A Large Market Model with Stochastic Volatility and Application to Pricing Credit Derivatives

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This thesis is dedicated to
my parents and my grandparents
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Abstract

We investigate a Large Market Model with Stochastic Volatility given by Heston dynamics. We introduce two correlated market drivers: one corresponding to the assets and one corresponding to the volatilities. By passing to the limit with the number of assets, we derive a Stochastic Partial Differential Equation (SPDE) governing the behaviour of the two dimensional limit density. We define the loss process as a functional of this density. We discretize the SPDE in order to price selected complex credit derivatives: Single Tranche Collateralized Debt Obligations (CDOs) and Forward Start CDOs. We study the impact of varying model parameters on the loss process and the pricing.
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Chapter 1

Introduction

Until the crisis of 2007 and 2008, we could observe a significant development in the market of credit derivatives. This is demonstrated in the chart below:

![Chart showing growth of the CDS market from 2001 to 2008. Source: ISDA.](chart.png)

Source: ISDA.

Figure 1.1: The growth of the CDS market

We observed high market demand for mathematical models, which would provide pricing formulas and techniques of hedging complicated instruments. Although there has been much blame for mathematicians in the press after the crisis, the mathematicians’ role in providing high quality risk management tools for credit products is still very important. To contribute to the existing literature, we consider extending the Large Market Model of credit instruments to incorporate stochastic volatility. Our purpose is to provide a realistic extension to the existing structural credit models of a large portfolio of credit derivatives.
The main reason for the gradual development of structural models has been the fact that there were no dynamics used in the most commonly used models. The risk management of credit derivatives has been problematic, if not impossible. Innovative models that would capture the risks involved in managing complex products are still in high demand. Before the increase in popularity of structural models, reduced-form frameworks had been used. Approaches based on choosing appropriate copulas provided simple derivative pricing tools that were easy to implement and to calibrate. And most importantly, their computational cost was very low. However, there were no dynamics used within these kinds of models. Pricing derivatives dependent in a complex way on time has been impossible, as models based on copulas provided only expectations of defaults for one time period. Hence, they would rather be applied to pricing products with maturity of a fixed time period.

The structural approach is especially interesting as it provides us with an intuition about the economic meaning of our framework. It also provides a simple link between credit and stock markets. Investors are able to hedge the spread risk by the reference component and the asset, thanks to the determined dynamics of the spread. We also determine the defaults within the model. It is important to outline the disadvantages of the structural framework which influence their applicability to real world instruments. The first issue is the predictability of credit actions. This is a result of perfect information in the market and the diffusive properties of underlying prices. It implies that contrary to empirical observations the spreads of credit products with short maturities are very close to zero. CreditGrades™ have been usually applied to solve this problem, which leads to the structural models becoming analytically very complicated. Various numerical approaches need to be used in order to deal with the lack of closed form pricing formulas. Calibrating the parameters of the model to market data then becomes a difficult task as well.

In our structural approach, we model the empirical measure of the underlying prices in a large pool of underlyings. The theoretical basis has been described in detail in [1], [2] and [4]. We extend the model introduced in [1] by combining the large market dynamics of the asset prices with the dynamics of a mean reverting Heston-type stochastic volatility variable. Both the asset dynamics and the volatility dynamics are linked by a market factor. The correlation between the market factor of the underlying and the market factor corresponding to the volatility variable is introduced.
The structural framework is based on the idea that a company defaults as soon as it decreases below a given barrier. Thus, we define the distance to default of each of the assets. We obtain a large market model by passing to the limit with the multidimensional model. We derive a Stochastic Partial Differential Equation governing the behaviour of the limit density as a result. This enables us to define the loss of our portfolio of assets evolving through time as a functional of the limit density. The value of credit derivative based on a large basket of assets is a function of the loss. By varying the parameters of the dynamics, we observe which segments of the model are important for pricing and risk management of complex credit instruments such as Single Tranche Collateralized Debt Obligations (STCDOs) and Forward Start CDOs.

We aim to develop this straightforward mathematical model and at the same time we stress that there are several flaws to our approach. The disadvantages would have to be evaluated before applying the model to real credit instruments. Problems might be encountered when applying structural models to products with short period maturities. The reason for this is that the diffusions of asset prices and volatilities comprise the basis for our analysis. The fitting of super senior tranche prices may be problematic as well, as we have only two diffusive factors in our model. There also might be an issue with locating an insufficient amount of risk in the most senior tranches. This model might however be a reasonable and more realistic extension of previously studied large market structural models for pricing complex credit products.
Chapter 2

The Large Market Model

In this dissertation, we study the extension of a large market model introduced in [1] with characteristics of a Heston Model. We assume that there are $N$ assets in the market with the corresponding probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The risk neutral dynamics of each of these assets are governed by some constant drift term, an idiosyncratic noise term and a common market noise term. We will interpret the last expression as the market factor which impacts all of the asset prices simultaneously. The idiosyncratic term impacts only one single asset price it corresponds to.

The volatility of each asset is a mean reverting process with two (analogous to the asset price) noise terms: a market term common for all volatilities and an idiosyncratic term, particular for each single volatility. The idea behind this is that the overall market volatility can increase and decrease, while companies have their own volatilities.

The value of an asset or the value of a company is given by the following Stochastic Differential Equation:

$$\frac{dS^i_t}{S^i_t} = \mu dt + \sigma \sqrt{Y^i_t} \rho dM^S_t + \sigma \sqrt{Y^i_t} \sqrt{1 - \rho^2} dW^i_t$$

(2.1)

where

$$dY^i_t = \kappa (\theta - Y^i_t) dt + \xi \eta \sqrt{Y^i_t} dM^Y_t + \xi \sqrt{1 - \eta^2} \sqrt{Y^i_t} d\tilde{W}^i_t$$

(2.2)

is the stochastic volatility variable. We have $i = 1, ..., N$. We assume that $t \in [0, T]$. 

4
Note:
We will refer to $Y^i_t$ as the volatility variable, however it is not the volatility of the $i$-th asset. We need to multiply it by the the scaling parameter $\sigma$ to obtain the $i$-th volatility: $\sigma \sqrt{Y^i_t}$.

$M^S, W^i, M^Y, \tilde{W}^i$ are all Brownian Motions. We interpret $M^S$ and $M^Y$ as market wide drivers influencing all assets and volatilities respectively. $W^i$ and $\tilde{W}^i$ can be interpreted respectively as idiosyncratic drivers of the asset price and the volatility. We assume that $dM^S dM^Y = \phi dt$ for some constant $\phi \in [-1, 1]$ and that any other pair of two Brownian Motions is uncorrelated.

We have introduced several constants in (2.1) and (2.2). The interpretation of these model parameters, which are all real constants, is the following:

- $\mu$ - the drift of the asset
- $\sigma$ - the scaling factor of the stochastic volatility
- $\rho$ - the correlation between the "market" noise and the idiosyncratic noise (of the asset price)
- $\theta$ - the volatility mean reversion level
- $\kappa$ - the speed of the mean reversion
- $\xi$ - the volatility of the volatility
- $\eta$ - the correlation between the volatility "market" noise and the idiosyncratic noise (of the stochastic volatility)
- $\phi$ - the correlation between the market-wide Brownian Motions: of the assets and of the volatility variables

In order to make our model more intuitive, we introduce the following:

- $\sigma_{mean}$ - the mean volatility of the model
- $\sigma_{max}$ - the maximum volatility of the model

The quantity $\sigma \sqrt{\theta}$ is the long term mean volatility in our model, as $Y^i_t$ fluctuates around the level $\theta$ and we calculate the volatility as $\sigma \sqrt{Y^i_t}$. Thus, we must have the equality: $\sigma \sqrt{\theta} = \sigma_{mean} \iff \theta = \left(\frac{\sigma_{mean}}{\sigma}\right)^2$.

Let $\epsilon$ be a fixed small positive number. Then we define:

$$B_\epsilon(\epsilon) = \inf\{B > 0 : \sup_i \mathbb{P}(\sup_{t \leq T} Y^i_t > B) \leq \epsilon\}$$
We interpret $B_y(\epsilon)$ as some upper bound for the $Y$ variable, although $Y$ is unbounded by construction in our model. Then the maximum model volatility is equal to $\sigma \sqrt{B_y(\epsilon)}$ and to $\sigma_{\text{max}}$ at the same time. Thus $\sigma = \frac{\sigma_{\text{max}}}{\sqrt{B_y(\epsilon)}}$. As we keep $B_y(\epsilon)$ fixed in our framework, we study the impact of varying $\sigma_{\text{max}}$ and $\sigma_{\text{mean}}$, which directly impact the values of $\sigma$ and $\theta$.

Note:
We set: $\sigma_{\text{max}} = 0.6, \sigma_{\text{mean}} = 0.2, \mu = 0.03, \rho = 0.3, \kappa = 1, \xi = 0.1, \eta = 0.3, \phi = 0.5$ as the standard collection of parameters. It is sensible to choose the annual risk free interest rate to be equal to 0.03, as the yield on the 10-year US Treasury is equal approximately to 0.03. Setting the mean long term volatility to be equal to 0.2 is also a realistic approximation, as this is the approximate value of the mean VIX index. For the same reason, we set the maximum model volatility to be equal to 0.6. We choose the values of the remaining parameters in such a way to ensure a realistic behaviour of the Loss process, which we introduce in this chapter. It is difficult to estimate their real values. We will study how varying each of the considered parameters impacts pricing selected complex credit instruments.

The distance to default
As our considered model falls in the group of structural models, below we introduce the following logarithm transformation of the asset price. This will significantly simplify our calculations when deriving the limit SPDE.

$$X^i_t = \log(S^i_t) - \log(B^i),$$
where $B^i$ is the (constant) $i$-th barrier. We interpret $X^i_t = 0$ as the default of $S^i$ at time $t$.

Then:

$$dX^i = \frac{dS^i}{S^i} - \frac{1}{2} \left( \frac{dS^i}{S^i} \right)^2 = (\mu - \frac{\sigma^2}{2} Y^i)dt + \sigma \sqrt{Y^i} \rho dM^S + \sigma \sqrt{Y^i} \sqrt{1 - \rho^2} dW^i$$

with

$$X^i_0 = x^i$$
$$\tau^i = \inf\{t : X^i_t = 0\}$$
$$X^i_t = 0 \quad \text{for all} \quad t \geq \tau^i$$
\( x^i \) are positive constants for all \( i \). \( \tau^i \) is the moment of the default of the \( i \)-th asset.

Now we can see that there are no \( X^i \) terms on the right side of the equation, whereas we did have the \( S^i \) factor in the lognormal dynamics for the asset price. This transformation is going to significantly simplify our further calculations.

**The limit measure**

We define the sequence of empirical measures \( \{\nu_{N,t}\}_{N \in \mathbb{N}} \) by setting:

\[
\nu_{N,t}(A) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X^i_t, Y^i_t)}(A) = \frac{1}{N} \sum_{i=1}^{N} 1_{(X^i_t, Y^i_t) \in A}
\]

for an arbitrary Borel set \( A \in \mathbb{R}^2 \).

We assume there exists a limit measure \( \nu_t \) such that:

\[
\lim_{N \to \infty} \nu_{N,t}((-\infty, \alpha] \times (-\infty, \beta]) = \nu_t((-\infty, \alpha] \times (-\infty, \beta])
\]

The behaviour of the density \( V_t \) corresponding to the limit measure is the main objective of our study. Then we can write \( \nu_t(dx, dy) = V_t(x, y) dxdy \) and integrate against \( \nu_t \).

**Empirical Densities**

We take \( N = 200 \) assets, the standard set of parameters and set the time frame to be \([0, 1]\). We take 50 grid points in the \( x \) direction with upper bound equal to 1. We take 50 grid points in the \( y \) direction with upper bound equal to 2.5. For all \( i = 1, \ldots, 200 \) we simulate both \( X^i \) and \( Y^i \) with 2000 timesteps using Euler’s discretization:

\[
\begin{align*}
Y^i_{t+\Delta t} &= Y^i_t + \kappa(\theta - Y^i_t) + \xi \eta \sqrt{Y^i_t} (M^Y_{t+\Delta t} - M^Y_t) + \xi \sqrt{1 - \eta^2} \sqrt{Y^i_t} (\tilde{W}^S_{t+\Delta t} - \tilde{W}^S_t) \\
X^i_{t+\Delta t} &= X^i_t + (\mu - \frac{\sigma^2}{2}) \Delta t + \sigma \sqrt{Y^i_t} \sqrt{1 - \rho^2} (W_{t+\Delta t} - W^i_t)
\end{align*}
\]

with

\[
Y^i_0 = 1 \quad \text{and} \quad X^i_0 = 0.4
\]

For all \( m, k \) we define the collection of sets:

\[
A_{m,k} = [m\Delta x, (m+1)\Delta x) \times [k\Delta y, (k+1)\Delta y).
\]
Assuming that $\Delta x$ and $\Delta y$ are sufficiently small numbers, we have:

$$P((X^i, Y^i) \in A_{m,k}) = \int_{A_{m,k}} V_{N,t}(x, y) dxdy$$

$$\approx V_{N,t}(m\Delta x, k\Delta y)\Delta x\Delta y$$

Hence:

$$V_{N,t}(m\Delta x, k\Delta y) \approx \frac{P((X^i, Y^i) \in A_{m,k})}{\Delta x\Delta y}$$

Therefore, we can calculate the value of the Empirical density with the formula:

$$V_{N,t}^{emp}(m\Delta x, k\Delta y) = \frac{1}{\Delta x\Delta y N} \sum_{i=1}^{N} \delta_{(X_i^t, Y_i^t)}(A_{m,k})$$

Below, we illustrate the behaviour of the 3-dimensional empirical density in time:

![Figure 2.1: Empirical Densities](image)
We can observe that with the passing of time the mass of the empirical distribution of \((X, Y)\) is moving towards the barrier value \(X = 0\). Also, the maximal value of the density decreases with time. The volatility variable seems to be concentrated more and more frequently in the lower half of the \([0, 2.5]\) interval with the passing of time.

Now we derive the dynamics of the limit density \(V_t\). They will be given by a Stochastic Partial Differential Equation. The derivation comprises the following steps: writing down the dynamics of a test function integrated against the \(N\)-th density \(V_{N,t}\), noticing that two of the terms in the equation vanish in the limit, integrating by parts and using the arbitrariness of the test function to obtain the Stochastic PDE. We present all of these steps below:

**The dynamics of test function integrated against the \(N\)-th measure**

We derive the dynamics of a test function \(f \in C^\infty_{\text{compact}}(\mathbb{R}^+ \times \mathbb{R}^+)\) (an infinitely differentiable function with compact support in \(\mathbb{R}^+ \times \mathbb{R}^+)\) integrated against the \(N\)-th measure. First for the measure \(\nu_t\) with the density \(V_t\) we define:

\[
\langle f, \nu_t \rangle = \int_{\mathbb{R}^+ \times \mathbb{R}^+} f(x, y) \nu_t(dx, dy) = \int_{\mathbb{R}^+ \times \mathbb{R}^+} f(x, y) V_t(x, y) dxdy
\]

Then, using \(<Z>_t\) as the quadratic variation of a process \(Z\), we have:

\[
d\langle f, V_{N,t} \rangle = d\left(\frac{1}{N} \sum_{i=1}^{N} f(X_i^t, Y_i^t)\right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial f}{\partial x} dX_i^t + \frac{\partial f}{\partial y} dY_i^t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} d <X_i^t>_t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} d <Y_i^t>_t + \frac{\partial^2 f}{\partial x \partial y} dX_i^t dY_i^t \right]
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial f}{\partial x}(\mu - \frac{\sigma^2}{2} Y_i^t) + \frac{\partial f}{\partial y} \kappa(\theta - Y_i^t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 Y_i^t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \xi^2 Y_i^t + \frac{\partial^2 f}{\partial x \partial y} \xi \eta Y_i^t \rho \phi \right] dt
\]

\[
+ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial x} \sigma \sqrt{Y_i^t} \rho dM^S + \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial y} \xi \eta \sqrt{Y_i^t} dM^Y
\]

\[
+ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial x} \sigma \sqrt{Y_i^t} \sqrt{1 - \rho^2} dW^i + \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial y} \xi \sqrt{1 - \eta^2} \sqrt{Y_i^t} d\tilde{W}^i
\]
In the following lemma, we prove that two sums of the expression written above vanish almost surely in the limit as $N \to \infty$.

**Lemma**

Let

$$E_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( \frac{\partial f}{\partial x} \right)^2 \sigma^2 Y_u^i \sqrt{1 - \rho^2} dW_u^i$$

$$\tilde{E}_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( \frac{\partial f}{\partial y} \right)^2 \xi^2 Y_u^i \sqrt{1 - \eta^2} d\tilde{W}_u^i$$

Then

$$E_{N,t} \xrightarrow{N \to \infty} 0$$

$$\tilde{E}_{N,t} \xrightarrow{N \to \infty} 0$$

almost surely.

**Proof.**

Both $E_{N,t}$ and $\tilde{E}_{N,t}$ are (local) martingales. By the independence of the Brownian Motions and by the fact that $f$ has bounded derivatives, we have (for some $C > 0$):

$$< E_{N} >_t = \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t \left( \frac{\partial f}{\partial x} \right)^2 \sigma^2 Y_u^i (1 - \rho^2) du$$

$$= \sigma^2(1 - \rho^2) \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t \left( \frac{\partial f}{\partial x} \right)^2 Y_u^i du$$

$$\leq \sigma^2(1 - \rho^2) C^2 \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t Y_u^i du$$

and similarly

$$< \tilde{E}_{N} >_t = \xi^2(1 - \eta^2) \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t \left( \frac{\partial f}{\partial y} \right)^2 Y_u^i du$$

$$\leq C^2 \xi^2(1 - \eta^2) \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t Y_u^i du$$
The dynamics of the i-th volatility is a Cox-Ross-Ingersoll (CIR) process:
\[
    dY_i^t = \kappa(\theta - Y_i^t)dt + \xi\sqrt{Y_i^t}dM_i^Y + \xi\sqrt{1 - \eta^2}\sqrt{Y_i^t}d\tilde{W}_i^t \\
    = \kappa(\theta - Y_i^t)dt + \xi\sqrt{Y_i^t}d\tilde{W}_i^t
\]
for \(\tilde{W}_i^t\) - a Brownian Motion such that \(d\tilde{W}_i^t = \eta dM_i^Y + \sqrt{1 - \eta^2}d\tilde{W}_i^t\)

We know that the mean of the CIR volatility variable is given by:
\[
    \mathbb{E}[Y_i^t] = Y_i^0e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) \leq Y_i^0e^{-0} + \theta(1 - (1 - \kappa t)) = Y_i^0 + \theta\kappa t
\]

As \(Y_i^t\) is by definition positive for all \(t \in [0, T]\), we can use Fubini’s theorem:
\[
    \mathbb{E}\int_0^t Y_i^u du = \int_0^t \mathbb{E}Y_i^u du = \int_0^t Y_i^0e^{-\kappa u} + \theta(1 - e^{-\kappa u})du \\
    = (Y_i^0 - \theta)\frac{1 - e^{-\kappa t}}{\kappa} + \theta t
\]

The last expression is bounded for all \(t \in [0, T]\). Therefore, we can apply the Strong Law of Large Numbers to obtain the following almost sure convergence of the considered stochastic integrals:
\[
    \frac{1}{N} \sum_{i=1}^N \int_0^t Y_i^u du \to \mathbb{E} \int_0^t Y_i^u du
\]

As \(\mathbb{E}\int_0^t Y_i^u du\) is bounded for all \(t\) we have:
\[
    <E_N>_{\geq} \leq \sigma^2(1 - \rho^2)C^2 \frac{1}{N} \frac{1}{N} \sum_{i=1}^N \int_0^t Y_i^u du \to 0
\]
\[
    <\tilde{E}_N>_{\geq} \leq C^2\xi^2(1 - \eta^2) \frac{1}{N} \frac{1}{N} \sum_{i=1}^N \int_0^t Y_i^u du \to 0
\]

\(E_{N,i}\) and \(\tilde{E}_{N,i}\) are (local) martingales with quadratic variations vanishing in the limit. Therefore, they must almost surely vanish themselves in the limit.
The dynamics of the test function integrated against the limit measure
Again, we consider the following dynamics:
\[
d<f, V_{N,t} > = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial f}{\partial x} (\mu - \frac{\sigma^2}{2} Y_i) + \frac{\partial f}{\partial y} \kappa (\theta - Y_i) \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 Y_i + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \xi^2 Y_i + \frac{\partial^2 f}{\partial x \partial y} \xi \eta \sigma Y_i \rho \phi \right] dt \\
+ \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial x} \sigma \sqrt{Y_i} \rho \right] dM^S + \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial y} \xi \sqrt{Y_i} \right] dM^Y \\
+ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial x} \sigma \sqrt{Y_i} \sqrt{1 - \rho^2} dW_i + \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial y} \xi \sqrt{1 - \eta^2} \sqrt{Y_i} d\tilde{W}_i
\]

We assume there exists a limit measure \( V_t \) such that \( V_{N,t} \rightarrow V_t \) in distribution. When we take \( N \rightarrow \infty \) the idiosyncratic terms \( dE_{N,t}, d\tilde{E}_{N,t} \) vanish and we obtain:

\[
d \int f(x, y) V_t(x, y) dx dy = d<f, V_t > \\
= \langle \left( \frac{\partial f}{\partial x} (\mu - \frac{\sigma^2}{2} Y_t) + \frac{\partial f}{\partial y} \kappa (\theta - Y_t) \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 Y_t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \xi^2 Y_t + \frac{\partial^2 f}{\partial x \partial y} \xi \eta \sigma Y_t \rho \phi \right), V_t \rangle dt \\
+ \langle \frac{\partial f}{\partial x} \sigma \sqrt{Y_t} \rho, V_t \rangle dM^S + \langle \frac{\partial f}{\partial y} \xi \sqrt{Y_t}, V_t \rangle dM^Y
\]

\[\square\]

The limit Stochastic Partial Differential Equation
The dynamics of the limit density \( V_t \) are governed by the following Stochastic Partial Differential Equation (SPDE):

\[
dV_t = \left[ -(\mu - \frac{\sigma^2}{2} y) \frac{\partial V_t}{\partial x} - \kappa \frac{\partial}{\partial y} ((\theta - y) V_t) \\
+ \frac{\sigma^2}{2} \frac{\partial^2 V_t}{\partial x^2} y + \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2} (y V_t) + \xi \eta \sigma \rho \phi \frac{\partial}{\partial y} \left( y \frac{\partial V_t}{\partial x} \right) \right] dt \\
- \sigma \rho \sqrt{y} \frac{\partial V_t}{\partial x} dM^S - \xi \eta \frac{\partial}{\partial y} (\sqrt{y} V_t) dM^Y
\]
with

$V_0(x, y)$ given

$V_t(0, y) = 0$ and $\lim_{x \to \infty} V_t(x, y) = 0$

$V_t(x, 0) = 0$ and $\lim_{y \to \infty} V_t(x, y) = 0$

\textbf{Proof.} In order to simplify notation, we write: $\int F = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} F(x, y) dxdy$ for any integrable function $F: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$.

By the fact that $f$ has compact support and by integration by parts:

$$\int \frac{\partial f}{\partial x} \left( \mu - \frac{\sigma^2}{2} y \right) V_t = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial f(x, y)}{\partial x} \left( \mu - \frac{\sigma^2}{2} y \right) V_t(x, y) dxdy$$

$$= \int_{\mathbb{R}^+} \left( \mu - \frac{\sigma^2}{2} y \right) \int_{\mathbb{R}^+} \frac{\partial f(x, y)}{\partial x} V_t(x, y) dxdy$$

$$= \int_{\mathbb{R}^+} \left( \mu - \frac{\sigma^2}{2} y \right) \left[ f(x, y) V_t(x, y) \right]_{x=\infty}^{x=-\infty} - \int_{\mathbb{R}^+} f(x, y) \frac{\partial V_t(x, y)}{\partial x} dx dy$$

$$= - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left( \mu - \frac{\sigma^2}{2} y \right) f(x, y) \frac{\partial V_t(x, y)}{\partial x} dx dy$$

$$= - \int \left( \mu - \frac{\sigma^2}{2} y \right) f \frac{\partial V_t}{\partial x}$$

$$\int \frac{\partial^2 f}{\partial x^2} y V_t = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial^2 f(x, y)}{\partial x^2} y V_t(x, y) dxdy$$

$$= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left[ \frac{\partial f}{\partial x} y \right]_{x=\infty}^{x=-\infty} - \int_{\mathbb{R}^+} \frac{\partial f(x, y)}{\partial x} y \frac{\partial V_t(x, y)}{\partial x} dx dy$$

$$= - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial f(x, y)}{\partial x} y \frac{\partial V_t(x, y)}{\partial x} dxdy$$

$$= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f y \frac{\partial^2 V_t(x, y)}{\partial x^2} dxdy = \int y f \frac{\partial^2 V_t}{\partial x^2}$$

$$\int \frac{\partial^2 f}{\partial x \partial y} y V_t = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial^2 f(x, y)}{\partial x \partial y} y V_t(x, y) dxdy$$

$$= \int_{\mathbb{R}^+} y \left[ \frac{\partial f}{\partial y} V_t(x, y) \right]_{x=\infty}^{x=-\infty} - \int_{\mathbb{R}^+} \frac{\partial f(y)}{\partial y} \frac{\partial V_t(x, y)}{\partial x} dx dy$$

$$= - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} y \frac{\partial f(y)}{\partial y} \frac{\partial V_t(x, y)}{\partial x} dy dx$$

$$= - \int_{\mathbb{R}^+} \left[ f \frac{\partial V_t(x, y)}{\partial x} \right]_{y=0}^{y=\infty} - \int_{\mathbb{R}} f \frac{\partial}{\partial y} \left( y \frac{\partial V_t(x, y)}{\partial x} \right) dy$$

$$= \int_{\mathbb{R}} \int f \frac{\partial}{\partial y} \left( y \frac{\partial V_t(x, y)}{\partial x} \right) dxdy = \int f \frac{\partial}{\partial y} \left( y \frac{\partial V_t}{\partial x} \right)$$
Similarly:

\[
\begin{align*}
\int \frac{\partial f}{\partial y}((\theta - y)V_i) & = - \int f \frac{\partial}{\partial y}((\theta - y)V_i) \\
\int \frac{\partial^2 f}{\partial y^2}yV_i & = \int f \frac{\partial^2}{\partial y^2}(yV_i) \\
\int \frac{\partial f}{\partial x}\sqrt{y}V_i & = - \int f \sqrt{y} \frac{\partial V_i}{\partial x} \\
\int \frac{\partial f}{\partial y}\sqrt{y}V_i & = - \int f \frac{\partial}{\partial y}(\sqrt{y}V_i) \\
\int \frac{\partial^2 f}{\partial y^2}yV_i & = \int f \frac{\partial^2}{\partial y^2}(yV_i)
\end{align*}
\]

Thus:

\[
\int f dV_i = \int f(x, y)dV_i(x, y)dxdy = d \int f(x, y)V_i(x, y)dxdy
\]

\[
= \left[ - \int f(\mu - \frac{\sigma^2}{2}y) \frac{\partial V_i}{\partial x} - \kappa \int f \frac{\partial}{\partial y}((\theta - y)V_i) \\
+ \frac{\sigma^2}{2} \int fy \frac{\partial^2 V_i}{\partial x^2} + \frac{\xi^2}{2} \int f \frac{\partial^2}{\partial y^2}(yV_i) + \xi \eta \rho \phi \int f \frac{\partial}{\partial y}(y \frac{\partial V_i}{\partial x}) \right] dt
\]

\[
- \sigma \rho \int f \sqrt{y} \frac{\partial V_i}{\partial x}dMS - \xi \eta \int f \frac{\partial}{\partial y}(\sqrt{y}V_i)dMY
\]

= \int f \left[ - (\mu - \frac{\sigma^2}{2}y) \frac{\partial V_i}{\partial x} - \kappa \frac{\partial}{\partial y}((\theta - y)V_i) \\
+ \frac{\sigma^2}{2} y \frac{\partial^2 V_i}{\partial x^2} + \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2}(yV_i) + \xi \eta \rho \phi \frac{\partial}{\partial y}(y \frac{\partial V_i}{\partial x}) \right] dt
\]

\[
- \sigma \rho \sqrt{y} \frac{\partial V_i}{\partial x}dMS - \xi \eta \frac{\partial}{\partial y}(\sqrt{y}V_i)dMY
\]

We can subtract the right side of the equation from the left side. As \( f \) is an arbitrary test function and the stochastic integral equals 0, we need the expression \( f \) is multiplied by to be equal to 0. Therefore, we obtain the desired SPDE:

\[
dV_i = \left[ - (\mu - \frac{\sigma^2}{2}y) \frac{\partial V_i}{\partial x} - \kappa \frac{\partial}{\partial y}((\theta - y)V_i) + \frac{\sigma^2}{2} y \frac{\partial^2 V_i}{\partial x^2} + \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2}(yV_i) \\
+ \xi \eta \rho \phi \frac{\partial}{\partial y}(y \frac{\partial V_i}{\partial x}) \right] dt - \sigma \rho \sqrt{y} \frac{\partial V_i}{\partial x}dMS - \xi \eta \frac{\partial}{\partial y}(\sqrt{y}V_i)dMY
\]
The Loss Function
We interpret \( \int V_t = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} V_t(x, y)dydx \) as the proportion of the market that has not defaulted up to time \( t \). Therefore, we define the lost function as:

\[
L_t = 1 - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} V_t(x, y)dydx
\]

with \( L_0 = 0 \), as our initial density integrates to 1.

Lemma
We have the following form of the loss function:

\[
L_t = \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^+} y \frac{\partial V_u(0, y)}{\partial x} dydu
\]

The interpretation of this formula is the following: the loss at time \( t \) is equal to the total flux across the boundary up to time \( t \). In other words, the loss equals the sum of all that is lost across the boundary up to time \( t \).

Proof.

\[
L_t = 1 - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} V_t(x, y)dydx = 1 - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} V_0(x, y)
\]

\[
+ \int_0^t \left[ - \left( \mu - \sigma^2 \frac{y}{2} \right) \frac{\partial V_u(x, y)}{\partial x} - \kappa \frac{\partial}{\partial y} \left( \left( \theta - y \right) V_u(x, y) \right)
\]

\[
+ \frac{\sigma^2}{2} \frac{\partial^2 V_u(x, y)}{\partial x^2}
\]

\[
+ \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2} \left( y \frac{\partial V_u(x, y)}{\partial x} + \xi \eta \sigma \rho \phi \frac{\partial}{\partial y} \left( y \frac{\partial V_u(x, y)}{\partial x} \right) \right) \right] du
\]

\[
- \int_0^t \sigma \rho \sqrt{y} \frac{\partial V_u(x, y)}{\partial x} dM_u^S - \int_0^t \xi \eta \frac{\partial}{\partial y} \left( \sqrt{y} V_u(x, y) \right) dM_u^Y dxdy
\]

\[
= L_0 + \int_0^t \left[ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left( \mu - \sigma^2 \frac{y}{2} \right) \frac{\partial V_u(x, y)}{\partial x} dxdy
\]

\[
+ \kappa \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial}{\partial y} \left( \left( \theta - y \right) V_u(x, y) \right) dx dxdy - \frac{\sigma^2}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial^2 V_u(x, y)}{\partial x^2} dxdy
\]

\[
- \frac{\xi^2}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial}{\partial y^2} \left( y \frac{\partial V_u(x, y)}{\partial x} \right) dxdy - \xi \eta \sigma \rho \phi \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial}{\partial y} \left( y \frac{\partial V_u(x, y)}{\partial x} \right) dxdy \right] du
\]

\[
+ \sigma \rho \int_0^t \left[ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \sqrt{y} \frac{\partial V_u(x, y)}{\partial x} dxdy \right] dM_u^S
\]

\[
+ \xi \eta \int_0^t \left[ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\partial}{\partial y} \left( \sqrt{y} V_u(x, y) \right) dxdy \right] dM_u^Y
\]
\[
\int_0^t \left[ \int_{\mathbb{R}^+} (\mu - \frac{\sigma^2}{2} y) [V_u(x, y)]_{x=0}^{y=\infty} \, dy + \kappa \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} [(\theta - y)V_u(x, y)]_{y=0}^{y=\infty} \, dx \right] \, du \\
- \frac{\sigma^2}{2} \int_{\mathbb{R}^+} [y \frac{\partial V_u(x, y)}{\partial x}]_{x=0}^{y=\infty} \, dy - \frac{\xi^2}{2} \int_{\mathbb{R}^+} \left[ \frac{\partial}{\partial y} (yV_u(x, y)) \right]_{y=0}^{y=\infty} \, dx \\
- \xi \eta \sigma \rho \phi \int_{\mathbb{R}^+} [y \frac{\partial V_u(x, y)}{\partial x}]_{y=0}^{y=\infty} \, dx + \sigma \rho \int_0^t \left[ \int_{\mathbb{R}^+} \sqrt{y} [V_u(x, y)]_{x=0}^{y=\infty} \, dy \right] dM_u^S \\
+ \xi \eta \int_0^t \left[ \int_{\mathbb{R}^+} [\sqrt{y} V_u(x, y)]_{y=0}^{y=\infty} \, dx \right] dM_u^Y \\
= \int_0^t \left[ \frac{\sigma^2}{2} \int_{\mathbb{R}^+} y \frac{\partial V_u(0, y)}{\partial x} \, dy - \frac{\xi^2}{2} \int_{\mathbb{R}^+} [V_u(x, y) + y \frac{\partial V_u(x, y)}{\partial y}]_{y=0}^{y=\infty} \, dx \right] \, du \\
= \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^+} y \frac{\partial V_u(0, y)}{\partial x} \, dy \, du \\
\]

Note:
We used the stochastic version of Fubini’s Theorem in the third equality. We also used the following facts/assumptions about the density functions:
\[
\lim_{x \to \infty} V_t(x, y) = \lim_{y \to \infty} V_t(x, y) = 0 \\
\lim_{y \to \infty} y V_t(x, y) = 0 \\
\lim_{x \to \infty} \frac{\partial V_t(x, y)}{\partial x} = 0 \\
\lim_{y \to \infty} \frac{\partial V_t(x, y)}{\partial y} = \lim_{y \to \infty} y \frac{\partial V_t(x, y)}{\partial x} = \lim_{y \to \infty} y \frac{\partial V_t(0, y)}{\partial x} = 0 \\
y \frac{\partial V_t(0, y)}{\partial x} \text{ exists for all } y.
\]

□

**Single Tranche Collateralised Debt Obligation (STCDO)**

A Collateralised Debt Obligation is a specific type of asset-backed security. Owning a CDO is equivalent to owning shares in a basket of risky assets, which are all based on credit. We distinguish several classes of securities within a CDO called tranches. The holders of Collateralised Debt Obligations receive payments in sequence of seniority, which is dependent on which tranche they have invested in. In the case of default of a small part of the collateral pool, the most junior tranche holders first lose money. The most senior tranche holders lose money only in case of the default of (almost) all components of the reference portfolio. As the risk borne by junior tranche holders is significantly higher than that of senior tranche holders, coupon payments vary by tranche. The most risky tranches pay the highest coupons and the safest ones will
pay the lowest. In the simplest case, there are only three tranches: senior, mezzanine and equity.

In this dissertation, we focus on a specific type of a CDO - a synthetic CDO (sCDO). The reference portfolio of a sCDO consists of Credit Default Swaps (CDS) only. The originator of the sCDO buys protection from the tranche holders and then sells it to investors and companies. A single tranche of this derivative may be considered as a tradable security. In our considerations, the reference portfolio of CDSs does not have to be created. In case of a transaction, the buyer and the seller of a STCDO exchange cash flows as if the whole sCDO existed. The collateral pool is only used as a reference so that we can determine these cash flows. The long position holder in the STCDO pays a regular fee, equal to the par spread of the tranche, to the short position holder. In case of losses incured on the reference portfolio, the seller pays the buyer an amount equal to the size of the loss corresponding to that tranche.

Two values define a STCDO: the attachment point \( a \) and the detachment point \( d \) \((d > a)\). From this we define the tranche notional as \( N_t = d - a \). We assume that all three are given as a percentage of the value of the whole collateral pool. The typical tranches of the CDS index iTraxx Europe are given by the following pairs of attachment and detachment points: \( 0 - 3\% \), \( 3 - 6\% \), \( 6 - 9\% \), \( 9 - 12\% \), \( 12 - 22\% \) and \( 22 - 100\% \). We will focus on these tranches in our simulations. Below we have a table with data of midprice quotes for 5-year iTraxx Europe tranches. The quote corresponding to the \( 0 - 3\% \) tranche is expressed in percents, while the rest of the values is expressed in basis points (per year).

<table>
<thead>
<tr>
<th>Date</th>
<th>Tranche 0-3%</th>
<th>Tranche 3-6%</th>
<th>Tranche 6-9%</th>
<th>Tranche 9-12%</th>
<th>Tranche 12-22%</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.01.2007</td>
<td>15.34%</td>
<td>41.59</td>
<td>11.95</td>
<td>5.6</td>
<td>2</td>
</tr>
<tr>
<td>31.01.2008</td>
<td>35.98%</td>
<td>316.9</td>
<td>212.4</td>
<td>140</td>
<td>73.6</td>
</tr>
<tr>
<td>30.01.2009</td>
<td>69.28%</td>
<td>1185.63</td>
<td>606.69</td>
<td>315.63</td>
<td>97.13</td>
</tr>
</tbody>
</table>

Source: [5].

As we have already mentioned, the buyer of a STCDO pays a premium to the seller for the protection. If the total loss in the reference portfolio is equal to a value less than \( d \), no cash flows take place. Assume we observe a cumulative loss \( L_t \) in the reference portfolio between the attachment and the detachment point: \( L_t \in [a, d] \).
Then, the investor holding a short position in the single tranche CDO pays the long position holder the value of the loss corresponding to the specified tranche, equal to \((L_t - d)\). \(N_t\) is then decreased by \((L_t - d)\). If we observe an excess of the cumulative loss above \(d\), then our new tranche notional is equal to 0.

From the description above we can write down the following formulas:

For the tranche loss:

\[
U_t = \begin{cases} 
0 & \text{for } L_t < a \\
L_t - a & \text{for } L_t \in [a, d] \\
d - a & \text{for } L_t > d
\end{cases} = (L_t - a)^+ - (L_t - d)^+
\]

For the outstanding tranche notional:

\[
N_t = \begin{cases} 
0 & \text{for } L_t > d \\
d - L_t & \text{for } L_t \in [a, d] \\
d - a & \text{for } L_t < a
\end{cases} = (d - L_t)^+ - (a - L_t)^+
\]

Pricing a Single Tranche CDO

The value of a STCDO is equal to the difference between the value of the fee leg (paid by the buyer) and the value of the protection leg (paid by the seller).

Let \(t_1, t_2, ..., t_N\) be the payment dates with \(t_i = i\Delta t\). We introduce the constant interest rate \(r\) and the value at \(t\) of 1 money unit invested at 0 in the bank account: \(B_t = e^{rt}\). Let \(s\) be the par spread of the given tranche. We will choose \(s\) in such a way, that the present value of the STCDO is equal to 0.

The short position holder has to pay the long position holder when the specified tranche has suffered losses. Thus, the value of such a single payment is equal to the expected change in the tranche loss: \(U_{t+\Delta t} - U_t\). We observe that for all \(i\) we have:

\[
U_{t_{i+1}} - U_{t_i} = N_{t_i} - N_{t_{i+1}}.
\]

Indeed we have:

\[
U_{t_{i+1}} - U_{t_i} = \begin{cases} 
0 & \text{for } L_{t_{i}} > d \\
d - L_{t_{i}} & \text{for } L_{t_{i}} \in [a, d], L_{t_{i+1}} > d \\
L_{t_{i+1}} - L_{t_{i}} & \text{for } L_{t_{i}}, L_{t_{i+1}} \in [a, d] \\
L_{t_{i+1}} - a & \text{for } L_{t_{i}} < a, L_{t_{i+1}} \in [a, d] \\
0 & \text{for } L_{t_{i+1}} < a
\end{cases} = N_{t_{i}} - N_{t_{i+1}}
\]
Hence, the value of the protection leg equals:

\[ P = \sum_{i=1}^{n} \frac{\mathbb{E}[U_{ti} - U_{t_{i-1}}]}{B_{ti}} = \sum_{i=1}^{n} \frac{\mathbb{E}[N_{t_{i-1}} - N_{ti}]}{B_{ti}} = \sum_{i=1}^{n} \frac{\mathbb{E}[N_{t_{i-1}} - N_{ti}]}{e^{ri\Delta t}} \]

The long position holder pays a par spread equal to \( s \) on the tranche notional \( N_t \) during all payment dates. Therefore, we calculate the value of the fee leg in the following way:

\[ sF = s \sum_{i=1}^{n} \frac{\Delta t\mathbb{E}[N_{ti}]}{B_{ti}} = s\Delta t \sum_{i=1}^{n} \frac{\mathbb{E}[N_{ti}]}{e^{ri\Delta t}} \]

Hence the value of the Single Tranche CDO equals \( sF - P \) and we choose \( s = \frac{P}{F} \) in order to make it equal to zero. For calculating both \( F \) and \( P \) we need the distribution \( N_t \), which we will obtain when we derive the behaviour of \( L_t \).

**Forward start CDOs (FSCDOs)**

Despite the fact that FSCDOs are rather illiquid products, their hedging and pricing is a popular topic among the research of academics specialized in mathematical finance. These exotic credit instruments are popular in short-period markets with large volatility, as the two sides of the contract do not exchange cashflows until a specified time (as it has the properties of a forward contract).

The holder of a Forward Start CDO is obliged to buy or sell protection on a STCDO for a given premium at a specified date in the future. We consider two versions of the FSCDO: one that resets the cumulative loss at the specified time in the future and a version without resetting.

The pricing technique for the non-resetting derivative is exactly the same as for the STCDO described before. We choose the par spread in such a way, that the present value of all the cash flows is zero. For this reason, we will price the STCDO and the resetting version of the FSCDO.

The resetting version takes the future loss into account only, ignoring the total loss up to a specified date. We give an example of how it works: the holder of the resetting FSCDO might be obliged to sell protection on a Single Tranche CDO with the attachment point \( a \) and detachment point \( d \) during a time period \([t_F, t_{F+1}]\) in the future and with a previously specified par spread equal to \( s \). Therefore, the Forward
contract expires at $t_F$, whereas the maturity of the Forward Starting CDO is $t_{F+1}$. At $t_F$ the FSCDO is converted into a STCDO with a new detachment point $d + L_{t_F}$ and a new attachment point $a + L_{t_F}$, where $L_{t_F}$ is the cumulative loss of the reference portfolio up to time $t_F$. In other words, the resetting type of the FSCDO is equivalent to the non-resetting version with a new detachment point $d + L_{t_F}$ and a new attachment point $a + L_{t_F}$. Because of the fact that $L_t$ is stochastic, the starting reference portfolio for the STCDO over the period $[t_F, t_{F+1}]$ is non-deterministic as well. Before the creation of the structural framework, the randomness of $L_t$ made the pricing of Forward Start CDOs extremely complicated. However, with our model, the pricing technique is fairly straightforward.
Chapter 3

Numerical implementation and results

In this chapter, we present the results of the model simulations. First, we introduce the family of initial densities. We describe in detail the finite difference discretization of the Stochastic Partial Differential Equation. Afterwards we examine the behaviour of the Loss Function process when varying selected parameters. We investigate how these changes impact the prices of Single Tranche CDOs and Forward Start CDOs.

Numerics for the density

Again, we study the Stochastic PDE:

\[
\begin{align*}
    dV_t &= \left[ -\left( \mu - \frac{\sigma^2}{2} y \right) \frac{\partial V_t}{\partial x} - \kappa \frac{\partial}{\partial y} ((\theta - y)V_t) \\
        &\quad + \frac{\sigma^2}{2} y \frac{\partial^2 V_t}{\partial x^2} + \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2} (yV_t) + \xi \eta \sigma \rho \phi \frac{\partial}{\partial y} \left( y \frac{\partial V_t}{\partial x} \right) \right] dt \\
        &\quad - \sigma \rho \sqrt{y} \frac{\partial V_t}{\partial x} dM^S - \xi \eta \frac{\partial}{\partial y} (\sqrt{y}V_t) dM^Y \\
    &= \left[ -\left( \mu - \frac{\sigma^2}{2} y \right) \frac{\partial V_t}{\partial x} + \kappa yV_t + \kappa (y - \theta) \frac{\partial V_t}{\partial y} + \frac{\sigma^2}{2} y \frac{\partial^2 V_t}{\partial x^2} \\
        &\quad + \frac{\xi^2}{2} \left( 2 \frac{\partial V_t}{\partial y} + y \frac{\partial^2 V_t}{\partial y^2} \right) + \xi \eta \sigma \rho \phi \left( \frac{\partial V_t}{\partial x} + y \frac{\partial^2 V_t}{\partial x \partial y} \right) \right] dt \\
        &\quad - \sigma \rho \sqrt{y} \frac{\partial V_t}{\partial x} dM^S - \xi \eta \left( \frac{V_t}{2 \sqrt{y}} + \sqrt{y} \frac{\partial V_t}{\partial y} \right) dM^Y
\end{align*}
\]

(3.1)

with

- \(V_0(x, y)\) given
- \(V_t(0, y) = 0\) and \(\lim_{x \to \infty} V_t(x, y) = 0\)
- \(V_t(x, 0) = 0\) and \(\lim_{y \to \infty} V_t(x, y) = 0\)
Discretization of the SPDE

We define our discretization on a uniform mesh in both spacial variables and time:

\[ x_m = m \Delta x \quad \text{for } m=1,...,B_x / \Delta x \]
\[ y_k = k \Delta x \quad \text{for } k=1,...,B_y / \Delta y \]
\[ t_l = l \Delta t \quad \text{for } l=1,...,T / \Delta t \]

where \( B_x = 1, B_y = 2.5 \) and \( T = 1 \) are upper bounds for \( x, y \) and \( t \) respectively. \( B_y \) is the discrete analogue of \( B_y(\epsilon) \) introduced before. We take 50 grid points in both the \( x \) and \( y \) directions and we will perform 2000 timesteps. Therefore, we have all \( \Delta x, \Delta y \) and \( \Delta t \) determined. We use the standard collection of parameters.

We start our discretization with defining the initial condition. In our analysis we set the following pyramid-shaped function as the initial density function \( V_0(x,y) \).

![Initial Density Function](image)

It is given by the formula:

\[ V_0(x,y) = H_\epsilon(x,y) \mathbb{1}_{(x,y) \in [0,\epsilon B_x] \times [0,\epsilon B_y]} \]

where
We apply the following boundary conditions:

\[ V(0, y_k) = V(y_k, 0) = 0 \text{ for all } k. \]
\[ V(x_m, 0) = V(x_m, -\frac{B_y}{B_x}) = 0 \text{ for all } m. \]

Now we discretize the Stochastic Partial Differential Equation (3.1):

\[
H(x, y) = \begin{cases}
\frac{6}{c^3 B_x^2 B_y} y & \text{for } y \leq \frac{B_y}{B_x} x \\ -\frac{6}{c^3 B_x^2 B_y} x + \frac{6}{c^3 B_x B_y} & \text{for } y < \frac{B_y}{B_x} x \text{ and } y > -\frac{B_y}{B_x} x + \epsilon B_y \\ -\frac{6}{c^3 B_x^2 B_y} y + \frac{6}{c^3 B_x B_y} & \text{for } y \geq \frac{B_y}{B_x} x \text{ and } y \geq -\frac{B_y}{B_x} x + \epsilon B_y \\ \frac{6}{c^3 B_x^2 B_y} x & \text{for } y > \frac{B_y}{B_x} x \text{ and } y < -\frac{B_y}{B_x} x + \epsilon B_y
\end{cases}
\]

We choose \( \epsilon \) to be equal to 0.4, as we assume that the initial distributions of both the distance to default and the volatility variable are concentrated close to zero.

\[
V_i(x_m, y_k) = V_i(x_m, y_k) + \left[ -\left( \mu - \frac{\sigma^2}{2} y_k \right) \frac{V_i(x_{m+1}, y_k) - V_i(x_m, y_k)}{\Delta x} + \frac{\kappa V_i(x_m, y_k)}{\Delta x} + \frac{\kappa (y_k - \theta) V_i(x_m, y_{k+1}) - V_i(x_m, y_k)}{\Delta x} \right]
\]
\[
+ \frac{\sigma^2}{2} y_k \frac{V_i(x_{m+1}, y_k) - 2V_i(x_m, y_k) + V_i(x_{m-1}, y_k)}{(\Delta x)^2} + \frac{\epsilon^2}{2} \frac{V_i(x_m, y_{k+1}) - V_i(x_m, y_k)}{\Delta y} + \frac{y_k V_i(x_m, y_{k+1}) - 2V_i(x_m, y_k) + V_i(x_m, y_{k-1})}{(\Delta y)^2}
\]
\[
+ \xi \eta \sigma \rho \phi \left( \frac{V_i(x_{m+1}, y_k) - V_i(x_m, y_k)}{\Delta x} + \frac{y_k V_i(x_{m+1}, y_{k+1}) - V_i(x_{m-1}, y_{k+1}) - V_i(x_{m+1}, y_{k-1}) - V_i(x_{m-1}, y_{k-1})}{4\Delta x \Delta y} \right) \Delta t
\]
\[
- \frac{\sigma \rho \sqrt{y_k} V_i(x_{m+1}, y_k) - V_i(x_m, y_k)}{\Delta x} (M_t^S + \Delta t - M_t^S)
\]
\[
- \frac{\xi \eta (V_i(x_m, y_k) + \sqrt{y_k} V_i(x_m, y_{k+1}) - V_i(x_m, y_k))}{2\sqrt{y_k}} (M_t^Y + \Delta t - M_t^Y)
\]

We apply the following boundary conditions:

\[ V_i(0, y_k) = V_i(y_k, 0) = 0 \text{ for all } k. \]
\[ V_i(x_m, 0) = V_i(x_m, -\frac{B_y}{B_x}) = 0 \text{ for all } m. \]
Below we graph the evolution of the density function in time for the standard parameters, the time period \([0, 1]\) and 2000 timesteps:

\[
\begin{align*}
\text{(a) After 500 time steps} & \\
\text{(b) After 1000 time steps} & \\
\text{(c) After 1500 time steps} & \\
\text{(d) After 2000 time steps} & \\
\end{align*}
\]

Figure 3.2: The evolution of the density in time for \(\epsilon = 0.4\)

It is clear that the maximal value of the density decreases with time. We may observe that the density is concentrated mainly on distance to default values which are very close to zero (definitely less than \(\epsilon = 0.4\)). This suggests that asymptotically the asset value does not move away significantly from the barrier, but it rather hovers closely to it. The volatility variable seems to move from being clustered close to 0 to being more or less symmetrically concentrated on most of the whole domain. The asymptotic behaviour of the volatility in our model is therefore significantly volatile itself, as the values of the function corresponding to 500 steps are rather spread out on the area \([0, 0.3] \times [0, 2.5]\). It is not clustered in any small subset of this rectangle. We can also notice that most of the mass of the density is lost on the \(X = 0\) axis.

We note one other interesting observation. Assume we set a different value of \(\epsilon\), for example \(\epsilon = 0.9\). The support of the initial density function would then be
[0, 0.9B_x] × [0, 0.9B_y] instead of [0, 0.4B_x] × [0, 0.4B_y]. In this setting, we would observe the same asymptotic behaviour of the density function as for ϵ = 0.4. That is, the amount of mass lost across the X = 0 axis increases in time and the Y variable seems to approach a symmetric distribution on the whole [0, 2.5] interval. Below we graph the time evolution of the density corresponding to ϵ = 0.9:

![Graphs of density evolution](image)

(a) After 500 time steps  
(b) After 1000 time steps  
(c) After 1500 time steps  
(d) After 2000 time steps

Figure 3.3: The evolution of the density in time for ϵ = 0.9

We conclude that it does not matter asymptotically which function of the pyramid shaped function family \( \{ V_\epsilon \}_{\epsilon \in (0,1)} \) we choose for the initial density.

**Numerics for the Loss Function**

Again, we study the Loss Function:

\[
L_t = 1 - \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} V_t(x, y) \, dx \, dy = \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^+} y \frac{\partial V_u(0, y)}{\partial x} \, dy \, du
\]
We discretize the loss function in the following way:

\[ L_{tk} = 1 - \sum_{m=1}^{n_x} \sum_{j=1}^{n_y} V_{tk}(x_k, y_j) \Delta x \Delta y \]

For the time period \([0, 1]\), 2000 timesteps and 50 grid points in both spatial directions, we study the dependence of the loss functions on the parameters. We set the standard collection of parameters, vary one of them and graph 7 paths of the loss functions corresponding to each of our choices.

![Figure 3.4: Varying \(\sigma_{\text{max}}\)](image)

We can observe that with the increase of \(\sigma_{\text{max}}\) the values of the paths of the Loss Function increase on average and become less volatile.

![Figure 3.5: Varying \(\mu\)](image)

We may conclude that with the increase of \(\mu\) the path values of the Loss function decrease on average and the variability remains rather constant. The higher the asset drift, the lesser the probability of default and hence the lesser the value of the loss.
With the increase of $\xi$ the values of the Loss Function increase on average and become more volatile. However, this dependency is not very clear from the graphs. The higher the volatility of the volatility, the more volatile the loss on a large basket of derivatives.

We can observe that with the increase of $\rho$ the variability of the loss process increases significantly and the paths become more "wiggly". We do not consider Loss Functions for $\rho \geq 0.6$, as for these values we have observed unsatisfactory behaviour of the Loss Function paths. This included extreme "wiggliness" and paths piecewise decreasing in time.

The impact of varying $\sigma_{mean}$, $\eta$, $\phi$ or $\kappa$ turned out to be insignificant. For this reason, we do not place the graphs corresponding to the changes in these parameters here.

We observe that the loss function paths are increasing functions of time on average. That is, more losses occur with the passing of time. We would expect this
property from a sensible model. The majority of the simulated paths are also concave functions of time. This means that in our model large losses occur early in time. The losses will become smaller with the passing of time, as we approach the default of the whole reference portfolio.

Below we graph empirical Probability Density Functions of the terminal values of the Loss process for varying parameters. We use 500 paths for each PDF. The values of the mean and standard deviation corresponding to each PDF are also presented.

![Graphs of empirical PDFs](image)

(a) Varying $\sigma_{max}$

(b) Varying $\mu$

Figure 3.8: Empirical PDFs for varying $\sigma_{max}$ and $\mu$

<table>
<thead>
<tr>
<th>$\sigma_{max}$</th>
<th>mean</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.086</td>
</tr>
<tr>
<td>0.6</td>
<td>0.39</td>
<td>0.097</td>
</tr>
<tr>
<td>0.7</td>
<td>0.46</td>
<td>0.098</td>
</tr>
<tr>
<td>0.8</td>
<td>0.53</td>
<td>0.098</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>mean</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.43</td>
<td>0.098</td>
</tr>
<tr>
<td>0.03</td>
<td>0.39</td>
<td>0.078</td>
</tr>
<tr>
<td>0.1</td>
<td>0.33</td>
<td>0.081</td>
</tr>
<tr>
<td>0.2</td>
<td>0.21</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 3.1: Mean and standard deviation for $\sigma_{max}$ and $\mu$

We can now confirm the relations suggested before. With the increase of $\sigma_{max}$ the average terminal value of the Loss Function and the standard deviation increase. The mean and the standard deviation decrease when we increase the value of $\mu$. 

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We can observe that there indeed is a slight increasing tendency in both the mean and the standard deviation of the Loss process terminal values when we increase the volatility of the volatility $\xi$. When increasing $\rho$ the average loss value remains rather constant, however the standard deviation increases significantly. This is clearly illustrated on the graph above.

**Pricing the STCDO**

In order to calculate the price of a Single Tranche CDO we use the formula:

$$ s = \frac{P}{F} = \frac{\sum_{i=1}^{n} \frac{\mathbb{E}[N_{t_{i+1}} - N_{t_{i}}]}{e^{r \Delta t}}}{\Delta t \sum_{i=1}^{n} \frac{\mathbb{E}[N_{t_{i}}]}{e^{r \Delta t}}} $$

We consider the typical collection of tranches of iTraxx Europe: 0 – 3%, 3 – 6%, 6 – 9%, 9 – 12%, 12 – 22% and 22 – 100%. We will calculate the par spreads for each of these tranches with varying model parameters. All we need for calculating the tranche spreads is the sequence $\mathbb{E}[N_{t_{i}}]$ for $i = 0, .., n$. 

---

**Figure 3.9:** Empirical PDFs for varying $\xi$ and $\rho$

**Table 3.2:** Mean and standard deviation for $\xi$ and $\rho$

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>mean</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.38</td>
<td>0.09</td>
</tr>
<tr>
<td>0.1</td>
<td>0.39</td>
<td>0.093</td>
</tr>
<tr>
<td>0.2</td>
<td>0.39</td>
<td>0.096</td>
</tr>
<tr>
<td>0.3</td>
<td>0.42</td>
<td>0.099</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>mean</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.39</td>
<td>0.008</td>
</tr>
<tr>
<td>0.1</td>
<td>0.39</td>
<td>0.027</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>0.055</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Assume we conduct $M$ Monte Carlo simulations. Then we have:

$$
\mathbb{E}[N_{t_i}] = \mathbb{E}[(d - L_{t_i})^+ - (a - L_{t_i})^+] \approx \frac{1}{M} \sum_{j=1}^{M} [(d - L_{t_i}^j)^+ - (a - L_{t_i}^j)^+]
$$

Where $L_{t_i}^j$ is the $j$–th sampled value of the Loss Function evaluated at time $t_i$.

The impact of varying the parameters on the pricing of STCDOs
We use 500 Monte Carlo samples to study the impact of varying each of the selected parameters on the pricing. We need to stress that there is some variability in our model. In order to illustrate this, we present the prices of 1-year STCDOs for the standard collection of parameters simulated three times. All tranches are quoted in basis points. Note that, we consider 1-year CDOs and hence the par spreads are significantly lower than the par spreads corresponding to e.g. 5-year CDOs.

Table 3.3: Par spreads of STCDOs for standard parameters

<table>
<thead>
<tr>
<th>Simulation no.</th>
<th>Tranche</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 – 3%</td>
</tr>
<tr>
<td>1</td>
<td>54.58</td>
</tr>
<tr>
<td>2</td>
<td>54.13</td>
</tr>
<tr>
<td>3</td>
<td>54.03</td>
</tr>
</tbody>
</table>

Now we present the prices of 1–year Single Tranche CDOs for varying $\sigma_{max}, \mu, \xi$ and $\rho$:

Table 3.4: Par spreads of STCDOs for varying $\sigma_{max}$

<table>
<thead>
<tr>
<th>$\sigma_{max}$</th>
<th>Tranche</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 – 3%</td>
</tr>
<tr>
<td>0.5</td>
<td>54.68</td>
</tr>
<tr>
<td>0.6</td>
<td>54.56</td>
</tr>
<tr>
<td>0.7</td>
<td>56.13</td>
</tr>
<tr>
<td>0.8</td>
<td>55.62</td>
</tr>
</tbody>
</table>

We can observe a slight increasing tendency of the STCDO prices, when we increase the value of $\sigma_{max}$. This is due to the average increase of the loss values.
Table 3.5: Par spreads of STCDOs for varying $\mu$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>0 - 3%</th>
<th>3 - 6%</th>
<th>6 - 9%</th>
<th>9 - 12%</th>
<th>12 - 22%</th>
<th>22 - 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>57.09</td>
<td>20.29</td>
<td>11.81</td>
<td>7.8</td>
<td>4.19</td>
<td>0.31</td>
</tr>
<tr>
<td>0.03</td>
<td>54.58</td>
<td>17.79</td>
<td>10.71</td>
<td>7.4</td>
<td>3.82</td>
<td>0.25</td>
</tr>
<tr>
<td>0.1</td>
<td>51.92</td>
<td>16.14</td>
<td>9.01</td>
<td>5.87</td>
<td>2.52</td>
<td>0.13</td>
</tr>
<tr>
<td>0.2</td>
<td>44.45</td>
<td>12.99</td>
<td>6.23</td>
<td>3.36</td>
<td>1.29</td>
<td>0.03</td>
</tr>
</tbody>
</table>

The impact of $\mu$ on the prices of Single Tranche CDOs is clear: the larger the $\mu$, the lower the price for all tranches. We could have expected this dependence, as the Loss Function paths decreased on average when we increased $\mu$.

Table 3.6: Par spreads of STCDOs for varying $\xi$

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>0 - 3%</th>
<th>3 - 6%</th>
<th>6 - 9%</th>
<th>9 - 12%</th>
<th>12 - 22%</th>
<th>22 - 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>54.96</td>
<td>18.67</td>
<td>10.88</td>
<td>7.26</td>
<td>3.8</td>
<td>0.23</td>
</tr>
<tr>
<td>0.1</td>
<td>56.27</td>
<td>19.38</td>
<td>10.72</td>
<td>7.21</td>
<td>3.97</td>
<td>0.28</td>
</tr>
<tr>
<td>0.2</td>
<td>58.63</td>
<td>18.64</td>
<td>11.19</td>
<td>7.27</td>
<td>3.9</td>
<td>0.27</td>
</tr>
<tr>
<td>0.3</td>
<td>61.46</td>
<td>19.83</td>
<td>11.45</td>
<td>7.87</td>
<td>4.06</td>
<td>0.29</td>
</tr>
</tbody>
</table>

The increase in $\xi$ implies the increase of the Loss Function values on average. This subsequently implies the average increase of the prices of STCDOs, which is illustrated in the table above.

Table 3.7: Par spreads of STCDOs for varying $\rho$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0 - 3%</th>
<th>3 - 6%</th>
<th>6 - 9%</th>
<th>9 - 12%</th>
<th>12 - 22%</th>
<th>22 - 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>75.45</td>
<td>23.08</td>
<td>12.97</td>
<td>8.74</td>
<td>4.7</td>
<td>0.25</td>
</tr>
<tr>
<td>0.2</td>
<td>72.79</td>
<td>22.77</td>
<td>12.66</td>
<td>8.56</td>
<td>4.63</td>
<td>0.25</td>
</tr>
<tr>
<td>0.4</td>
<td>28.43</td>
<td>9.66</td>
<td>6.99</td>
<td>4.51</td>
<td>2.52</td>
<td>0.27</td>
</tr>
<tr>
<td>0.5</td>
<td>17.22</td>
<td>7.33</td>
<td>4.76</td>
<td>3.77</td>
<td>2.05</td>
<td>0.28</td>
</tr>
</tbody>
</table>

The prices of the Single Tranche CDOs decrease with the increase of $\rho$ for all tranches, apart from the senior tranche. This is an effect of increased variability of the Loss Function paths.
Pricing the resetting FSCDO

In order to price a resetting FSCDO, we define the effective forward loss at time $t_i$ as:

$$
\hat{L}_{ti} = L_{ti} - L_{tF}
$$

The outstanding forward tranche notional is defined in a similar way as before:

$$
\hat{N}_{ti} = (d - \hat{L}_{ti})^+ - (a - \hat{L}_{ti})^+
$$

We apply the same pricing formula as for the STCDO to obtain the forward par spread.

$$
\mathbb{E}[\hat{N}_{ti}] = \mathbb{E}[(d - \hat{L}_{ti})^+ - (a - \hat{L}_{ti})^+] \approx \frac{1}{M} \sum_{j=1}^{M} [(d - \hat{L}_{tj})^+ - (a - \hat{L}_{tj})^+]
$$

The impact of varying the parameters on the pricing of FSCDOs

Below, we present the prices of 1 year Forward Start CDOs with the specified date of 1 month. We investigate how changes in $\sigma_{max}, \mu, \xi$ and $\rho$ impact the pricing:

<table>
<thead>
<tr>
<th>$\sigma_{max}$</th>
<th>0 – 3%</th>
<th>3 – 6%</th>
<th>6 – 9%</th>
<th>9 – 12%</th>
<th>12 – 22%</th>
<th>22 – 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>7.18</td>
<td>4.94</td>
<td>3.19</td>
<td>2.3</td>
<td>1.21</td>
<td>0.05</td>
</tr>
<tr>
<td>0.6</td>
<td>7.93</td>
<td>5.55</td>
<td>4.15</td>
<td>3.26</td>
<td>2.02</td>
<td>0.12</td>
</tr>
<tr>
<td>0.7</td>
<td>8.21</td>
<td>6.33</td>
<td>4.9</td>
<td>3.9</td>
<td>2.47</td>
<td>0.19</td>
</tr>
<tr>
<td>0.8</td>
<td>8.57</td>
<td>6.63</td>
<td>5.41</td>
<td>4.49</td>
<td>2.93</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Due to the fact that the loss values increase on average when we increase the value of $\sigma_{max}$, we can observe a clear increasing tendency of the FSCDO prices.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>0 – 3%</th>
<th>3 – 6%</th>
<th>6 – 9%</th>
<th>9 – 12%</th>
<th>12 – 22%</th>
<th>22 – 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.02</td>
<td>5.92</td>
<td>4.55</td>
<td>3.65</td>
<td>2.36</td>
<td>0.16</td>
</tr>
<tr>
<td>0.03</td>
<td>7.93</td>
<td>5.55</td>
<td>4.15</td>
<td>3.26</td>
<td>2.02</td>
<td>0.12</td>
</tr>
<tr>
<td>0.1</td>
<td>7.45</td>
<td>5</td>
<td>3.48</td>
<td>2.35</td>
<td>1.16</td>
<td>0.04</td>
</tr>
<tr>
<td>0.2</td>
<td>6.19</td>
<td>3.42</td>
<td>1.84</td>
<td>1.06</td>
<td>0.36</td>
<td>0.005</td>
</tr>
</tbody>
</table>
The impact of $\mu$ on the prices of Forward Start CDOs is clear: the larger the $\mu$, the lower the price for all tranches. We could have expected this dependence, as the Loss Function path values decreased on average when we increased $\mu$.

<table>
<thead>
<tr>
<th>$\xi$ (Tranche)</th>
<th>0 – 3%</th>
<th>3 – 6%</th>
<th>6 – 9%</th>
<th>9 – 12%</th>
<th>12 – 22%</th>
<th>22 – 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.98</td>
<td>5.69</td>
<td>4.29</td>
<td>3.1</td>
<td>1.81</td>
<td>0.11</td>
</tr>
<tr>
<td>0.1</td>
<td>7.93</td>
<td>5.55</td>
<td>4.15</td>
<td>3.26</td>
<td>2.02</td>
<td>0.12</td>
</tr>
<tr>
<td>0.2</td>
<td>7.88</td>
<td>5.64</td>
<td>4.19</td>
<td>3.28</td>
<td>1.97</td>
<td>0.13</td>
</tr>
<tr>
<td>0.3</td>
<td>7.95</td>
<td>5.52</td>
<td>4.23</td>
<td>3.36</td>
<td>2.03</td>
<td>0.14</td>
</tr>
</tbody>
</table>

According to our analysis, the increase in $\xi$ implies the increase of the Loss Function values on average. This subsequently should imply an average increase of the prices of FSCDOs, which is true for the 3 most senior tranches. However we cannot observe such a relationship on the remaining tranches. This is probably due to the amount of randomness in our model.

<table>
<thead>
<tr>
<th>$\rho$ (Tranche)</th>
<th>0 – 3%</th>
<th>3 – 6%</th>
<th>6 – 9%</th>
<th>9 – 12%</th>
<th>12 – 22%</th>
<th>22 – 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.41</td>
<td>6.24</td>
<td>4.85</td>
<td>3.83</td>
<td>2.37</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>8.23</td>
<td>5.97</td>
<td>4.55</td>
<td>3.55</td>
<td>2.16</td>
<td>0.11</td>
</tr>
<tr>
<td>0.4</td>
<td>7.11</td>
<td>4.78</td>
<td>3.67</td>
<td>2.78</td>
<td>1.65</td>
<td>0.13</td>
</tr>
<tr>
<td>0.5</td>
<td>6.2</td>
<td>4.02</td>
<td>3.01</td>
<td>2.41</td>
<td>1.42</td>
<td>0.14</td>
</tr>
</tbody>
</table>

The prices of the Forward Start CDOs decrease with the increase of $\rho$ for all tranches, but the most senior tranche. We made the same observation for the STCDO prices.
Chapter 4

Conclusions

In this dissertation, we investigated the large market model with Stochastic Volatility determined by the Heston framework. After examining several combinations of model parameters, we decided to set the following as the standard collection of parameters: $\sigma_{\text{max}} = 0.6, \sigma_{\text{mean}} = 0.2, \mu = 0.03, \rho = 0.3, \kappa = 1, \xi = 0.1, \eta = 0.3, \phi = 0.5$. We obtained a two dimensional Stochastic Partial Differential Equation. This enabled us to define the loss of a basket of credit derivatives as a functional of the solution to the SPDE. We simulated our model using a Finite Difference discretization and Monte Carlo methods. We calculated prices of Single Tranche Collateral Debt Obligations (STCDOs) and Forward Start CDOs for varying model parameters.

According to our results it appears that for the chosen family of initial density functions $\{V_0^\epsilon\}_{\epsilon \in (0,1]}$, it does not matter asymptotically which function we choose. For a choice of two significantly different values of epsilon we observed a similar behaviour of the density process in time.

The loss function paths have to be increasing functions of time. The reason for this is that once a company, which is a part of the credit reference portfolio, defaults it does not re-enter the market. Our results have confirmed this. We could also observe a concave behaviour of the majority of the paths. Therefore, large losses occur early in time in our model.

The impact of varying $\sigma_{\text{mean}}, \eta, \phi$ or $\kappa$ on the Loss process turned out to be insignificant for pricing both STCDOs and FSCDOs.
The increase of $\sigma_{\text{max}}$ implied an increase of the values of the Loss paths on average. It decreased the variability as well. Therefore, the prices of both STCDOs and FSCDOs increased on average. This effect has not been very clear however for the Single Tranche CDO prices.

The increase of $\mu$ implied a clear decrease of the values of the Loss paths on average. As a consequence, the prices of both STCDOs and FSCDOs decreased.

The increase of $\xi$ implied an increase of the values of the Loss paths on average. Thus, the prices of STCDOs increased as a consequence. This dependency turned out to be also true for the prices corresponding to the 3 most senior tranches of the FSCDO. For the other tranches, we cannot make such a conclusion, which is most probably due to the model noise.

The increase of $\rho$ implied a clear increase of the variability of the Loss process. Therefore, we observed an evident decrease of the STCDO and FSCDO prices on average.

The future research based on our study could focus on calibrating the model to market data or extending the framework by a jump-diffusion or Levy process. One might also consider making model parameters, especially the ones having significant impact on the pricing, dependent on the loss. These solutions would ensure that there is more risk located in the most senior tranches, which would increase the values of corresponding par spreads.
Appendix A
Matlab Code

%% EVOLUTION OF HISTOGRAMS %%

N=200;

xBound=1; xMeshNr=50; dx=xBound/xMeshNr;
yBound =2.5; yMeshNr=50; dy=yBound/yMeshNr;

TimeHorizon =1; TimeSteps =2000; dt=TimeHorizon/TimeSteps;

LTmeanvol =0.2;
VolatilityMax =0.6; sigma=VolatilityMax/sqrt(yBound);
theta=LTmeanvol^2/sigma^2;

% Heston Model parameters
miu=0.03; rho = 0.3; K=1; xi=0.1; eta =0.3; phi =0.5;

Y=zeros(TimeSteps,N);
X=zeros(TimeSteps,N);

for i=1:N
Y(1,i)=1;
X(1,i)=0.4;
end

for t=1:TimeSteps−1
Yincr = sqrt(dt)*normrnd(0,1);
YincrIndep = sqrt(dt)*normrnd(0,1);
YincrIdiosyncr = sqrt(dt)*normrnd(0,1);
XincrIdiosyncr = sqrt(dt)*normrnd(0,1);

Y(t+1,i)=Y(t,i)+K*(theta−Y(t,i))*dt +
xi*eta*sqrt(Y(t,i))*Yincr+xi*sqrt(1−eta^2)*
YincrIdiosyncr;
Y(t+1,i) = \min(\max(Y(t+1,i),0),yBound) ;

X(t+1,i) = X(t,i) + (\miu - \sigma^2/2 \cdot Y(t,i)) \cdot dt +
\sigma \cdot \sqrt{Y(t,i)} \cdot \rho \cdot (\phi \cdot Yincr + \sqrt{1-\phi^2} \cdot YincrIndep) +
\sigma \cdot \sqrt{Y(t,i)} \cdot \sqrt{YincrIndep} ;

X(t+1,i) = \min(\max(X(t+1,i),0),xBound) ;

end

V = zeros(TimeSteps, xMeshNr, yMeshNr);
for i=1:4
    tt = floor(TimeSteps*i/4);
    for m=1:xMeshNr
        for k=1:yMeshNr
            if X(tt,1)>(m-1)*dx && X(tt,1)<(m)*dx &&
                Y(tt,1)>(k-1)*dy && Y(tt,1)<(k)*dy
                V(tt,m,k) = V(tt,m,k) + 1/(N*dx*dy);
            end
        end
    end
end

MyHistogram = zeros(xMeshNr, yMeshNr);
for i=1:4
    tt = floor(TimeSteps*i/4);
    for m=1:xMeshNr
        for k=1:yMeshNr
            MyHistogram(m,k) = V(tt,m,k);
        end
    end
end

figure(i)
colormap(0.9*[1 1 1])
bar3(MyHistogram);
xlabel('X', 'FontSize', 18, 'FontWeight', 'bold', 'Color', 'k');
ylabel('Y', 'FontSize', 18, 'FontWeight', 'bold', 'Color', 'k');
set(gca, 'XTickLabel', [xBound/5, xBound * 2/5, xBound * 3/5, xBound * 4/5, xBound]);
set(gca, 'YTickLabel', [0, yBound/5, yBound * 2/5, yBound * 3/5, yBound * 4/5, yBound]);
end

% % % % % % % % % % % % % % EVOLUTION OF DENSITIES % % % % % % % % % % % % % %

%% initialFunction = the pyramid function
epsilon = 0.4;
xBoundInitial = epsilon * xBound;
xMeshNrInitial = epsilon * xMeshNr;
yBoundInitial = epsilon * yBound;
yMeshNrInitial = epsilon * yMeshNr;

initialFunction = zeros(xMeshNrInitial+1, yMeshNrInitial+1);
for x=1:xMeshNrInitial+1
    for y=1:yMeshNrInitial+1

        if (y-1)/yMeshNrInitial*epsilon*yBoundInitial <=
           -yBoundInitial/xBoundInitial*(x-1)/xMeshNrInitial*
           epsilon*xBoundInitial + epsilon*yBoundInitial &&
           (y-1)/yMeshNrInitial*epsilon*yBoundInitial <=
           yBoundInitial/xBoundInitial*(x-1)/xMeshNrInitial*
           epsilon*xBoundInitial

           initialFunction(x, y) = (y-1)/yMeshNrInitial*
           epsilon*yBoundInitial*
           6/(epsilon^3*xBoundInitial*yBoundInitial^2);

        end

        if (y-1)/yMeshNrInitial*epsilon*yBoundInitial >
           -yBoundInitial/xBoundInitial*(x-1)/xMeshNrInitial*
           epsilon*xBoundInitial + epsilon*yBoundInitial &&

    end
end

end

\[(y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} < y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}}\]

\begin{align*}
\text{initialFunction}(x,y) &= -6/ \varepsilon^3 x_{\text{BoundInitial}}^2 y_{\text{BoundInitial}} (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + 6/ \varepsilon^2 x_{\text{BoundInitial}} y_{\text{BoundInitial}} ;
\end{align*}

\text{end}

if \((y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} \geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} \) &&
\((y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} \geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} \)
\text{initialFunction}(x,y) &= -6/ \varepsilon^3 x_{\text{BoundInitial}}^2 y_{\text{BoundInitial}} (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + 6/ \varepsilon^2 x_{\text{BoundInitial}} y_{\text{BoundInitial}} ;
\text{end}

endif

\text{end}

\text{Density} = \text{zeros}(\text{TimeSteps}+1,\text{MeshNr}+1, \text{MeshNr}+1) ;

%%%3

\text{if } (y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} \geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} &&
\((y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} \geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} \)
\text{initialFunction}(x,y) &= -6/ \varepsilon^3 x_{\text{BoundInitial}}^2 y_{\text{BoundInitial}} (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + 6/ \varepsilon^2 x_{\text{BoundInitial}} y_{\text{BoundInitial}} ;
\text{end}

endif

%%%4

\text{if } (y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} 
\geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} \) &&
\((y-1)/y_{\text{MeshNrInitial}} \times \varepsilon \times y_{\text{BoundInitial}} 
\geq -y_{\text{BoundInitial}}/x_{\text{BoundInitial}} \times (x-1)/x_{\text{MeshNrInitial}} \times \varepsilon \times x_{\text{BoundInitial}} + \varepsilon \times y_{\text{BoundInitial}} \)
\text{initialFunction}(x,y) &= 6/ \varepsilon^3 x_{\text{BoundInitial}}^2 y_{\text{BoundInitial}} ;
\text{end}

end

\text{end}

%%%The sequence of densities:
\text{Density} = \text{zeros}(\text{TimeSteps}+1,\text{MeshNr}+1, \text{MeshNr}+1) ;

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\begin{verbatim}
trialstart = zeros(xMeshNr+1, yMeshNr+1);
for x=1:xMeshNrInitial+1
  for y=1:yMeshNrInitial+1
    trialstart(x,y) = initialFunction(x,y)/sum(sum(initialFunction));
  end
end
time = 0:dt:TimeHorizon;
Density(1,:,:)=trialstart;

for t=1:TimeSteps
  MSincr=sqrt(dt)*normrnd(0,1);
  MSincrIndep = sqrt(dt)*normrnd(0,1);
  MYincr = phi*MSincr + sqrt(1-phi^2)*MSincrIndep;
  for y=2:yMeshNr
    for x=2:xMeshNr
      %Drift Terms
      ddxDriftT = -(miu - 0.5*sigma^2*(y-1)/yMeshNr*yBound)*
                   (Density(t,x+1,y)-Density(t,x,y))/(dx);
      ddyDriftT = K*Density(t,x,y) +
                  K*((y-1)/yMeshNr*yBound-theta)*
                   (Density(t,x,y+1)-Density(t,x,y))/(dy);
      d2dx2DriftT = 0.5*sigma^2*(y-1)/yMeshNr*yBound*
                    (Density(t,x+1,y)-2*Density(t,x,y)+Density(t,x-1,y))/(dx^2);
      d2dy2DriftT = 0.5*xi^2*
                    (2*(Density(t,x,y+1)-Density(t,x,y))/(dy) + (y-1)/yMeshNr*yBound*
                     (Density(t,x,y+1)-2*Density(t,x,y)+Density(t,x,y-1))/(dy^2));
      MixedDriftT = xi*eta*sigma*rho*phi*
                    ((Density(t,x+1,y) - Density(t,x,y))/(dx) +
                     (y-1)/yMeshNr*yBound*(Density(t,x+1,y+1)-Density(t,x-1,y+1)-
                     Density(t,x+1,y-1)+Density(t,x-1,y-1))/(4*dx*dy));
\end{verbatim}
%Noise Terms

ddxNoiseT = - sigma*rho*sqrt(((y-1)/yMeshNr*yBound)*
(Density(t,x+1,y)-Density(t,x,y))/(dx));

ddyNoiseT = -xi*eta*
(0.5*Density(t,x,y)/sqrt(((y-1)/yMeshNr*yBound)*
+sqrt(((y-1)/yMeshNr*yBound)*
(Density(t,x,y+1)-Density(t,x,y))/(dy)));

value=Density(t,x,y) + (ddxDriftT + ddyDriftT + d2dx2DriftT +
d2dy2DriftT + MixedDriftT)*dt +
ddxNoiseT*MSincr +ddyNoiseT*MYincr;

Density(t+1,x,y) = max(value,0);
end
end
end

[X,Y] = meshgrid(0:dx:xBound, 0:dy:yBound);
figure(1)
surf(X,Y, trialstart);
xlabel('X', 'FontSize',18, 'FontWeight','bold', 'Color','k');
ylabel('Y', 'FontSize',18, 'FontWeight','bold', 'Color','k');

for i=2:5
MyDensity = zeros(xMeshNr+1, yMeshNr+1);
for y=2:yMeshNr
for x=2:xMeshNr
MyDensity(x,y) = Density(floor(0.25*(i-1)*TimeSteps),x,y);
end
end

figure(i)
surf(X,Y, MyDensity);
xlabel('X', 'FontSize',18, 'FontWeight','bold', 'Color','k');
ylabel('Y', 'FontSize',18, 'FontWeight','bold', 'Color','k');
end

% % % % % % % % % % % % % % LOSS FUNCTIONS % % % % % % % % % % % % %

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%The sequence of densities:
DensityNEW = zeros(xMeshNr+1, yMeshNr+1);
DensityOLD = zeros(xMeshNr+1, yMeshNr+1);

NrLossPaths = 7;
LossFunction = zeros(NrLossPaths, TimeSteps+1);

time = 0:dt:TimeHorizon;

for p=1:NrLossPaths
    DensityNEW = trialstart;
    for t=1:TimeSteps
        DensityOLD = DensityNEW;
        MSincr = sqrt(dt)*normrnd(0,1);
        MSincrIndep = sqrt(dt)*normrnd(0,1);
        MYincr = phi*MSincr + sqrt(1−phi^2)*MSincrIndep;
        for y=2:yMeshNr
            for x=2:xMeshNr
                %Drift Terms
                ddxDriftT = -(miu − 0.5*sigmaˆ2*(y−1)/yMeshNr*yBound)∗
                        (DensityOLD(x+1,y)−DensityOLD(x,y))/(dx);
                ddyDriftT = K*DensityOLD(x,y) +
                        K*((y−1)/yMeshNr*yBound−theta)∗
                        (DensityOLD(x,y+1)−DensityOLD(x,y))/(dy);
                d2dx2DriftT = 0.5*sigma^2*(y−1)/yMeshNr*yBound∗
                        (DensityOLD(x+1,y)−2*DensityOLD(x,y)+
                        DensityOLD(x−1,y))/(dx^2);
                d2dy2DriftT = 0.5*xi^2*
                        (2*(DensityOLD(x,y+1)−DensityOLD(x,y))/(dy)+
                        (y−1)/yMeshNr*yBound∗
                        (DensityOLD(x,y+1)−2*DensityOLD(x,y)+
                        DensityOLD(x,y−1))/(dy^2));
                MixedDriftT = xi*eta*sigma*rho*phi∗
                ((DensityOLD(x+1,y)−DensityOLD(x,y))/(dx)+
                (y−1)/yMeshNr*yBound∗
                (DensityOLD(x+1,y+1)−DensityOLD(x−1,y+1)−
                DensityOLD(x+1,y−1)+DensityOLD(x−1,y−1))/(4*dx*dy));
            end
        end
    end
% Noise Terms

\[ ddx_{\text{NoiseT}} = - \sigma \rho \sqrt{\frac{(y-1)/y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}{y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}} \times \frac{(\text{Density}_{\text{OLD}}(x+1,y) - \text{Density}_{\text{OLD}}(x,y))}{(dx)}; \]

\[ ddy_{\text{NoiseT}} = -\xi \eta \left( 0.5 \times \frac{\text{Density}_{\text{OLD}}(x,y)}{\sqrt{\frac{(y-1)/y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}{y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}}} + \sqrt{\frac{(y-1)/y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}{y_{\text{Mesh Nr}} \cdot y_{\text{Bound}}}} \times \frac{(\text{Density}_{\text{OLD}}(x,y+1) - \text{Density}_{\text{OLD}}(x,y))}{(dy)} \right); \]

\[ \text{value} = \text{Density}_{\text{OLD}}(x,y) + (ddx_{\text{DriftT}} + ddy_{\text{DriftT}} + d^2dx^2_{\text{DriftT}} + d^2dy^2_{\text{DriftT}} + \text{Mixed}_{\text{DriftT}}) \times dt + ddx_{\text{NoiseT}} \times M_{\text{Sincr}} + ddy_{\text{NoiseT}} \times M_{\text{Yincr}}; \]

\[ \text{Density}_{\text{NEW}}(x,y) = \max(\text{value},0); \]

end
end

LossFunction(p, t+1) = 1 - \text{sum}(\text{sum}(\text{Density}_{\text{NEW}}));

end
end

figure(1)
plot(time, LossFunction(1,:), 'k');
hold on
plot(time, LossFunction(2,:), 'g');
hold on
plot(time, LossFunction(3,:), 'b');
hold on
plot(time, LossFunction(4,:), 'y');
hold on
plot(time, LossFunction(5,:), 'm');
hold on
plot(time, LossFunction(6,:), 'c');
hold on
plot(time, LossFunction(7,:), 'r');
xlabel('t', 'FontSize', 18, 'FontWeight', 'bold', 'Color', 'k');
ylabel('Loss', 'FontSize', 18, 'FontWeight', 'bold', 'Color', 'k');
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% Densities:
DensityOLD = zeros(xMeshNr+1, yMeshNr+1);
DensityNEW = zeros(xMeshNr+1, yMeshNr+1);
NrLossPaths = 500;
FinalDistribution = zeros(1, NrLossPaths);

for p=1:NrLossPaths
    DensityNEW = trialstart;
    for t=1:TimeSteps
        DensityOLD = DensityNEW;
        MSincr = sqrt(dt)*normrnd(0,1);
        MSincrIndep = sqrt(dt)*normrnd(0,1);
        MYincr = phi*MSincr + sqrt(1-phi^2)*MSincrIndep;
        for y=2:yMeshNr
            for x=2:xMeshNr
                % Drift Terms
                ddxDriftT = -(mu - 0.5*sigma^2*(y-1)/yMeshNr*yBound)*
                    (DensityOLD(x+1,y)-DensityOLD(x,y))/(dx);
                ddyDriftT = K* DensityOLD(x,y) +
                    K*((y-1)/yMeshNr*yBound-theta)*
                    (DensityOLD(x,y+1)-DensityOLD(x,y))/(dy);
                d2dx2DriftT = 0.5*sigma^2*(y-1)/yMeshNr*yBound*
                    (DensityOLD(x+1,y)-2*DensityOLD(x,y)+
                        DensityOLD(x-1,y))/(dx^2);
                d2dy2DriftT = 0.5*xi^2*2*(DensityOLD(x,y+1)-
                    DensityOLD(x,y))/(dy)+ (y-1)/yMeshNr*yBound*
                    (DensityOLD(x,y+1)-2*DensityOLD(x,y)+
                        DensityOLD(x,y-1))/(dy^2));
                MixedDriftT = xi*eta*sigma*rho*phi*
                    ((DensityOLD(x+1,y) - DensityOLD(x,y)))/(dx)+
                    (y-1)/yMeshNr*yBound*(DensityOLD(x+1,y+1)-
                        DensityOLD(x,y+1))*/
        end
    end
end

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DensityOLD (x-1,y+1) - DensityOLD (x+1,y-1) +
DensityOLD (x-1,y-1) + (4*dx*dy) ;

% Noise Terms
ddxNoiseT = - sigma*rho*sqrt ((y-1)/yMeshNr*yBound)*
(DensityOLD(x+1,y)-DensityOLD (x,y))/(dx);

ddyNoiseT = - xi*eta*
(0.5*DensityOLD(x,y)/sqrt ((y-1)/yMeshNr*yBound)
+sqrt ((y-1)/yMeshNr*yBound)*
(DensityOLD(x,y+1)-DensityOLD (x,y))/(dy));

value = DensityOLD (x,y) + (ddxDriftT + ddyDriftT +
d2dx2DriftT + d2dy2DriftT + MixedDriftT)*dt +
ddxNoiseT*MSincr +ddyNoiseT*MYincr;

DensityNEW(x,y) = max(value,0);

end

end

end

FinalDistribution (p) = 1 - sum(sum(DensityNEW));
end

[f,x] = ksdensity (FinalDistribution);
plot(x,f,'k');
theMean = mean(FinalDistribution)
TheStdDev = std(FinalDistribution)

% % % % % % % % % % % % % % PRICING STCDOs % % % % % % % % % % % % %

nrSimulations = 500;
The final density:

DensityNEW = zeros(xMeshNr+1, yMeshNr+1);
DensityOLD = zeros(xMeshNr+1, yMeshNr+1);

%attachment and detachment points: 1–most senior, 6–equity
attachpts = zeros(1,6);
detachpts = zeros(1,6);
attachpts(1) = 0.22;
attachpts(2) = 0.12;
attachpts(3) = 0.09;
attachpts(4) = 0.06;
attachpts(5) = 0.03;
attachpts(6) = 0;

detachpts(1) = 1;
detachpts(2) = 0.22;
detachpts(3) = 0.12;
detachpts(4) = 0.09;
detachpts(5) = 0.06;
detachpts(6) = 0.03;

s = zeros(1,6); %vector of par spreads
r = miu; %interest rate

for tranche = 1:6
    LossFunction = zeros(1, TimeSteps+1);
    OutstandingTrancheNotionalValue = zeros(1, TimeSteps+1);
    OutstandingTrancheNotionalValue(1) =
        detachpts(tranche) − attachpts(tranche);

    for i = 1:nrSimulations
        DensityNEW = trialstart;
        for t = 1:TimeSteps
            DensityOLD = DensityNEW;
            MSincr = sqrt(dt)∗normrnd(0,1);
            MSincrIndep = sqrt(dt)∗normrnd(0,1);
MYincr = phi*MSincr + sqrt(1-phi^2)*MSincrIndep;

for y=2:yMeshNr
for x=2:xMeshNr

%Drift Terms
ddxDriftT = -(miu - 0.5*sigma^2*(y-1)/yMeshNr*yBound)*
(DensityOLD(x+1,y)-DensityOLD(x,y))/(dx);

ddyDriftT = K*(y-1)/yMeshNr*yBound-theta*
(DensityOLD(x,y+1)-DensityOLD(x,y))/(dy);

d2dx2DriftT = 0.5*sigma^2*(y-1)/yMeshNr*yBound*
(DensityOLD(x+1,y)-2*DensityOLD(x,y)+
DensityOLD(x-1,y))/(dx^2);

d2dy2DriftT = 0.5*xi^2*
(2*(DensityOLD(x,y+1)-DensityOLD(x,y))/(dy)+
(y-1)/yMeshNr*yBound*
(DensityOLD(x,y+1)-2*DensityOLD(x,y)+
DensityOLD(x,y-1))/(dy^2));

MixedDriftT = xi*eta*sigma*rho*phi*
((DensityOLD(x+1,y) - DensityOLD(x,y))/(dx)+
(y-1)/yMeshNr*yBound*(DensityOLD(x+1,y+1)-DensityOLD(x-1,y+1)
- DensityOLD(x+1,y-1)+DensityOLD(x-1,y-1))/(4*dx*dy));

%Noise Terms
ddxNoiseT = -sigma*rho*sqrt((y-1)/yMeshNr*yBound)*
(DensityOLD(x+1,y)-DensityOLD(x,y))/(dx);

ddyNoiseT = -xi*eta*
(0.5*DensityOLD(x,y)/sqrt((y-1)/yMeshNr*yBound)
+sqrt((y-1)/yMeshNr*yBound)*(DensityOLD(x,y+1)-
DensityOLD(x,y))/(dy));

value=DensityOLD(x,y) + (ddxDriftT + ddyDriftT + d2dx2DriftT
+ d2dy2DriftT + MixedDriftT)*dt + ddxNoiseT*MSincr +
ddyNoiseT*MYincr;

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DensityNEW(x, y) = max(value, 0);

end
end

LossFunction(t+1) = 1 - sum(sum(DensityNEW));
OutstandingTrancheNotionalValue(t+1) =
OutstandingTrancheNotionalValue(t+1) +
(max(detachpts(tranche) - LossFunction(t+1), 0) -
max(attachpts(tranche) - LossFunction(t+1), 0))/nrSimulations;
end
end

P=0; F=0;
for i=1:TimeSteps
P=P+(OutstandingTrancheNotionalValue(i) -
OutstandingTrancheNotionalValue(i+1))/exp(r*(i+1)*dt);
F=F+dt*OutstandingTrancheNotionalValue(i+1)/exp(r*(i+1)*dt);
end

s(tranche)=P/F;
end
s=vpa(s,5)

% % % % % % % % % % % % % % PRICING FSCDOs % % % % % % % % % % % % %

%%Date for theFSCDO (ie 1 month)
EpsilonDate = 1/12;
DateNr = max(floor(EpsilonDate*TimeSteps),1);

DensityNEW = zeros(xMeshNr+1, yMeshNr+1);
DensityOLD = zeros(xMeshNr+1, yMeshNr+1);

for tranche=1:6
    LossFunction = zeros(1, TimeSteps+1);
end
LossFunctionFS = zeros(1,TimeSteps+1);
NotionalFS = zeros(1,TimeSteps+1);
ExpectedNotionalFS = zeros(1,TimeSteps+1);

for i=1:nrSimulations
    DensityNEW=trialstart;
    for t=1:TimeSteps
        DensityOLD=DensityNEW;
        MSincr=sqrt(dt)*normrnd(0,1);
        MSincrIndep = sqrt(dt)*normrnd(0,1);
        MYincr =phi*MSincr + sqrt(1-phi^2)*MSincrIndep;
        for y=2:yMeshNr
            for x=2:xMeshNr
                %Drift Terms
                ddxDriftT = -(miu - 0.5*sigma^2*(y-1)/yMeshNr*yBound)*
                (DensityOLD(x+1,y)-DensityOLD(x,y))/(dx);
                ddyDriftT = K* DensityOLD(x,y) +
                K*((y-1)/yMeshNr*yBound-theta)*
                (DensityOLD(x,y+1)-DensityOLD(x,y))/(dy);
                d2dx2DriftT = 0.5*sigma^2*(y-1)/yMeshNr*yBound*
                (DensityOLD(x+1,y)-2*DensityOLD(x,y)+
                DensityOLD(x-1,y))/(dx^2);
                d2dy2DriftT = 0.5*xi^2*(2*(DensityOLD(x,y+1)-
                DensityOLD(x,y))/(dy)+(y-1)/yMeshNr*yBound*  
                (DensityOLD(x,y+1)-2*DensityOLD(x,y)+
                DensityOLD(x,y-1))/(dy^2));
                MixedDriftT = xi*eta*sigma*rho*phi*
                ((DensityOLD(x+1,y) - DensityOLD(x,y))/(dx)+
                (y-1)/yMeshNr*yBound*(DensityOLD(x+1,y+1)-DensityOLD(x-1,y+1)  
                -DensityOLD(x+1,y-1)+DensityOLD(x-1,y-1))/(4*dx*dy));

        %Noise Terms
        ddxNoiseT = - sigma*rho*sqrt((y-1)/yMeshNr*yBound)*

    end
end
\[
\frac{\text{Density}_{\text{OLD}}(x+1,y) - \text{Density}_{\text{OLD}}(x,y)}{dx};
\]
\[
\frac{\text{Density}_{\text{OLD}}(x,y)}{\sqrt{(y-1)/y_{\text{Mesh Nr}}*y_{\text{Bound}}}} + \sqrt{(y-1)/y_{\text{Mesh Nr}}*y_{\text{Bound}})*
\]
\[
\frac{\text{Density}_{\text{OLD}}(x,y+1) - \text{Density}_{\text{OLD}}(x,y)}{dy};
\]
\[
\text{value} = \text{Density}_{\text{OLD}}(x,y) + (\text{ddxDriftT} + \text{ddyDriftT} + \text{d2dx2DriftT} + \text{MixedDriftT})*dt + \text{ddxNoiseT}*\text{MSincr} + \text{ddyNoiseT}*\text{MYincr};
\]
\[
\text{DensityNEW}(x,y) = \max(\text{value},0);
\]
\[
\text{LossFunction}(t+1) = 1 - \sum(\sum(\text{DensityNEW}));
\]
\[
\text{for} \ t = 1: \text{TimeSteps+1}
\]
\[
\text{LossFunctionFS}(t) = \text{LossFunction}(t) - \text{LossFunction}(	ext{DateNr});
\]
\[
\text{NotionalFS}(t) = \max(\text{detachpts} \ (\text{tranche}) - \text{LossFunctionFS}(t),0) - \max(\text{attachpts} \ (\text{tranche}) - \text{LossFunctionFS}(t),0);
\]
\[
\text{ExpectedNotionalFS}(t) = \text{ExpectedNotionalFS}(t) + \text{NotionalFS}(t)/\text{nrSimulations};
\]
\[
\text{end}
\]
\[
\text{end}
\]
\[
P=0; \quad F=0;
\]
\[
\text{for} \ i = 1: \text{TimeSteps}
\]
\[
P=P+(\text{ExpectedNotionalFS} \ (i) - \text{ExpectedNotionalFS}(i+1))/\exp(r*(i+1)*dt);
\]
\[
F=F+dt*\text{ExpectedNotionalFS}(i+1)/\exp(r*(i+1)*dt);
\]
\[
\text{end}
\]
\[
\text{s(\text{tranche}) = P/F};
\]
\[
\text{end}
\]
\[
s=\text{vpa}(s,5)
\]
Bibliography


