conclusion

In this thesis we have proved that the 80% rule first derived in Li and Zhou [2] is still valid if the interest rate is a uniformly bounded stochastic process satisfying certain conditions, which are that the SDE of the corresponding instantaneous forward rate need to have the volatility term to be a deterministic function in time. In the end, we have carried out simulations to demonstrate and implement the rule. The first discussion point is how to find a real uniformly bounded interest rate process. In the thesis, we assumed that there exists deterministic volatility functions $\sigma_1(\cdot)$ of the forward rate such that the interest rate is uniformly bounded, but in practice how to find such $\sigma_1$ functions remains a question. Note that the translation between the interest rate and the forward rate from one to the other is generally very complex, and Equation (3.6) shows the level of complexity even if the interest rate has a fairly simple stochastic behaviour. On the other hand, the question can be posed in the opposite direction, which is whether the uniform boundedness condition imposed on the interest rate is too strict and whether it may be loosened. As shown in the numerical application section, a mean-reverting but unbounded interest rate process still yielded the desired result.

The second point is how the fluctuations of the interest rate really influence the mean-variance portfolio strategy and the 80% rule. Figure (3.1) showed that for $\gamma(T)$ greater than 0.5, the probability function is an increasing function of $\gamma(T)$, which means that the probability increases with greater difference between risk premium $\theta(t)$ and the bond volatility $v(t, T)$. However, in Li and Zhou [2] the $v(t, T)$ term does not exist. This implies that the volatility of the bond makes it more difficult for the strategy to reach higher target-reaching probability, which seems reasonable.

The last point concerns the discrepancy between theory and the reality. As the rule states, there is always a probability of at least 80% that the target is reached before
or at the terminal time regardless of the level of the targeted return, but in reality, this is almost impossible. Indeed, it is hard to imagine how an investor can realise a 10 000% return within one year in applying the optimal strategy. The problem, in fact, lies with the possible number of times of rebalancing on the portfolio. In theory, trading is continuous and the portfolio is consequently rebalanced on a continuous basis, which thus allows an infinitely many times of rebalancing. On the contrary in reality or in numerical analysis, trading is discrete and the times of rebalancing can only be finite, which prevents high target returns from being reached. For the same reason, a conspicuous shift down of around 1% of the real probability points from the theoretical curve is observed in Figure (3.1) due to the defaults originating from the discretisation scheme.

Many interesting open questions still remain unsolved for the topic. For instance, what will happen if stochastic dynamics of the volatilities of the risky assets are introduced, how the appreciation rates are measured in the real markets, how the mean-variance portfolio model will change if the trading of the Zero Coupon Bonds is allowed, etc. More delicate analysis will be required to answer these questions.
Bibliography


Appendix A

Matlab Code for Section 3.2.1

The following Matlab code is for generating Figure (3.1).

```matlab
1 n = 20; % number of scenarios
2 gamm = ones(n, 1); % vector of gamma(T) values
3 prob = ones(n, 1); % vector of probabilities of reaching the target
4
5 M = 10000; % number of simulations for each scenario
6
7 d = 1.1; % target return
8 N = 250; % number of time steps
9 T = 1; % maturity
10 dt = T/N; % timesteps
11
12 beta = -2; % beta in the interest rate SDE
13
14 P0 = @(x) (exp(-0.03*x/T)); % bond price curve at time 0
15 f0 = 0.03/T; % initial forward rate curve
16
17 sig1 = 0.01*ones(10,1); % interest rate volatility vector
18 v = @(x) -1/beta*(exp(beta*(T-x))-1)*sig1; % bond volatility vector
19 b = @(T) norm(sig1, 2)^2/beta*(exp(2*beta*T)-exp(beta*T)) ... 
20 -beta*(f0+0.5*(norm(sig1, 2)/beta*(exp(beta*T)-1))^2);
21 % the varying mean of the interest rate
22
23 sig = 0.2*(0.8*eye(10)+0.2*ones(10,10)); % equity covariance matrix
24 for sce = 1:n
25 B = (1+0.1*sce)*[.07 .07 .08 .08 .09 .09 .1 .1 .11 .11]';
26 % the B vector
27
28 theta = sig\B; % risk premium vector
29 theta2 = @(x) norm(theta-v(x),2)^2;
30 % the norm squared of the adjusted theta
31
32 gamm(sce) = 0;
```
for k = 1:1000 % integration
    gamm(sce) = gamm(sce)+1/1000*theta2(k*T/1000);
end

counter = 0;

for sim = 1:M
    t = 0; % time
    x = zeros(N+1, 1);
    % portfolio value process, starting from 1 at time 0
    x(1) = 1;
    P = zeros(N+1, 1); % bond prices
    P(1) = P0(T);
    r = zeros(N+1, 1); % interest rates
    r(1) = 0.03;
    lambda = 1/(1-exp(gamm(sce)))*(d-x(1)/P0(T));

    u = -inv((sig*sig'))*((B-2*sig*v(0))*(x(1) ... + (lambda-d)*P(1)) + sig*v(0)*(lambda-d)*P(1));
    % control process

    dW = zeros(10,1); % noise increment under measure P
    target_reached = false;

    for k = 1:N
        dW = sqrt(dt)*randn(10,1);
        P(k+1) = P(k) + (r(k)*P(k) + theta'*v(t)*P(k))*dt ... + P(k)*v(t)'*dW;
        x(k+1) = x(k) + (r(k)*x(k) + B'*u)*dt + u'*sig*dW;
        if x(k+1) >= d*P(k+1);
            target_reached = true;
            break;
        end
        t = t+dt;
        u = -inv((sig*sig'))*((B-2*sig*v(t))*(x(k+1) ... + (lambda-d)*P(k+1)) + sig*v(t)*(lambda-d)*P(k+1));
        r(k+1) = r(k) + (b(t)+beta*r(k))*dt + sig1'*(theta*dt + dW);
    end
    if target_reached == true
        counter = counter+1;
    end
end
prob(sce) = counter/M;

end

plot(gamm, prob, 'r*')
hold on

% The following code is for the drafting of the theoretical curve
ymax = 3;
precision = 0.01;
NN = ymax/precision;
cumulprob = @(x) ncf(0.5*sqrt(x))+exp(3*x)*ncf(-2.5*sqrt(x));

yy = 0:precision:ymax;
zz = zeros(NN+1, 1);

for k = 1:NN+1
    zz(k) = cumulprob(yy(k));
end

plot(yy, zz);
legend('Simulation', 'Theoretical Curve')
title('Probability of Target Reaching')
xlabel('Value of gamma(T)')
ylabel('Probability')
Appendix B

Matlab Code for Section 3.2.2

The following Matlab code is for generating Figure (3.2).

```matlab
Y = 40; % number of years
M = 10000; % number of simulations

d = 1.1;
% target annual return
N = 250;
% number of timesteps in each year (250 trading days)
T = 1; % maturity
dt = T/N; % timesteps

r = 0.03;

sig = 0.2*(0.8*eye(10)+0.2*ones(10,10)); % equity covariance matrix

B = [.08 .08 .09 .09 .10 .10 .11 .11 .12 .12]'; % the B vector

theta = sig\B; % risk premium vector
theta2 = @(x) norm(theta, 2)^2;
% the norm squared of the adjusted theta

gamm = 0; % gamm stands for gamma(T)
for k = 1:1000 % integration
    gamm = gamm+1/1000*theta2(k*T/1000);
end

pens = zeros(M, 1);
count1 = 0; % counter for processes ending at over 54
count2 = 0; % counter for processes ending at over 27
for sim = 1:M
```

27
sal = 1; % annual salary
rr = 0.2; % proportion of salary put into pension fund
addi = sal*rr; % annual addition to the fund

x = zeros(N*Y+1, 1); % wealth process

for y = 1:Y
    t = 0; % time
    x(N*(y-1)+1) = x(N*(y-1)+1)+addi;
    dl = d*x(N*(y-1)+1); % target of year y
    lambda = (dl-x(N*(y-1)+1)*exp(r*T)*exp(-gamm))/(1-exp(-gamm));
    u = (x(N*(y-1)+1)-lambda*exp((t-T)*r))*(-inv((sig*sig')))*B;
    % control process
    dW = zeros(10,1); % noise increment under measure P
    for k = 1:N
        dW = sqrt(dt)*randn(10,1);
        x(N*(y-1)+k+1) = x(N*(y-1)+k)+r*x(N*(y-1)+k)+B'*u*dt+u'*sig*dW;
        t = t+dt;
    end
    if x(N*(y-1)+k+1) >= dl*exp(r*(t-T))
        for kk = k+2:N+1
            x(N*(y-1)+kk) = x(N*(y-1)+kk-1)*exp(r*dt);
        end;
        break;
    else
        u = -inv((sig*sig'))*B*(x(N*(y-1)+k+1) ... -lambda*exp((t-T)*r));
    end
    sal = sal*exp(r*T);
    addi = sal*rr;
end

pens(sim) = x(end);

if pens(sim) >= 27
    count2=count2+1;
end

if pens(sim) >= 54
    plot(0:dt:Y, x, '-g')
    hold on
    count1=count1+1;
else
    plot(0:dt:Y, x, '-b')
    hold on
end
end
mean(pens)

depot=zeros(Y*N+1, 1);
sal = 1;
rr = 0.2;
addi = sal*rr;
for y = 1:Y
    depot(N*(y-1)+1) = depot(N*(y-1)+1)+addi;
    for k = 1:N
        depot(N*(y-1)+k+1) = depot(N*(y-1)+k)*exp(r*dt);
    end
    sal = sal*exp(r*T);
    addi = sal*rr;
end

plot(0:dt:40, depot, 'r-', 'LineWidth',2)
xlabel('Time (in year)')
ylabel('Wealth')
title('Pension fund following optimal strategy vs. Bank deposit')
toc