Fractional Kelly Strategy and Time Consistent Investment: A Statistical Analysis

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1 Introduction

When building a portfolio, there are entire libraries dedicated to the rules you should follow when selecting stocks to buy, estimating their future variations, diversifying your portfolio, etc... All of which miss a primordial factor: If you are disposed to invest a certain amount, is it advisable to invest all of it or should you prefer fractions of it? The Kelly criterion aims at answering this question and, generally speaking, at judging the size of the bets you should make.

In [1], S.Kallblad, J. Oblój and T. Zariphopoulou exposed a framework allowing an investor with a logarithmic utility function to apply the Kelly criterion to their portfolio. In the financial conception of it, the Kelly criterion consists in investing the optimal proportion of the wealth dedicated to the portfolio in the stock(s) the agent has decided to invest in. This framework is also time consistent, it takes into account neither the starting time of the portfolio, the time horizon nor the distance between the two. It is optimal wether these are decided or not when building the portfolio in the first place. Finally this strategy also takes into account model uncertainty and allows the investor to adapt her investment to her confidence in the market and her trust in her own analysis of it. This specificity also provides the investor with a mean to assess performance with respect to expectations as well as profit. These features make the strategy performant at any point in time and, hence, very desirable.

However this framework relies on a specific modelisation of the stock and requires adaptation to be readily applicable. Since the adaptation to an existing stock requires the specification of several characteristics of the portfolio itself, it is necessary to test it in a theoretic context. In order to test the performance of the portfolio in a general case, the model needs to have its parameters specified to be close to what an investor would expect of a stock behaviour on the long run. From these tests it will be possible to directly implement the portfolio as precisely as possible on a stock or selection of stocks.

In the first and second sections I will briefly describe the original theory on which the rest of this thesis is based and calibrate the model in order for it to be as close as possible to realistic values. The third section is dedicated to exploring different portfolios, show how efficient they are or can be and optimise them in order for them to give the best results under constraints. The fourth section is aimed at evaluating the impact of an inaccurate model on the strategy. The final section is a back-testing of the portfolio.
2 The theory behind the portfolio

The whole model exposed here is thoroughly explained in [1]. In this paragraph, I will only give the underlying theory to the strategies studied in the sections thereafter.

2.1 The environment

The market considered has a classic form. The stocks are given in \((S^0, S) = (S^0_t, S^1_t, \ldots, S^d_t)_{t \in [0, \infty)}\), a \((d+1)\)-dimensional semi-martingale on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) where \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)}\) is the usual set of filtration for the geometric Brownian motion. All assets are considered discounted so we set \(S^0 \equiv 1\). And we have the following dynamics for \(S\):

The portfolio process \(\pi = (\pi_t)_{t \in [0, \infty)}\) is an \(\mathbb{F}\)-predictable process said to be an admissible portfolio if \(\pi\) is \(S\)-integrable and there exists a constant \(a > 0\) such that the wealth process given by

\[
X^\pi_t = \int_0^t \pi_u dS_u
\]

is bounded from below by \(-a\), for all \(t \in [0, T]\). The set is denoted \(\mathcal{A}_{bd}\).

For all \(T > 0\) we denote the set of equivalent local martingale measures \(\mathbb{Q}\) equivalent to \(\mathbb{P}|_{\mathcal{F}_T}\) on \([0, T]\) as \(\mathcal{M}_T^e\). The corresponding set of density processes are denoted by \(\mathcal{Z}_T^e\)

\[
\mathcal{Z}_T^e = \left\{ Z = \frac{d\mathbb{Q}}{d\mathbb{P}|_{\mathcal{F}_T}} : \mathbb{Q} \in \mathcal{M}_T^e \right\}
\]

2.2 Robust forward performance criteria

A robust forward performance criteria is composed of a utility random field \(U\) and a penalty function \(\gamma\). This criteria determines how the agent will decide her investment by considering the expected value of the utility of her future wealth where \(U(\omega, x, t)\) is the value the agent attributes to the satisfaction her wealth procures her in function of the events leading to \(t\). The uncertainty of the agent regarding her estimation of the market is included in \(\gamma_{t,T}(\mathbb{Q})\). These two factors combined are used for comparison of different strategies coming from different
analyses of the market. The following definition is the one given in [1] for the random field:

**Definition 2.1** (random field). A random field is a mapping $U : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ which is measurable with respect to the product of the optimal $\sigma$-algebra on $\Omega \times [0, \infty)$ and $\mathcal{B}(\mathbb{R})$. A Utility random field is a random field which satisfies the following conditions:

i) for all $t \in [0, \infty)$, the mapping $x \to U(\omega, x, t)$ is $\mathbb{P}$-a.s. a strictly concave and increasing $C^1(\mathbb{R})$-function which satisfies the Inada conditions

$$\lim_{x \to 0} \frac{\partial}{\partial x} U(\omega, x, t) = \infty$$

$$\lim_{x \to \infty} \frac{\partial}{\partial x} U(\omega, x, t) = 0;$$

ii) For all $x \in \mathbb{R}$, the mapping $t \to U(\omega, x, t)$ is cadlag on $[0, \infty)$;

iii) For each $x \in \mathbb{R}$ and $T \in [0, \infty)$, $U(\cdot, x, T) \in L^1(\mathcal{F}_T)$.

And the penalty function is defined as follows:

**Definition 2.2** (penalty function). For given $t \leq T < \infty$, a mapping $\gamma_{t,T} : \Omega \times \{ Q : Q|_{\mathcal{F}_T} \} \to \mathbb{R}_+ \cup \{ \infty \}$, is called a penalty function if

i) $\gamma_{t,T}$ is $\mathcal{F}_t$-measurable;

ii) $Q \to \gamma_{t,T}(Q)$ is convex a.s.;

iii) for $\kappa \in L^\infty(\mathcal{F}_t)$, $Q \to \mathbb{E}[\kappa \gamma_{t,T}(Q)]$ is weakly lower semi-continuous on $\{ Q : Q \sim \mathbb{P}|_{\mathcal{F}_T} \}$.

Moreover, for a given utility random field $U(x, t)$, we say that $(\gamma_{t,T})$, $0 \leq t \leq T < \infty$, is an admissible family of penalty functions if for all $T > 0$ and $x \in \mathbb{R}$, $\mathbb{E}^0[U(x, T)]$ is well defined in $\mathbb{R} \cup \{ \infty \}$ for all $Q \in \mathcal{Q}_{t,T}$, $t \leq T$, denotes the following set of measures on $\mathcal{F}_T$:

$$\mathcal{Q}_{t,T} := \{ Q : Q \sim \mathbb{P}|_{\mathcal{F}_T} \text{ and } \gamma_{t,T}(Q) < \infty \text{ a.s.} \}.$$

Finally, from the above definitions the robust forward criteria definition is obtained:

**Definition 2.3** (robust forward criterion). Let $U$ be a utility random field and $\gamma$ an admissible family of penalty functions. Then the value field associated with
$U$ and $\gamma$ is a family of mappings $\{u(\cdot; t, T) : 0 \leq t \leq T < \infty\}$, with $u(\cdot; t, T) : L^\infty(F_t; \mathbb{R} \cup \{\infty\})$ given by

$$u(\xi; t, T) := \sup_{\pi \in \mathcal{A}_{bd}} \inf_{Q \in \mathcal{Q}_{t,T}} \left\{ \mathbb{E}^Q \left[ U\left( \xi + \int_t^T \pi_s dS_s, T \right) | \mathcal{F}_t \right] + \gamma_{t,T}(Q) \right\}, \text{ for } \xi \in L^\infty(F_t).$$

We say that the combination of a utility random field and a family of penalty functions is a robust forward criterion (or self generating) if

$$U(\xi, t) = u(\xi; t, T), \text{ a.s.,}$$

for all $0 \leq t \leq T < \infty$ and all $\xi \in L^\infty(F_t)$.

In the other sections of this paper I will use a logarithmic robust forward criterion which is only valid for positive wealths and will require some precautions in order to avoid inconsistencies.

### 2.3 Adapting the theory to the Brownian setup

The geometric Brownian motion style dynamics are widely used as a good approximation of existing stocks and it has been used in the simulation section of this essay. Because of this, we can start specifying the theory for this case, but the results here will remain theoretical. We are still considering a market as described in the first subsection. In a filtration generated by a $\mathbb{P}$-Brownian Motion $W_t$, $t > 0$ there is a riskless asset $S_0^0 \equiv 0$, one risky asset $S$ with the following dynamics:

$$dS_t = \sigma S_t (\lambda_t dt + dW_t) \quad (1)$$

With $\sigma_t$ an $\mathbb{F}$-measurable, $\mathbb{P}$ a.s. positive process and $\lambda_t$, $t \leq 0$ the market price of risk.

These dynamics are not known to the investor which estimates a model $\hat{\mathbb{P}}$ for which the density process is given as:

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \mathcal{E} \left( \int_0^T (\hat{\lambda}_s - \lambda_s) dW_s \right)_T$$

Hence the dynamics the investor supposes to be true are

$$dS_t = \sigma S_t (\hat{\lambda}_t dt + d\hat{W}_t)$$

where $\sigma$ is a constant and $\hat{\lambda}_t$ is a bounded predictable process which is the best estimate the agent has for the original market price of risk. $\hat{W}$ is an $\mathbb{F}$-Brownian motion but $\mathbb{F}$ may be larger than $\hat{\mathbb{F}}$ the filtration of $\hat{W}$. 

2.4 The logarithmic robust criteria

This specific adaptation of the theory is the criteria that will be used and analysed in the later parts of this paper. The log-investor as it is called is an investor that uses a logarithmic utility function to assess performance. The typical form of the utility function is

\[ U(x) = \ln(x) \]

As the function \( x \to \ln(x) \) is strictly increasing the utility will observe the same order as the wealth and a supremum for one is a supremum for the other. The function also satisfies the Inada conditions, i.e. the partial derivative in \( x \) goes to \( \infty \) when \( x \) goes to \( 0 \) and to 0 when \( x \) goes to \( \infty \). It is strictly continuous on \((0, \infty)\), and hence this type of utility function is a random field.

The portfolio considered here is a proportion of the total wealth of the investor that he owns in stocks. Since the assets are all discounted it is the equivalent of the classic risky/riskless repartition portfolio, taking into account the possible borrowing with no interest rate. The wealth process will have the following dynamics under \( \hat{\mathbb{P}} \)

\[ dX_t^\pi = \pi_t X_t^\pi \sigma_t (\hat{\lambda}_t dt + d\hat{W}_t), \quad X_0 = x. \]

The set of admissible strategies is

\[ \mathcal{A} := \{ \pi : (\pi_t) \text{ adapted, } (X_t^\pi \text{ well-defined and } X_t^\pi > 0 \text{ a.s. for all } t > 0 \}. \]

We will also denote \( \mathcal{A}_t^x \) the admissible strategies starting with \( X_0 = x \) and \( \mathcal{A}_t^x \) the strategies on \([t, \infty)\) starting with \( X_t^x = x \).

\( \hat{\mathbb{Q}} \) is a measure equivalent to \( \hat{\mathbb{P}} \) the agent’s estimate of the true dynamics, \( \mathbb{P} \). It is defined by

\[ \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}|_{F_t} := \mathcal{E}(\int_0^t \eta_s d\hat{W}_s)_t \]

where \((\eta_t)_{t \geq 0} \) is an \( \mathbb{F} \)-measurable process such that \( \int_0^T \eta_s^2 ds < \infty \) a.s. for all \( T > 0 \).

The penalty function used in this case will be quadratic and is defined as
\[
\gamma_{t,T}(Q^n) = \begin{cases} 
\mathbb{E}^{Q^n}[\int_t^T \frac{\delta_t}{2} |\eta_u|^2 du | \mathcal{F}_t] & \text{if } \mathbb{E}^{Q^n}[\int_t^T \hat{\lambda}_u^2 du] < \infty \\
+\infty & \text{otherwise}
\end{cases}
\]

where \((\delta_t)\) is a subjective non-negative process that specifies the strength of penalisation. The penalisation coefficient \(\delta\) is set by the investor in function of how confident she is in \(\hat{P}\), her estimate. Simply put, the higher \(\delta\) is the more confident the investor is, in the sense that it increases the penalty on estimates that are completely wrong.

The next proposition is given in [1] and determines the optimal investment choice for this type of investor.

**Proposition 1.** Given the investor's choice of \((\hat{\lambda}_t)\) and \((\delta_t)\) as above, let

\[
\bar{\eta} := -\frac{\hat{\lambda}_t}{1 + \delta_t} \\
\text{and } \bar{\pi}_t := \frac{\delta_t}{1 + \delta} \frac{\hat{\lambda}_t}{\sigma},
\]

and

\[
U(x,t) := \ln(x) - \frac{1}{2} \int_0^t \frac{\delta_s}{1 + \delta_s} \hat{\lambda}_s^2 ds, t \geq 0, X \in \mathbb{R}_+.
\]

Recall that the penalty function is given as specified above and assume that \(\bar{\eta} \in Q_{0,T} \) for \(T > 0\). Then, for all \(0 \leq t \leq T < \infty\),

\[
U(x,t) = \sup_{\pi \in A_t^\mathbb{Q}} \inf_{\eta \in Q_{t,T}} \mathbb{E}^\pi [U(X^\pi_T, T) + \gamma_{t,T}(Q^n)| \mathcal{F}_t],
\]

and the optimum that provides equality is given by the pair \((\bar{\eta}, \bar{\pi})\) specified above.

It is interesting to observe here that the agent is maximising her utility which takes into account her perception of the events until now as well as her wealth. Both parameters directly influence the values of her utility. If she is uncertain of her estimate, a lower performance will still provide a decent utility. On the other hand, a high absolute estimate of the growth will require an increase in wealth to provide the same utility.
2.5 Discussion

Although this is a very mathematical theory in the sense that it does not actually provide any readily usable tools to build a portfolio, it does provide a great insight concerning which parameters should be taken into account when building a portfolio. To be precise, it does not provide any information on which assets one should choose and add to ones portfolio. What it does provide is a framework which, once the asset choice is made, is easy to follow and allows the investor to perfect her investment choice.

However, even with a full understanding of this theory, quite a lot of the strategy is left to the subjectivity of the investor. The perfect example is the $\delta$ that influences the strength of penalisation. Some more insight is given about it in the original publication but in general it could easily be a reasoned gut feeling out of which you build the portfolio.

But even before actually building the portfolio, there is some analysis to do on the stock in which they wish to invest, in order to fit the model to her needs.

3 Calibrating

In order to analyse the performance of this strategy, I studied data from actual stocks and implemented a stock with dynamics that aim at being as close as possible to reality. Since the log-investor (referring to the type of utility function used here) tends to be a long-term investor I studied data from almost a century ago up until now. As considering only one tendency over such a long period would be impossible, it seems more appropriate to consider a stock with a drift-changing process which is what I did here. The steps are described thereafter.

3.1 The model

In this section as well as the two sections after I implemented a stock following the equation (1) using a constant $\sigma$ and, in order to give the versatility required to simulate several decades of a stock, $(\lambda_t)_{t \in [0, \infty)}$ is simulated by a Markov chain.

The different states of the Markov chain are $(a_1, ..., a_n) \in \mathbb{R}^n$, the state changes happen at each jump of a Poisson process $(N_t)_{t \in [0, \infty)}$ with fixed intensity $\nu$ and the transition matrix $\Sigma \in \mathbb{R}^{n \times n}$ that controls the transition from one state of the chain to the other has the form $\Sigma = (p_{i,j})_{1 \leq i, j \leq n}$ where

$$\mathbb{P}(\lambda_t = a_j | \lambda_{t^-} = a_i, N_t = N_{t^-} + 1) = p_{i,j}$$
The filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ in these sections is the natural filtration of $S$ which is generated by the Poisson process $(N_t)_{t \in [0, \infty)}$ and $(W_t)_{t \in [0, \infty)}$ the Brownian motion.

### 3.1.1 The trends

It is one of the basic assumptions of the Kelly criterion that “the odds are consistent with the probabilities of occurrence” [2] i.e. that the observed proportions of events reflect their probability to occur. In that spirit, I used data of the S&P500 and the logarithm of its discounted price to infer my model.

![Figure 1: 90 years of discounted log-returns on the S&P500](image)

This graph allowed me to determine the different possible values of the market price of risk for the model. This graph is provided by Eckhard Platen in [3] and some other data present in [4], still from Platen, was used beforehand to narrow down the range of possible values.

Since this calibrating is an approximation and although precision is important, here a graph brings more directly useful data than the original daily closing values.

As observed on the graph, recurring long term trends are as follows: a slow upwards trend which is dominant from 1940 to 1970 and from 1982 to 1990, an
almost flat trend (see end of the 1970’s) and a very slow downward trend that one can observe twice in the 1940’s. Finally there are bubble-like and crisis-like steep slopes that can be observed in the 1930’s, 1970’s and all the way through the graph everytime the values spike up or down.

The actual values are obviously going to change from one case to the other but the following approximations along with the randomness of the Brownian Motion generate similar curves.

Table 1: Trends used for simulation

<table>
<thead>
<tr>
<th></th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.3</td>
<td>-0.05</td>
<td>0.005</td>
<td>0.09</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Note: For experiment’s sake I have tested different values for these, some mainly downwards, the impact on the portfolio is trivial enough not to be demonstrated but is explained later on. As one can notice these values tend to be mostly going upwards (see also the Markov chain transition matrix). This goes in accordance with the fact that the S&P500 is a very reliable index composed of some of the most trustworthy stocks.

3.1.2 Trend changes

I used \( \nu = 3 \) for the parameter of the Poisson process. This value gives the stock the necessary versatility while maintaining the focus on long term trends. The transition matrix described before was adapted to the analysis of the S&P 500 as follows:

Table 2: Trend transition matrix as described in the model

<table>
<thead>
<tr>
<th>probability from ( \lambda ) to</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
<td>0.05</td>
<td>0.2</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0.075</td>
<td>0.125</td>
<td>0.125</td>
<td>0.6</td>
<td>0.075</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>0.075</td>
<td>0.125</td>
<td>0.125</td>
<td>0.6</td>
<td>0.075</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>0.075</td>
<td>0.125</td>
<td>0.125</td>
<td>0.6</td>
<td>0.075</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0.05</td>
<td>0</td>
<td>0</td>
<td>0.35</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Here the methods require some explanation. For a start the transition densities when starting from \( \mu_2, \mu_3 \) or \( \mu_4 \) are the same. The reason for that is simple:
they are extremely difficult to delimit whether you are using the graph or the actual numerics. The volatility can easily make one get confused for the other. As such they are considered as default states and are the most common states. The values come from the apparent average time spent in those respective states.

One can observe that \( \mu_1 \) and \( \mu_5 \) are minor states as they are quite rare. This is intentional. It keeps actual bubbles and crises to an average between two and four noticeable ones on a 20-year period, which is representative of the past 40 years at least.

When in \( \mu_1 \) or \( \mu_5 \) these states are more obvious and, thus, better documented. It is easy to identify the different possible states after a bubble or a crisis and the proportions give the respective probabilities. Considering then the binomial law composed by the events of staying in this state and changing state and the average time spent in these states at once gives us the full set of probabilities. One can also observe on the graph the high chance of going from one extreme state to the other. This is represented here by a limited communication with other states.

3.2 The agents

The theory explained in section 2 leaves space for interpretation. An investor using it can have her own preferences on how she chooses to implement it and that is what this section aims to explore. I have studied a number of variations of a few different types of investors. Each of which is characterised by her views on the market (what she thinks the trend will be) her confidence in her own estimate represented by the strength of the penalisation and the frequency at which she chooses to update her analysis and portfolio.

3.2.1 The fixed portfolio

This first agent has the most simple approach of the market: she only has one view of the market from the start and never changes it. She can be more or less confident in her estimate but in general the portfolio does not change over time.

\[
\pi_t \equiv \frac{\bar{\lambda}}{\sigma \delta + 1} \tag{3}
\]

The logic behind this behaviour is the following: the agent has a very long term view of the market and her investment. She has chosen how much she wants
to invest, most likely in an index that is representative of the economy, and she will keep to this strategy. Since the index should perform well in the end there is no reason to change the portfolio during ups or downs and it might actually be more of a risk than she is willing to take.

### 3.2.2 The sensible analysis

This agent will try to estimate the trend of the market and adapt her portfolio in accordance to her results. The bottom line is estimating $\mu$ by regression on the market data, adapt the confidence to the variance and choose a strategy. This is what would be expected of a realistic agent. She can have various degrees of knowledge or intelligence, or be more or less willing to adapt her portfolio often (see comparison section).

$$
\pi_t = \frac{\hat{\lambda}_t}{\sigma} \frac{\delta}{\delta + 1}
$$

(4)

Two main versions of this agent are studied here: one has a set of trend values and estimates which is the closest to reality. The second has no preconceived idea on the possible trend values and only adapts the portfolio to his calculation of the recent drift.

### 3.2.3 The omniscient agent

This is the perfect agent. She has full knowledge of the actual model except for the values the Brownian motion will take. In particular she knows when the drift changes and to what value. She immediately adapts her portfolio and is entirely sure of her information. It is the Merton strategy

$$
\pi_t = \frac{\lambda_t}{\sigma}
$$

(5)

Here since the agent is certain of the values $\delta$ is infinite and does not appear in the portfolio.

Although this agent is completely unrealistic (or could represent a form of insider trading) she represents a very good demonstration of the impact information can have on the performance of the portfolio. This agent cannot be outperformed by any one using the other strategies, at least not on the long run.
3.2.4 The random portfolio

This agent is more of a sanity check than a proper investment strategy. Every now and then, she rolls a die and decides on a trend she is going to consider as the actual trend. Then she builds her portfolio accordingly but with a limited confidence in her estimate.

\[ \pi_t = \frac{\lambda_t}{\sigma} \frac{\delta}{\delta + 1} \]  

(6)

Where \( \lambda_t \) is randomly picked from \((a_1, ..., a_5)\) the states of the markov chain following a discrete uniform distribution. The value for \( \lambda_t \) changes at consistent intervals.

This agent can prove that not just any strategy can perform well by investing in an index with good returns. It will also prove that the Kelly strategy can limit losses even with the most unlikely strategy there is, only by building a random portfolio and admitting that you do not trust your guess.

3.3 The delta

This part of the calibration process is intentionally relayed to the end of this section. Indeed, the \( \delta \) is the part of the model that is actually subjective and the best course of action would probably be to try out various approaches of it and decide the one you prefer or, if the impact is visible, the one that brings the most obvious or the most consistent profit difference.

The \( \delta \) is meant to reflect the confidence and for most agents described before, since they are either extremely confident or base their portfolio on a general impression (not to mention randomness), it does not take a calculation since they do not have access to the data for it. On the other hand the sensible agents will try to have a sensible \( \delta \) too.

Ideally the agent should build a confidence interval. With enough data it is fairly easy to use the central limit theorem, and use the variance or \( R^2 \) in the case of a regression to get a 95% confidence interval, for example. Then \( \delta \) should be proportional to the value of the penalty function on the edges of the interval (cf. definition of the penalty function in the previous section).

The subjectivity and derivation method are not the only issues when building a \( \delta \). For a start, when trying to build a strategy out of existing stocks, the investor
should probably take into account that her analysis of the market remains an educated guess. For example, the investor supposes that the asset follows a log-Brownian distribution with changing drift. Not only might the assumption of one driving Brownian motion be off, since there may be several at play at various times and only one at others. But also the source of randomness might well have a different law – a T-student law for example – which is not necessarily integrable and would present some mathematical issues.

But if this is taken into account when choosing the strength of penalisation, then, even during periods where the inaccuracy becomes visible and may interfere with the profit, decreasing $\delta$ would limit the loss risk.

4 Performance comparison and optimisation

Since so many approaches of the Kelly strategy are possible, some testing is required in order to assess the impact of each factor and which rules an investor should apply over others. For a start I have studied the impact of the choice of $\lambda_t$ and $\delta_t$ in different cases so that an intuition could be built from it. Once the investor has an intuition on how to use every parameter, she should start defining more complex strategies and try to optimise them. This is the aim of the second part of this section.

4.1 Getting a feel for the strategy

Since no numerics have actually been done so far, starting with very simple comparisons seemed important. The example named above as the fixed portfolio gives us this opportunity.

This first figure illustrates the behaviour of a fixed portfolio. As one can see from the stochastic differential equation of the portfolio the variations of the wealth will be proportional to those of the asset. These first two figures also show that when kept to fixed proportions there is no difference between changing the estimation of the market and changing your confidence in your estimate. However, and this is going to be my next point, changing how much your portfolio is sensitive to ups and downs should be the role of the $\delta$, the $\hat{\lambda}$ should always try to get as close as possible to market value.

The third figure here illustrates the case of the random portfolio and aims at showing the importance of the choice of the strength of penalisation. In this example, every hundredth business day, the investor randomly decides which trend he thinks is starting and he invests accordingly. A very poor strategy but that
Figure 2: The fixed portfolio with different estimates of the trend

First example of the performance of a portfolio as in (3) and the impact of changing the estimated trend. There is no interest rate here so $\mu = \lambda$ illustrates perfectly the impact of $\delta$.

As this figure shows, a random investment can work for a time, a long time even. The investor with a high confidence coefficient performs decently on the first years. She even outperforms the index for a time. Then the $\delta$ shows it’s importance. The agent barely knows anything about the market here, but one of them is conscient of it and limits her investment while the other agent is purely gambling. As would be expected, the cautious agent ends up doing much better than the reckless one. This also implies that even one of the worst possible strategies can cut its losses by lowering the strength of the penalisation when in doubt. This fractional Kelly strategy and its reduction of the losses is going to be illustrated later on in a more realistic investment strategy.

Finally, in order to show the importance of the value of $(\hat{\lambda}_t)$, the final graph illustrates the performance of the agent that knows every deterministic part of the process that generates the stock, i.e. the only parameter unknown to her is
the Brownian motion and the value it is going to take. She is the perfect Kelly investor. As she has no doubt her \( \delta_t \) is infinitie and the impact of this parameter has been completely cancelled on the next graph.

It is worth noting here that the graph chosen is actually a poor performance, on an average performance the portfolio would reach value of the order \( 10^4 \) to \( 10^5 \) but the value of the stock would not be visible anymore. This is the most one could expect of the Kelly strategy and it shows that getting the correct estimate of the trend at the right time is crucial for performance. This is what the optimisation section will be about.

Note: I mentionned before the impact of downward trends on performance. As it is easily deduced from the graphs here, a portfolio with a lower \( \delta \), or, in the fixed portfolio case, a negative \( \lambda \) will be much less affected or will even outperform the stock. See the poor performance on graph 2 for reference.
Figure 4: The random portfolio, $\delta$ comparison

This is the portfolio described in (6) illustrating the importance of adapting the $\delta$ to reduce the losses in case of low confidence in the estimate.

4.2 Strategy Optimisation

Now that we know how the parameters that we need to specify influence the performance of the portfolio, implementing a way of estimating the trend and specifying a delta, I will treat both these issues in that order.

The first method for approximating the model studied here is a sensible agent, as described before, but with an idea of the values the trend can take. With these values and the data of the stock, this agent runs a log-regression on the latter and chooses the value that is closest to the one she found. Two issues arise: How often should she adjust her portfolio, and how much should the investor run the regression on.

Concerning the rate at which the agent should refresh her analysis of the market, since she does not know the trend change dates but knows the average of
the duration between changes is four months, she should not keep her portfolio unchanged longer than this.

After simulation with a variety of durations between portfolio change, an optimum appears around three months for the optimal duration. Since the change of portfolio creates a ridiculously high variance after 20 years and given all the parameters at work it is not possible to give a precise number but it is definitely very close to a quarter and shorter than four months.

Note: The graphs in this section are averages on 10000 simulation.
Figure 6: Performance with different refreshing rates

Optimisation of the frequency at which the portfolio is adjusted to a new regression in the case of the sensible investment (4) with a set of pre-established values for $\lambda$. The performance curve reaches a high point around three months. Average on 1000 simulations

Carlo estimation the second figure shows the impact of the length of regression on the performance of the portfolio. Here the ideal value clearly appears to be four months.

Since it would be suboptimal to use random values for the trends the investor supposes to be correct, here the agent with these performances knows the exact values the trend can take. Unfortunately you cannot assume this from an everyday investor. This investor would need a different way of estimating the model.

The next agent still uses a log-regression to estimate the model, except that this time he does not have any specific idea of the actual values of the trend and uses the regression coefficient for his estimation of the model. The only safety, since on such a large sample some periods will give completely unreasonable val-
Figure 7: Performance with different regressions

Optimisation of the length of the regression window of the sensible investor (4) with a set of pre-established values for \( \lambda \).

The best value is for four months, a longer window would be suboptimal most of the time. Average on 10000 simulations

ues, this investor will decide that if the value is not possible, he will suppose that something is off and become very wary of the market. In this case his \( \delta \) will become very small, as well as his investment.

The figures of simulation of this agent show that the optimal values for the length of regression and the pace at which the investor renews her portfolio are the same as before.

As one can see here the objective performance can vary a lot from one simulation to the other, even with a large sample. The only way to assess these strategies is to directly confront them on the same simulation. However, one can already suppose a lot from these four graphs. To begin with, it is obvious that as soon as the strategy is sensible and results from an actual analysis of the market, the real differences only appear after a few years, which confirms the focus of this
strategy and of the log investor in general. Secondly it is clear that the average performance of these different portfolios is reasonably stable which would suggest that this performance is reasonably independent of the performance of the stock which is the best case we could hope for on a long term basis. Finally these two last graphs compared to the previous two graphs would suggest that the difference in performance between the investor that actually knows the possible trends and the one that only takes his best guess are fairly similar.

For the next graph I have taken the optimal values of both the regression length and the portfolio adjustment pace in order to compare both these strategies at their respective optimum performance. It is clear from this graph that both strategies perform quite similarly.

The last parameter to optimise is the delta. So far and in order to best assess the quality of the trend estimation, every simulation was with a fixed $\delta$ common to every portfolio. Now in order to use a sensible delta we first need a formula for it. The parameters we need to take into account are the type of penalty function (quadratic here), the variance of the trend estimate (which $\delta$ should be inversely proportional to), and the confidence interval for our estimate of the trend. Here I used

$$\delta_t := \frac{n}{1.96^2} \times \frac{1}{\hat{\sigma}^2} \times K$$
Figure 10: Comparison of both regression portfolios

Comparisons of two sensible investments (4). One with knowledge of the possible values for $\lambda_t$ and the other one without.

Average on 10000 simulation

Where $\hat{\sigma}^2$ is the variance of the estimate, $n$ is the number of days the regression coefficient is computed from and $K$ is a proportion factor that the investor should adjust. Here the investor should be cautious and $n$ should only increase the confidence if it is increased by a more precise sampling and not if it is a regression over a longer period.

On this figure the modification of delta is inconclusive, partly because the regression here actually gives a good estimate of the true variance and is very stable. Adding the smoothing out of the average makes the two curves almost identical, except for a different growth factor. However, here the $\sigma_t$ is fixed, I will, later on expose a case of changing $\sigma$ and adding a changing delta there should make a difference.

The last issue of this is the variance. The $\sigma$ of the stock is used in the $\pi_t$ of the portfolio, but on an actual stock the investor does not know the original $\sigma$. She can however estimate it and that is the point of this last graph.
Comparison of the performance of two sensible agents as in (4) one with a \( \delta \) adapted to the estimated \( \sigma \) at each portfolio adjustment and the other with a fixed \( \delta \). The fixed \( \delta \) is greater on average than the adapted one, probably due to a slight recurrent estimation error on \( \sigma \). Average on 10000 simulations.

From this figure we can conclude that, luckily, the difference in performance between the agent who knows the volatility of the model and uses it to adjust his portfolio does not outperform by much the one that estimates the volatility. This means that when applying to real world data, using the variance of the daily log return can give a very precise estimate of the actual variance in the model and will not impact negatively on performance.

4.3 Analysis of the utility function

As the general aim of any portfolio is to do well on the market, the analysis before was performed on a discounted stock and compared how well these portfolios performed in monetary units, virtually: dollars. It is however important to recall that the investor’s portfolio choice is based on the utility function (2).

The graph 13 shows the difference in utility between three of the agents exposed before. As it would be expected, on average the Merton investor does very well and is quite satisfied. She does much better than the sensible agent whose
adjustments are always delayed, since the regression is her only mean to estimate the market price of risk.

Here the third curve is the one that justifies the precedent analysis. The third agent, with a random portfolio, has a higher utility than the sensible agent. This comes from the fact that she has a very low $\delta$ compared to the one of the sensible investor and since she has low expectations, she is quite satisfied regardless of how well she is doing. Recall that the random agent often performs very poorly and should not be considered a better strategy than the sensible agent. However, these are two very different strategies and this graph shows that the utility function should only be used to decide between two similar portfolios.

Now that the matter of getting an optimal analysis of the market and being able to use it in the construction of our portfolio has been dealt with, two new questions arise:
Figure 13: Comparison between Utility functions

In this graph the value of the utility function of the Merton investor (5), the random investor (6) and the sensible agent (4) are compared. Note that the Y-axis values are only informal and having a negative utility does not change anything since the only point of the utility function is to reflect preferences. These values would be positive with higher starting wealth.

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- would this work as well on real data as it does on simulated data?
- how much would a confusion on the actual model impact the performance?

Or in other words, would this strategy perform decently if the distribution is not exactly log-brownian?

5 Sensibility to model misspecification

Even though the geometric Brownian motion is one of the most widely used dynamics to simulate the price of a stock, it is quite often criticised as being inaccurate. Among the most common suggestions for better approximation are the
possibility of a T-student law instead of a log-normal law, or the fact that the volatility is not necessarily constant. This section is a risk management point and is informal.

The different variations that I implemented behave as follows:

**The T-Student law**
It has the same dynamics as in 1 except that during the simulation the normal distribution is replaced by a T-student distribution with 30 degrees of freedom. This possibility was described by Professor S. Cohen in his lecture about risk measures [5] as being extremely difficult to statistically differentiate from a classic Brownian motion.

With 30 degrees of freedom the T-student law is easily confused with a normal law, the main difference being that its distribution has thick tails and allows the variable to take high values more often.

**The Ornstein-Uhlenbeck volatility**
Here the stock has the same dynamics as before, except that the $\sigma_t$ has its own SDE, which is said to be mean-reverting

$$d\sigma_t = \theta(\bar{\sigma} - \sigma_t)dt + \nu dW_t^2$$

where $(W_t^2)$ is a new brownian motion independent of the one driving the asset dynamics.

In these simulations I used the following values : $\bar{\sigma} = 1/3$, $\theta = 0.2$ and $\nu = 0.03$. With these values the variance almost always remains close to the mean of $1/3$.

Note that with this type of asset, estimating the $\sigma_t$ of using an adapted delta should make much more sense since the volatility is not fixed anymore, especially here where the mean reverting coefficient is quite weak, the time it will take to come back to the mean value will be relatively long and can have a long-term impact.

The first two graphs of this section represent one realisation of each of these processes, the fourth is an average over 10000 realisations of each of these processes.

From the first three graph, one can see that the general behaviour is very similar and it would be hard to tell either from a graph or actual data the type of dynamics behind these processes. Even the average on the fourth graph confirms that there is very little difference between these dynamics in realisation but this unsuspected difference might affect the portfolio a lot.
5.1 Performance comparison

On a punctual simulation, performance could go either way. Randomness, increased by the changing portfolio proportion can push performance up or down depending on the circumstances and one simulation is not representative. The graphs in this section come from long Monte Carlo simulations similar to the strategy optimisation section.

In order to have comparable values, for the Student process an inversion of cumulative distribution function method was used here. After generation of one random variable uniformly distributed on \([0, 1]\) this method gives the value depending on whether the process uses a normal distribution or a T-student distribution (still with 30 degrees of freedom); with the latter, values will tend to be more extreme. For the Ornstein-Uhlenbeck volatility, the same Brownian Motion was used, only with a different volatility.

As we can see here, the performance difference can be problematic. Without being completely out of bounds the final value of the portfolio with a T-student distribution is around 20 percent lower than its log-brownian equivalent which means it might perform worse than the asset itself. However, on shorter periods and during calmer times the performance stays very decent. It is only the fact that values with low probability seem to have been obtained repeatedly around the end of the observation that have heavily impacted the portfolio. Although
this was to be expected, it means that high volatility times require a lower $\delta$ to prevent this to happen.

In order to stress the importance of frequently re-evaluating the volatility, the agent investing in the stock with mean reverting volatility only knows the average value of $\sigma$. However, here it might be because the volatility changes are very subtle (which still tends to have important impacts) but the performance does not seem to be affected. This is a very reassuring result as it means that even a wrong estimate of the variance, as long as it is close enough, will not be a problem.

5.2 Improving performance in these situations

In the previous section the investor acted as if the model had not changed and saw her performance plummet (except for the Ornstein-Uhlenbeck case). It might be useful to reintroduce the adapted $\delta$ and the volatility estimation and see if this time, when the model does not behave as smoothly as it should, it reduces the impact.
Figure 17: Performance difference with Student

Performance difference between a portfolio on a stock driven by the classic stochastic differential equation and the stock with a T-student law in the implementation. Obtained by inversion of the cumulative distribution function and with the same drift in order to maintain the general behaviour. Average on 1000 simulations

On the student stock the difference is barely noticeable. The performance is almost the same, the main difference can be found around the spikes in the curve. With the adjusted parameters the spikes in the curve are much smoother. This is probably an improvement. As a long term investor one of the main aims is to avoid important losses rather than seek high punctual profits with high risk. As a long term investor it is sensible to look for consistent growth as a consequent loss will hinder your future profits.

On the stock with changing volatility the conclusion is the opposite. The implementation of return-adjusted parameters in the portfolio does not smooth the curve anymore, worse it actually make it more unpredictable. However, even with its drastic ups and downs, the portfolio manages to outperform the version with fixed parameters which suggests a consequent improvement on the strategy and, since this is an average, a consistent outperforming of the original strategy. The adjusted parameters here are the way to go. And since adjusted parameters are all an actual investor will have access to, this is very encouraging concerning how well an investor can perform when trying to implement this strategy.

To conclude this section it is clear that even though a misspecification in the model will alter the portfolio performance, using consistently adjusted parameters...
can prevent this from having dramatic consequences on the performance and will even, sometimes, improve on the original performance.

The last test this strategy has to pass is resisting the test on data from a real stock.
Figure 19: Implementation of adapted parameters on a student driven stock
Comparison of performance on a stock with T-Student in the implementation between an investor with access to the $\sigma$ of the model and one that estimates it when adapting the portfolio, both in $\sigma$ and $\delta$. Average on 1000 simulations
Figure 20: Implementation of adapted parameters on a stock with Ornstein-Uhlenbeck volatility

Comparison of performance on a stock with Ornstein-Uhlenbeck volatility between an investor with access to the $\sigma$ of the model and one that estimates it when adapting the portfolio, both in $\sigma$ and $\delta$. Average on 1000 simulations.
6 Test on actual data

Since the final aim of any strategy is to be applied on a stock exchange rather than simulations it is only fitting to evaluate the performance on a meaningful index. One of the most representative indices is the S&P500 and it is at the same time the stock the previous simulation were trying to recreate, and it is the index on which I tested the portfolio.

The first graph shows the performance of the portfolio while keeping the same length for the regression and the same refreshing rate for the portfolio. Since the optimal values for these depended on the average rate at which the trend changed, it might not be directly adaptable to a real index.

Figure 21: The portfolio used on the S&P 500

Adaptation to the values of the S&P500 (discounted by the annual treasury bond, data from the Federal Reserve of the United States) since 1955 of the portfolio (4) optimised as in the previous section with regression window of four months and adjustment of the portfolio every three months.
It becomes obvious from the results that these values are probably not optimal and should be adapted again. Even though with suboptimal values the result is still there and the index is outperformed with very little effort per year, the index is not always outperformed by much and adapting the portfolio differently might make more of a difference. It is worth noting here that even though the performance is not always outstanding, the portfolio tends to be more stable than the index which is a good sign.

On the next graphs I compared different values for the number of days on which the regression is calculated, and the number of days between two portfolio updates.

From these graphs the optimal values are clearly every two months for the adjustment pace of the portfolio and four months for the period on which the regression is applied. This may be only specific to these circumstances but since it is the best portfolio on 60 years of data it is likely to be the best version on longer or shorter durations.

Compared to the version we had previously, this portfolio reaches around $120 (discounted) where the previous did not exceed $100 which means that this is efficient optimisation. However it would be interesting to know what is the maximum this portfolio has to offer on this data. To this end the last graph represents the performance of an investor that has a precise idea of what the two months to come have to offer (named foresight investor) compared to the optimised portfolio, as
described before. The foresight investor has access to a regression on data that is not currently available and which is slightly different from the omniscient investor in the simulation section before. Here it will include data from the random part of the model, but this is the closest it gets to the previous example.

![Figure 24: Optimised compared to foresight portfolio](image)

Comparison of the portfolio with the best results on the window 1955-2014 with the portfolio of an investor able to have a precise intuition on the next two months.

The difference in performance is striking. However, this investor is only a very theoretic idea of the best a log investor could hope for and has only example value. As with insider trading for example, the difference between the optimal investor and the foresight investor is the capacity to take advantage of sudden and drastic market changes.

There would probably be something to say concerning the investor’s reaction to crises and sudden market bubbles, in other words it would be interesting to make him react to trend reversals faster than on the next 3-months deadline. However, this investor shows already a more-than-decent increase on the return
of the S&P 500 (more than 20% when comparing figures 21 and 23) and that with only four trades a year, one stock choice for the duration of the investment and no reason to worry about the market performance between trades.

6.1 Utility function on real data

Since the utility function \(2\) is what determines the felt difference between two strategies, I compared the utility provided by both the optimised portfolio and the portfolio with intuition.

Figure 25: Utility comparison between the optimised portfolio and the foresight portfolio

Comparison of the portfolio with the best results on the window 1955-2014 with the portfolio of an investor able to have a precise intuition on the next two months. This time they are compared based on their utility function.

As the graph shows the difference in utility is minimal and this shows that since the optimised investor has less information she is not necessarily dissatisfied
with the results of her portfolio, even though an agent with more information does better.

7 Conclusion

The Kelly strategy is one of the numerous utility-based portfolio building strategies. One of its most appreciated features, and probably the one that makes it popular among investors, is its optimality regardless of the investment duration. It is the best strategy even if the investor has not yet decided when she will sell her portfolio.

In this paper I have used statistical tools to approximate as closely as possible the typical behaviour of the S&P 500 and used it to test and optimise the strategy as much as possible. I have also shown that the fact that the model is not perfectly accurate is either easily corrected or of little impact on average. Finally I have used this analysis to adapt the strategy to the S&P 500 and show that even when poorly executed it improves the result of the index.

The analysis of which I gave an example is simple to perform on any stock, index or even portfolio. Given that the Kelly strategy allows the investor to outperform an index by balancing the ratio of money in the stock compared to the money invested or borrowed, this is applicable to any portfolio and can allow the investor utilising this strategy to greatly improve their results.

Ultimately the aim of this thesis is not so much to try and give a portfolio outperforming the S&P 500 but rather to give a simple, ready-to-use, analysis and decision framework allowing an investor to perfect the composition of her portfolio.
Appendices

The code presented here is only a small fraction of what was used, the rest is available on demand.

Main

%This is the main code for one simulation.
%For the average the same type of loop was used in a %monte-carlo simulation.
%This is an example with Agent3 the random portfolio.
S0=100;
mu = [−0.3,−0.05,0.005,0.09,0.3];
markovMat=[0.75,0,0.05,0.2; 0.075,0.125,0.125,0.6,0.075; 0.075,0.125,0.125,0.6,0.075; 0.075,0.125,0.125,0.6,0.075; 0.05,0,0.35,0.6];
mean = 0.09;
sigma = 1/4;

L = 3; %We’re considering an average of 3 trend changes per year
T = 20; % Studying on a year
SpY=120;
N = SpY*T; %number of time steps : 250 trading days per year
h = 1/SpY;
jumpmax = 20*T; % considering that there will be less than 20 jumps % per year seems enough
poissonTimesVec =poissonTimesGene(L, jumpmax);
X0=100;

jumpcount = 1;
St = S0;
Svec = zeros(N+1,1);
Svec(1) = S0;

% Slogvec=zeros(N+1,1);
Slogvec(1)=log(S0);
X13=zeros(N+1,1);  
X13(1) = X0;  
X14=zeros(N+1,1);  
X14(1) = X0;  

for k = 2:N+1
    brown = randn*sqrt(h);  

    if (k*h >= poissonTimesVec(jumpcount))
        mean = meanChange(rand,mu,mean,markovMat);  
        jumptime = k*h  
        jumpcount = jumpcount+1;  
    end
    St = St*(1 + mean*h + sigma*brown);  
    Svec(k) = St;  

    Agent41=Agent3(Agent42(2),0.5,k,h, brown, X14(k-1), mu,mean, sigma,0)  
    X14(k)=Agent41(1);  
    Agent42=Agent3(Agent42(2),2,k,h, brown, X13(k-1), mu,mean, sigma,0);  
    X13(k)=Agent42(1);  
end

Timevec = 0:h:T;  
plot(Timevec,Svec,'linewidth', 0.1)
hold on;
plot(Timevec,X13,'r','linewidth', 0.1);
hold on;
plot(Timevec,X14,'g','linewidth', 0.1);
xlabel('t');
legend('St','delta = 2','delta = 0.5');
hold off;

Regression

function recentMV = recentMV(step, reglength, S, SpY) %input:
% the element up to which the regression is calculated,
% the number of ellements the regression is calculated on,
% the vector of logvalues of the stock and the number of
% steps per year to give annualised values
% gives a mean of the recent annualized return and the variance for it
if step>reglength
    X=zeros(reglength,1);
    for i = 1:reglength
        X(i) = S(step-reglength+i)-S(step-reglength+i-1);
    end
else
    X=zeros(step,1);
    X(1) = S(1);
    for i=2:step
        X(i)=S(i)-S(i-1);
    end
end
% gives mu
    recentMV(1)=mean(X)*SpY+var(X)*SpY/2;
% gives sigma
    recentMV(2)=sqrt(var(X)*SpY);
end

Code for the sensible agent

function nextX = Agent4(step, h, brown, Slog, Xt,
sigma, mean, delta, sigmahat, realMean, SpY)

    if mod(step,60)==0 && step>240
        MV = recentMV(step-1,80,Slog,SpY);
mean = MV(1);
delta = 2; % this version had a fixed delta
sigmahat = MV(2); % but only used data from the stock
end

% security for unreasonable regression values
if abs(mean) > 0.7
    mean = 0.1;
    delta = 0.1;
end

nextX(1) = PortfolioStep(mean/sigmahat, realMean, delta, h,
                          brown, sigma, Xt);
nextX(2) = mean;
nextX(3) = delta;
nextX(4) = sigmahat;
end

Code for the change of trend

function mean = meanChange(x, mu, mean, mat)
% here we will change from one value of mu to the other. the values will be:
% Mu1 = 0.1, Mu2 = 0.2, Mu3 = 0.0, Mu4 = 0.4, Mu5 = -0.2
% x is generated as a uniform variable

if mean == mu(2)
    if x < mat(2,1)
        mean = mu(1);
    elseif x < mat(2,1) + mat(2,2)
        mean = mu(2);
    elseif x < mat(2,1) + mat(2,2) + mat(2,3)
        mean = mu(3);
    elseif x < mat(2,1) + mat(2,2) + mat(2,3) + mat(2,4)
        mean = mu(4);
    else
        mean = mu(5);
    end
endif mean == mu(1)

    if x < mat(1,1)
mean = \mu(1);
elseif x<\text{mat}(1,1)+\text{mat}(1,2)
    mean = \mu(2);
elseif x<\text{mat}(1,1)+\text{mat}(1,2)+\text{mat}(1,3)
    mean = \mu(3);
elseif x<\text{mat}(1,1)+\text{mat}(1,2)+\text{mat}(1,3)+\text{mat}(1,4)
    mean = \mu(4);
else
    mean = \mu(5);
end

elseif mean == \mu(3)
    if x<\text{mat}(3,1)
        mean = \mu(1);
    elseif x<\text{mat}(3,1)+\text{mat}(3,2)
        mean = \mu(2);
    elseif x<\text{mat}(3,1)+\text{mat}(3,2)+\text{mat}(3,3)
        mean = \mu(3);
    elseif x<\text{mat}(3,1)+\text{mat}(3,2)+\text{mat}(3,3)+\text{mat}(3,4)
        mean = \mu(4);
    else
        mean = \mu(5);
    end

elseif mean == \mu(5)
    if x<\text{mat}(5,1)
        mean = \mu(1);
    elseif x<\text{mat}(5,1)+\text{mat}(5,2)
        mean = \mu(2);
    elseif x<\text{mat}(5,1)+\text{mat}(5,2)+\text{mat}(5,3)
        mean = \mu(3);
    elseif x<\text{mat}(5,1)+\text{mat}(5,2)+\text{mat}(5,3)+\text{mat}(5,4)
        mean = \mu(4);
    else
        mean = \mu(5);
    end

else
    if x<\text{mat}(4,1)
        mean = \mu(1);
    elseif x<\text{mat}(4,1)+\text{mat}(4,2)
        mean = \mu(2);
    elseif x<\text{mat}(4,1)+\text{mat}(4,2)+\text{mat}(4,3)
        mean = \mu(3);
    elseif x<\text{mat}(4,1)+\text{mat}(4,2)+\text{mat}(4,3)+\text{mat}(4,4)
        mean = \mu(4);
    else
        mean = \mu(5);
    end

end
mean = mu(2);
else if x < mat(4,1) + mat(4,2) + mat(4,3)
mean = mu(3);
else if x < mat(4,1) + mat(4,2) + mat(4,3) + mat(4,4)
mean = mu(4);
else
mean = mu(5);
end
end

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