On sensitivities in utility-based hedging
and parameter uncertainty

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Abstract

We study utility-based hedging and indifference pricing in a basis risk model where we wish to hedge a claim on a non-traded asset by trading a correlated traded asset. We examine the sensitivities of such trading strategies to relevant model parameters and we give first-order approximation formulae of the partial derivatives of the indifference price and hedging strategy with respect to these parameters. We also study the effect of the drift parameter mis-estimation on the performance of the optimal strategy.
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Chapter 1

Introduction

1.1 Problem description and thesis contribution

In a complete market, all risk associated with a contingent claim can be hedged away by trading a suitable perfectly replicating portfolio. This goes back to the seminal work of Black and Scholes [1] and Merton [2]. The result is that claim prices are discounted expectations of the payoff under a unique risk neutral measure, which renders discounted traded asset prices martingales.

Some real life markets are incomplete and options on non-traded assets are common such as weather derivatives, stock indices and sports betting. In an incomplete market, by definition, not all risk can be eliminated. Risk preferences must therefore play a role. Hence, optimal hedging becomes synonymous with optimal investment. This means that indifference prices and hedging strategies must unavoidably show dependence on drift parameters of assets and not just on volatilities. This is to be contrasted with the complete market situation (e.g. Black-Scholes) where we only need knowledge of the volatilities. The drift dependence is problematic since it is not possible to obtain an accurate
estimate of the drift as discussed by Rogers [3] and Monoyios [4].

One way to introduce risk preferences is by embedding the valuation and hedging problem for a claim into a utility maximization framework. This leads to the concept of indifference valuation and hedging. In this paradigm, an agent computes prices of claims which renders his expected utility to be unchanged relative to a problem without the claims.

This thesis is an analysis of the sensitivities of utility-based valuation and hedging to model inputs such as drifts, volatilities and correlation. While drift uncertainty is more serious than volatility and correlation uncertainty, it does not necessarily mean that the sensitivity of such strategies to drift parameters is larger than their sensitivities to the other parameters. Hence, we study the sensitivities of indifference valuation and hedging to errors in estimating the drift parameters and we compare it to their respective sensitivities to mis-estimation of the volatilities in the basis risk model.

### 1.2 Literature review

The idea of utility-based trading strategies and indifference pricing goes back to Hodges and Neuberger [5]. They used utility maximization to derive an optimal hedging strategy and pricing of contingent claims in the context of markets with transaction costs. Later, the same pricing procedure has been used in other incomplete market models such as basis risk models with a traded and a non-traded lognormal asset [6–8], and stochastic volatility models [9].

An alternative approach to utility maximization is the quadratic hedging approach where the agent finds the optimal hedging strategy that minimizes, for instance, the variance of the terminal hedging error [10]. The main drawback of such approaches is that they give equal weights to losses and gains from a
given hedging strategy.
It was shown in [11] that, when all model parameters are known, the optimal utility-based hedging strategy has a superior performance to that of the Black-Scholes strategy in terms of the distribution of the terminal hedging error. However, it turns out that both the indifference valuation and hedging are dependent on the drifts of the underlying assets. This is also the case for the optimal Markowitz [12] and Merton [2] strategies. Monoyios [4] showed that drift parameter mis-estimation in utility-based strategies can be destructive depending on the sign of the mis-estimation which we will cover more thoroughly in Chapter 3.

1.3 Thesis overview

The thesis is organized as follows. In Chapter 2, we briefly describe the basis risk model and we define important concepts in utility-based pricing and hedging. We also recall some key results for indifference pricing, optimal hedging, and residual risk mostly known from literature [4, 11] and we provide more insight about these results such as their dependence on correlation. We also examine and interpret the performance of the optimal strategy under different correlation regimes (low correlation, medium correlation and high correlation) and we compare it to the performance of the naive Black-Scholes type strategy. In Chapter 3, we briefly explain the challenge in obtaining an accurate estimate of the drift parameter and why it is more complicated than estimating volatilities. We then provide general expressions for the partial derivatives of the first order approximation of the indifference price and hedging strategy to model parameters and we specialize these expressions to the case when the contin-
gent claim is a put option on the non-traded asset. We also concretely study these sensitivities by plotting them for sensible choice of model parameters. Finally, we study the effect of parameter mis-estimation on the performance of the optimal strategy and we show how the sign of the mis-estimation plays a big role in the effect it has on the performance. We derive the dynamics of the residual risk process when parameter mis-estimation is involved, and we demonstrate numerically how the performance of the utility-based hedging strategy is affected if the agent uses erroneous parameters.
Chapter 2

The Basis Risk Model

2.1 Model description and important definitions

We will use the same notation in [11]. We have one traded asset $S$ and a non-traded asset $Y$. Their dynamics are given by:

$$dS_t = S_t(\mu dt + \sigma dW_t),$$  \hspace{1cm} (2.1)

$$dY_t = Y_t(\mu_0 dt + \eta dW_t^0),$$  \hspace{1cm} (2.2)

for $0 \leq t \leq T$. The driving Brownian motion of the non-traded asset $Y$ is

$$W_t^0 = \rho W_t + \sqrt{(1-\rho^2)}W_t^\perp,$$

where $W_t^\perp$ is a Brownian motion independent of $W_t$, and $\rho \in [-1,1]$ is the correlation parameter between the two assets.

The parameters $\mu, \mu_0, \sigma > 0, \eta > 0$ are constants representing the drifts and volatilities of both assets. We assume that the risk free interest rate is a
constant $r \geq 0$. The trading strategy is a stochastic process $\pi = (\pi_t)_{0 \leq t \leq T}$ which represents the amount of money invested in the traded stock at time $t$.

The wealth process dynamics are given by

$$dX_t = (rX_t + \pi_t (\mu - r)) \, dt + \pi_t \sigma \, dW_t$$

We assume in the following that the agent uses an exponential utility function

$$U(x) = -\exp(-\gamma x),$$

where $\gamma > 0$ is the constant absolute risk aversion parameter. We also assume that we have a European contingent claim on the non-traded asset $Y$ with bounded payoff $h(Y_T)$ at maturity. The agent wants to maximize his expected utility from terminal wealth and from trading $n$ claims.

$$u(n)(x, y, t) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{E} \left[ U (X_T + nh (Y_T)) \mid X_t = x, Y_t = y \right],$$

where $\mathcal{A}(x)$ is the set of admissible strategies starting with a time $t$ wealth $x \geq 0$. A strategy is called admissible if it satisfies $\int_0^T \pi_t^2 \, dt < \infty$. We denote by $\pi^{*,n} = (\pi^{*,n}_t)_{0 \leq t \leq T}$ the optimal strategy that achieves the supremum in (2.4).

**Definition 2.1 (Indifference Price)** The indifference price per claim $p^{(n)}(t, x, y)$ is defined by

$$u^{(n)}(t, x - np^{(n)}(t, x, y), y) = u^{(0)}(t, x, y).$$

Hence, the indifference price is the amount of money paid per claim from the initial wealth $x$, such that the expected terminal utility when trading the $n$ claims is equal to expected terminal utility when no claims are traded and starting from the same initial wealth.
**Definition 2.2 (Optimal hedging strategy)** We define the optimal hedging strategy for trading \( n \) claims as follows

\[
\pi_{h,n}^t = \pi_{t}^{*,n} - \pi_{t}^{*,0}, \quad 0 \leq t \leq T.
\]

In other words, the hedging strategy is defined as the difference between the optimal trading strategy when \( n \) claims are traded and the optimal strategy when no claims are traded. While this definition is natural, it is well known that indifference pricing leads to a non-linear pricing rule (as we will show in Section 2.2). This might raise some doubts regarding the linearity in the definition of the hedging strategy. Nevertheless, we will follow this definition throughout.

**Definition 2.3 (Minimal martingale measure)** The minimal martingale measure \( Q^M \) is given by

\[
\frac{dQ^M}{dP} |_{\mathcal{F}_t} = \mathcal{E}(-\lambda W)_t, \quad 0 \leq t \leq T, \tag{2.5}
\]

where

\[
\lambda = \frac{\mu - r}{\sigma}
\]

and \( \mathcal{E}(\cdot) \) denotes the stochastic exponential.

\( Q^M \in \mathcal{M} \) where \( \mathcal{M} \) is the set of equivalent martingale measures \( Q \) with finite relative entropy with respect to \( P \).

In general, equivalent martingale measures \( Q \in \mathcal{M} \) have density processes

\[
Z_t := \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \mathcal{E}(-\lambda W - \Psi W^\perp)_t,
\]
where $\Psi.\mathcal{W}^\perp := \int_0^t \Psi_s \mathcal{W}_s^\perp \, ds$, and $\Psi$ is any adapted process such that:

- $\int_0^T \Psi_t \, dt < \infty$.
- $Z$ is a martingale.

The minimal martingale measure, which corresponds to $\Psi = 0$, will appear in some of the equations in later sections. While all martingale measures in the basis risk model eliminate the drift in the discounted traded asset price, different choices of $\Psi$ deal differently with the drift part in the non-traded asset. In particular, the minimal martingale measure completely ignores to the drift part and any risk in the non-traded asset, hence the name.

### 2.2 Valuation and hedging

In this section, we will only include the final expressions of the indifference price, optimal hedging strategy, and residual risk process. Interested readers are encouraged to read relevant sections in [4, 11] for the complete derivation of these results.

**Lemma 2.1 (Indifference price)** The indifference price starting at time $t \in [0, T]$ with a wealth $x$ and a non-traded asset price $y$, and using the exponential utility in (2.3) is given by

$$p^{(n)}(t, x, y) \equiv p^{(n)}(t, y) = -\frac{e^{-r(T-t)}}{\gamma n(1 - \rho^2)} \log \mathbb{E}^{\mathbb{Q}^M} \left[ e^{-\gamma n(1 - \rho^2)h(Y_T)} \mid Y_t = y \right]. \quad (2.6)$$

Note that the indifference price does not depend on the initial wealth $x$. This is a well-known property of indifference price under the exponential utility. Indifference prices under other utility functions might depend on the initial
wealth. The indifference price has a volume-scaling property (see, for instance, Mania and Schweizer [13, Proposition 5]), i.e., \( p^{(n)}(t, y, \gamma) = p^{(1)}(t, y, n\gamma) \) where \( p^{(n)}(t, y, \gamma) \) is the indifference price for the exponential utility with risk aversion parameter \( \gamma \). As mentioned earlier, we can clearly see the utility-based pricing in this case is not linear unlike risk-neutral pricing methods. Hence, the price of a single claim varies depending on the amount purchased/sold as shown in Fig. 2.1. In fact, the theoretical dependence on \( n \) suggests that the ask price \( (n < 0) \) is greater than the bid price \( (n > 0) \). This is a typical behavior observed in real trades, since traders want to buy at a low price and sell at a high price.

Moreover, we observe another interesting dependency between price and correlation. The pricing of a single claim becomes more linear (flat curve vs. \( n \)) as the correlation \( \rho \) approaches 1. In fact, in the limit as \( \rho \to 1 \), the market be-
comes almost complete, the minimal martingale measure becomes the unique equivalent martingale measure and hence the indifference price approaches the Black-Scholes price of the claim which is linear. We can see that more rigorously from the indifference price formula in (2.6). By doing a first order Taylor expansion of the exponential inside the expectation and then of the logarithm outside it, we obtain that

\[ p^{(n)}(t, y) \to e^{-r(T-t)}E^{Q^M} [h(Y_T) \mid Y_t = y] \]

\[ := p^M(t, y), \]

where \( p^M(t, y) \) is the marginal price which we will define now.

**Lemma 2.2 (Marginal price)** The marginal price of a contingent claim on an asset \( Y \) with value \( y \) at time \( t \) is given by

\[ p^M(t, y) = e^{-r(T-t)}E^{Q^M} [h(Y_T) \mid Y_t = y] \]  \hspace{1cm} (2.7)

The original definition of the marginal price goes back to Davis [14] and corresponds to indifference prices with small \( n \). In fact, the marginal price is the first order term in the power series expansion of the indifference price \( p^{(n)}(t, y) \). It is a good approximation of the indifference price when the term \( \gamma n(1 - \rho^2)h(Y_T) \) is small. This is shown in Fig. 2.2 where we can see that the difference between the indifference price and the marginal price is negligible when \( \rho \) is close to 1 (\( \gamma n(1 - \rho^2)h(Y_T) \ll 1 \)). Another observation is that the indifference ask price is always larger than the marginal price, whereas the bid price is smaller than the marginal price. In the limit when \( |\rho| \to 1 \), the we get the Black-Scholes price. The difference between the indifference price and the
Figure 2.2: The difference between the indifference price of a single put option $p^{(1)}$ and the marginal price $p^M$ with respect to the initial spot price. The parameters are in Table 2.1 and the strike of the put option is $K = 100$.

marginal price increases when the absolute value of the correlation coefficient $|\rho|$ decreases.

**Lemma 2.3 (Optimal trading and hedging strategies)** The optimal trading strategy $\pi^{*,n}(t, Y_t)$ for buying $n$ contingent claims on the non-traded asset $Y$ with value $y$ at time $t$ is given by

$$
\pi^{*,n}(t, y) = e^{-r(T-t)} \frac{\lambda}{\sigma} - \frac{n\rho y}{\sigma} p_y^{(n)}(t, y),
$$

where

$$
p_y^{(n)}(t, y) = \frac{\partial}{\partial y} p_y^{(n)}(t, y).
$$

Applying Definition 2.2, the expression of the optimal hedging strategy is

$$
\pi^{h,n}(t, y) = -\frac{n\rho y}{\sigma} p_y^{(n)}(t, y).
$$

We can clearly see the effect of the correlation coefficient $\rho$ on the trading and hedging strategy. As one would expect, when the traded and non-traded
assets are uncorrelated, the optimal hedging strategy is zero meaning that the contingent claim on \( Y \) is unhedgeable by trading \( S \). On the other extreme when \(|\rho| \to 1\), the hedging strategy reduces to the perfect delta hedge meaning that the contingent claim on \( Y \) can be fully replicated/hedged by a trading strategy on the asset \( S \). Consistently with our previous remarks, the hedging strategy is not linear in the number of traded claims in general. It becomes linear when indifference pricing is linear when \(|\rho| \to 1\) (\( P_y^{(n)} \) is the only non-linear term in the expression of the optimal hedging strategy).

2.3 Residual risk process

Another stochastic process of interest for us is the residual risk process. We assume that we trade \( n \) claims at \( t = 0 \) for a price \( p^{(n)}(0, Y_0) \) per claim and we invest the proceeds in the optimal hedging strategy in (2.9). Hence, the initial value of the hedging portfolio is \( X_0 = -np^{(n)}(0, Y_0) \). The residual risk process
$R_t$ is our position at time $t$. It is given by:

$$R_t = X_t + np^{(n)}(t, Y_t). \quad (2.10)$$

It was shown in [4] that the indifference price satisfies the following PDE

$$p_t^{(n)} + (\mu_0 - \rho \eta \lambda) y p_y^{(n)} + \frac{1}{2} \eta^2 y^2 p_{yy}^{(n)} = r p^{(n)} + \frac{1}{2} n \eta^2 e^{-r(T-t)} \gamma (1 - \rho^2) y^2 (p_y^{(n)})^2$$

$$\quad (2.11)$$

The PDE can be obtained from (2.6) by using Feynman-Kac and Itô’s formula and the dynamics under $Q^M$.

Applying Itô’s formula to (2.10) and using the PDE in (2.11), we obtain the following SDE for $R_t$

$$\begin{cases} 
    dR_t = \left[ r R_t + \frac{1}{2} \rho e^{-r(T-t)} \gamma (1 - \rho^2) \eta Y_t (p_y^{(n)}(t, Y_t))^2 \right] dt \\
    + n \eta \sqrt{1 - \rho^2} Y_t (p_y^{(n)}(t, Y_t)) dW_t \\
    R_0 = 0.
\end{cases} \quad (2.12)$$

The full derivation can be found in [4]. When $\rho \to 1$, $R_t \to 0$ since the optimal hedging strategy perfectly hedges the claim. The terminal residual risk is

$$R_T = X_T + np^{(n)}(T, Y_T) = X_T + n h(Y_T). \quad (2.13)$$

Alternatively, if we assume that $\rho \to 1$ regardless of its actual value and trade $n$ claims at their Black-Scholes price $v(0, Y_0)$ at $t = 0$ and invest the proceeds of the trade in the naive Black-Scholes hedging strategy

$$\pi^{h, BS}(t, y) = \lim_{\rho \to 1} \pi^{h, n}(t, y)$$


then it can be shown that the naive residual error process follows the SDE

\[
\begin{align*}
\frac{dR_t^{\text{naive}}}{R_t^{\text{naive}}} &= [rR_t^{\text{naive}} + n\eta Y_t(\theta - \lambda)v_y(t, y)]dt \\
&\quad + n\eta Y_t v_y(t, Y_t) \left[ (\rho - 1)dW_t + \sqrt{1 - \rho^2}dW_t^\perp \right]
\end{align*}
\] (2.14)

where

\[v_y(t, y) = \frac{\partial v(t, y)}{\partial y}\]

and

\[\theta = \frac{\mu_0 - r}{\eta}.
\]

When \(\rho = 1\), the market is complete and hence according to the fundamental theorems of asset pricing (FTAP 1 and 2), there exist a unique equivalent martingale measure \(Q\) under which the discounted asset values are \(Q\)-martingales. Hence, a necessary condition for no arbitrage is that \(\theta = \lambda\) for both assets to be discounted martingales under the unique risk neutral measure \(Q\). Hence in that case, we can see that \(R_t^{\text{naive}} = 0\) for all \(t\). This is not surprising, since we expect \(R_t^{\text{naive}} \to R_t\) as \(\rho \to 1\) and when \(\rho = 1\), the market becomes complete and we can perfectly hedge contingent claims on \(Y\) leaving no residual errors.

There is an underlying assumption in both SDEs (2.12) and (2.14) that hedging is done continuously. This is not the case in practice where typically traders rebalance their positions at discrete intervals. However, the SDE gives an easily computable approximation of the true residual error. The effects of discrete hedging will not be analysed here and could be an interesting direction. In the Merton problem, Rogers [3] shows that this effect is small compared to parameter uncertainty. Thus, we will neglect the discretization error.
We will use the statistics for the simulated terminal residual errors $R_T$ and $R_T^{(naive)}$ to compare the performance of the two strategies. The histogram in

![Histogram](image)

Figure 2.4: Terminal residual error distribution for an at-the-money put option with spot price $Y_0 = 100$ using (a) the optimal hedging strategy, (b) the naive Black-Scholes strategy. The parameters are in Table 2.1, the correlation coefficient is $\rho = 0.85$.

Fig. 2.4(a) is generated from 50,000 samples of the terminal residual error for a correlation coefficient $\rho = 0.85$, resulting from the optimal hedging strategy to hedge a short position in a put option, while the histogram in Fig. 2.4(b) corresponds to the terminal residual error from the naive hedging strategy.
These effects have been reported by Monoyios in [4, 11]. Comparing both histograms, we can immediately see that profits are more frequent when optimal hedging is used. This is shown more explicitly in Table 2.2, where we tabulate significant statistics of the terminal residual process (mean, standard deviation and median). We first observe that we consistently get a smaller standard deviation in the terminal residual process resulting from the optimal strategy. In other words, the optimal strategy does not result in very large profits or very large losses as frequently as the naive strategy.

<table>
<thead>
<tr>
<th>Correlation Coefficient $\rho$</th>
<th>Optimal Strategy</th>
<th>Naive Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E[R_T]$</td>
<td>$sd(R_T)$</td>
</tr>
<tr>
<td>$\rho = 0.1$</td>
<td>0.6911</td>
<td>11.2180</td>
</tr>
<tr>
<td>$\rho = 0.65$</td>
<td>0.3870</td>
<td>9.4929</td>
</tr>
<tr>
<td>$\rho = 0.85$</td>
<td>0.2225</td>
<td>6.8384</td>
</tr>
<tr>
<td>$\rho = 0.98$</td>
<td>0.0551</td>
<td>2.6716</td>
</tr>
</tbody>
</table>

Table 2.2: Statistics for $R_T$ and $R_T^{(naive)}$ obtained from $N = 5 \times 10^4$ simulations. The simulation parameters are in Table 2.1.

Moreover, the median terminal residual error from the optimal hedging is also consistently higher (by a factor of approximately 2) than that from the naive hedging. This suggests that the optimal hedging strategy results in small positive gains more frequently than the naive strategy. Finally, the mean of the terminal residual error from the optimal strategy is higher than that from the naive strategy which is almost 0. This can also be observed from Fig. 2.4. The histogram obtained from the naive strategy is almost symmetric around 0, suggesting a mean terminal residual error of 0. On the other hand, the histogram obtained from the optimal strategy is more skewed towards positive
values, hence the positive bias in the mean of the terminal residual error. There is also a clear relationship between the correlation coefficient and the terminal residual process. As $\rho$ increases, both naive and optimal hedging become more efficient. In other words, the hedging error becomes smaller when the traded asset becomes more correlated with the non-traded one. In the limit, when $\rho = 1$, we get perfect hedging and $R_T = 0$ almost surely. This relationship is clearly illustrated in Table 2.2. In fact, we observe that the mean, standard deviation and median of the terminal residual process, resulting from the optimal hedging strategy, are all decreasing with $\rho$. Similarly, the standard deviation and the median of the terminal residual process are also decreasing with $\rho$.

The above observations suggest that the optimal hedging strategy outperforms the naive Black-Scholes strategy. However, this comparison is not always fair since the indifference price is not necessarily equal to the Black-Scholes price and in certain cases the indifference price per sold (purchased) claim is higher (lower) than its Black-Scholes price. Hence, there is a bold underlying assumption that another party is willing to execute the trade at $t = 0$ at a less convenient price (lower price if selling and higher price if buying).

Another important issue we will be investigating in the next chapter is the dependence of the indifference price and optimal hedging strategy on the drift parameters of the traded and non-traded asset. Since it is impossible to obtain an accurate estimate of the drift, it is important to compute the sensitivities of the indifference price, hedging strategy, terminal residual process and expected terminal utility with respect to the drift parameters, and compare them to the respective sensitivities with respect to the volatility $\sigma$. 
Chapter 3

Sensitivity analysis in utility-based hedging

We have seen that, in idealised conditions of perfect knowledge of all relevant model parameters, the optimal utility-based hedging strategy has a statistically superior performance to that of the Black-Scholes hedging strategy. In fact, the optimal strategy reduces the variance of the terminal residual risk and increases the likelihood of getting profits, albeit small, over losses. However, the optimal strategy requires knowledge of the drift parameters $\mu$ and $\mu_0$ of both the traded and non-traded asset, $S$ and $Y$, in addition to their respective volatilities $\sigma$ and $\eta$ and correlation coefficient $\rho$. On the other hand, the naive Black-Scholes strategy only depends on the volatilities. Obtaining an accurate estimate of the drifts is very difficult, even impossible sometimes whereas an accurate estimate of the volatilities can be obtained with less observations. In fact, for the traded asset

$$d[S]_t = \sigma S_t \, dt,$$

where $[S]_t$ denotes the quadratic variation of $S$ until time $t$. Therefore, observing $S_t$ and $[S]_t$ gives us an estimate of $\sigma^2$ and the error in the estimate
only comes from the sampling error. We can estimate \( \eta \) in a similar way. As for the correlation coefficient \( \rho \), we have that

\[
d[S, Y]_t = \rho \sigma \eta S_t Y_t \, dt,
\]

where \([S, Y]_t\) is the quadratic covariation of \( S \) and \( Y \). Hence, having estimated \( \sigma \) and \( \eta \) we can as easily estimate \( \rho \) from observing \([S, Y]_t\), \( S_t \) and \( Y_t \).

On the other hand, assuming that we have an estimate of \( \sigma \), the best estimate for the market price of risk \( \lambda = \frac{\mu - r}{\sigma} \) is given by

\[
\bar{\lambda}_t = \frac{1}{t} \int_0^t \frac{dS_s}{\sigma S_s} = \lambda + \frac{W_t}{t}.
\]

Hence, the estimate of \( \lambda \) is normally distributed \( \bar{\lambda}_t \sim N(\lambda, \frac{1}{t}) \) [4], and so is estimate \( \bar{\mu} \) of \( \mu \) (\( \bar{\mu} \sim N(\mu, \frac{\sigma^2}{t}) \)). Note that the error in the estimate comes from both the sampling error and the Monte Carlo error. From here it can be shown that to obtain an estimate within 0.05 from the actual value of \( \mu \) with a 95% confidence interval for \( \sigma = 0.2 \), we need \( t \approx 62 \) years. This gives us an idea about the big difficulty in obtaining an accurate estimate of \( \mu \).

Hence, we are interested in investigating the sensitivity of the optimal strategy to estimation errors in the drift parameters, and compare it to its sensitivity to the volatility mis-estimation. We assume that an accurate estimate of the correlation coefficient can be obtained.
3.1 Sensitivity of the optimal strategy to model parameters

We first provide in Table 3.1 the general expression of the partial derivatives of the marginal price and the marginal hedging strategy with respect to $\sigma, \eta, \mu$ and $\mu_0$ for any European claim with payoff $h(Y_T)$ at maturity $T$. The marginal price is defined in Lemma 2.2. The marginal hedging strategy $\pi^M$ is the first order approximation of $\pi^{h,n}$, given by

$$
\pi^M(t, y) = \frac{\rho n y}{\sigma} \frac{\partial}{\partial y} p^M(t, y) = \frac{\rho n y}{\sigma} p^M_y(t, y),
$$

where $p^M_y(t, y) := \frac{\partial}{\partial y} p^M(t, y)$.

<table>
<thead>
<tr>
<th>Marginal Price $p^M(t, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \sigma} p^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \eta} p^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \mu} p^M(t, y)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Marginal hedging strategy $\pi^M(t, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \sigma} \pi^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \eta} \pi^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \mu} \pi^M(t, y)$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \mu_0} \pi^M(t, y)$</td>
</tr>
</tbody>
</table>

Table 3.1: Sensitivities of the marginal price and marginal hedging strategy to model parameters $\sigma, \eta, \mu$ and $\mu_0$. 

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We assume in Table 3.1 that the payoff is twice differentiable. This assumption does not always hold. In fact, the payoff for vanilla options is piecewise continuous and differentiable almost everywhere except at the strike. However, the second derivative can be defined as a dirac measure.

In Table 3.2, we give closed form expressions of these sensitivities for a European put option on $Y$ with strike $K$. In the table and throughout the rest of the thesis, $N(\cdot)$ denotes the standard cumulative normal distribution and $d_1$ is given by

$$d_1 = \frac{\log \left( \frac{y}{K} \right) + (r - q + \frac{\sigma^2}{2})(T - t)}{\eta \sqrt{T - t}},$$

where

$$q = r - \mu_0 + \eta \rho \lambda.$$
It is immediately clear from Table 3.2 that both the marginal price and hedging strategy become less sensitive to the drift and the volatility of the traded asset as the correlation coefficient decreases. This is expected since the smaller the correlation, the smaller the dependence of the indifference price and hedging strategy on the traded asset and hence the smaller their sensitivities to the drift and volatility of the traded asset.

Figure 3.1: Sensitivities of the indifference price of an at-the-money put option, struck at 100, with the model parameters values in Table 2.1, to (a) $\sigma$, (b) $\eta$, (c) $\mu$ and (d) $\mu_0$. 

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While the formulae in Tables 3.1 and 3.2 are useful to have, we are interested in having a more concrete understanding of these sensitivities. To this end, we plot them for an at-the-money put option with strike $K = 100$ and the parameters in Table 2.1.

Figure 3.1(a) plots $\frac{\partial}{\partial \sigma} p^{(-1)}(0, Y_0)$ for different values of $\sigma$. The first observation is that this derivative is always negative. The reason for that is the following. The traded asset becomes a more effective hedging instrument when its volatility increases. Which means that the European claim on the non-traded asset can be hedged more efficiently using the traded asset, making it less risky. Hence, the selling price of the contingent claim is expected to decrease when the volatility of the traded asset increases. Moreover, we can see that the partial derivative decreases in absolute value with respect to $\sigma$. This behaviour partially reflect the risk aversion tendency that is introduced by the utility function. In other words, increasing the volatility of the non-traded asset by a fixed amount makes the trader more aggressive and hence sells the claim at a lower price. On the other hand, decreasing $\sigma$ by the same amount worries the trader and makes him sell the claim at a higher price. However, the change in price is higher when the trader is worried than it is when he is being aggressive. This suggests that the trader is risk averse, which is not surprising at all given that we are assuming that the trader is using a constant absolute risk aversion utility function in the exponential utility.

In Fig. 3.1(b), we plot $\frac{\partial}{\partial \eta} p^{(-1)}(0, Y_0)$ for different values of $\eta$. We omitted the closed-form expression from Table 3.2 for convenience. As we can see from the plot, the correlation coefficient $\rho$ does not have a remarkable contribution to the partial derivative with respect to $\eta$, which is not the case for the partial derivatives with respect to $\mu$ and $\sigma$. More importantly, we notice that the
partial derivative with respect to \( \eta \) is always positive, which is also the case of the Vega of a put option in the Black-Scholes model (sensitivity of the Black-Scholes price of a put option to the volatility of the underlying). The reason for positivity is the same. When the volatility of the underlying non-traded asset increases, the put option becomes riskier which increases its selling price. Moreover, the curve is strictly increasing highlighting the trader’s risk aversion. The reasoning goes along the same line as the one given for the first plot. When the volatility of the underlying increases by a certain amount \( \Delta \eta \), the trader is worried and raises the price by an amount \( \Delta p_1 \). On the other hand, if \( \eta \) decreases, the claim is less risky and hence the trader becomes more aggressive and reduces the value of the claim by \( \Delta p_2 \). Because the trader is risk averse, we expect \( \Delta p_1 > \Delta p_2 \), which is illustrated in Fig. 3.1(b).

Figures 3.1(c) and (d) illustrate the same concepts discussed above. Essentially, when the drift of the traded asset decreases, that asset becomes a more effective hedging instrument which encourages the trader to become more aggressive and sells the claim at a lower price, resulting in a positive partial derivative with respect to \( \mu \). The opposite is true for the drift of the non-traded asset. A higher drift results in more effective hedging and a lower price, hence the negative partial derivative with respect to \( \mu_0 \).

In Figure 3.2(a), we plot the partial derivative of the optimal hedging strategy with respect to \( \sigma \). We can immediately see that the partial derivative is always negative. This is expected since the larger the volatility of the non-traded asset the more effective it becomes as a hedging instruments. This means that we need to invest less money in \( S \) in order to optimally hedge the risk in the contingent claim. The opposite is true in Fig. 3.2(b). The larger the volatility of
the underlying, the riskier the contingent claim and hence more money needs to be invested in the traded asset to hedge the risk. Indeed, the partial derivative of the hedging strategy with respect to $\eta$ is always positive. The exact same reasoning can be used to explain the positivity of the partial derivative with respect to $\mu$ in Fig. 3.2(c) and the negativity of the partial derivative with respect to $\mu_0$ in Fig. 3.2(d).

We are also interested in how sensitive the indifference price and hedging strategy are to the model parameters. In the medium to high correlation regimes ($\rho = 0.65, 0.98$ respectively), the absolute value of the sensitivity of the indifference price to $\eta$ is, roughly, at least twice as much as its absolute sensitivity to $\sigma$, and is in the same order as its sensitivity to the drift parameters $\mu$ and $\mu_0$. Since the estimation of the drift parameters is less accurate than the volatilities’ estimates, we expect that the error from parameter mis-estimation is dominated by the errors from the drift parameters mis-estimation. We will see in the next section that in certain scenarios the errors cancel and have a benign effect on the price and hedging strategy. But in other cases, the effects can be drastic.

On the other hand, the absolute sensitivity of the hedging strategy to $\sigma$ is at least twice as big as its absolute sensitivity to $\eta$, and roughly 5 times bigger than its sensitivity to the drift parameters. Hence, unless the drift estimates are 5 times less accurate than the volatility estimates the hedging strategy, the main component of the effect of parameter mis-estimation on the hedging strategy will be coming from the mis-estimation of $\sigma$. We also investigate how the model parameters affect the terminal utility and residual error. To this end, we simulate 100,000 independent Brownian paths, half of them are used as paths for $W_t$ and the other half for $W_t^\perp$. We use these Brownian paths to simulate the SDEs for the wealth process $X_t$ and the residual risk.
Figure 3.2: Sensitivities of the optimal hedging strategy of an at-the-money put option, struck at 100, with the model parameters values in Table 2.1, to (a) $\sigma$, (b) $\eta$, (c) $\mu$ and (d) $\mu_0$. 
process $R_t$. We investigate the effect of the model parameters on the mean and variance of the terminal utility and the mean, variance and median of the terminal residual wealth. We can use a similar reasoning to the one we used for the sensitivities of the price and hedging to explain the plots in Figs. 3.3 and 3.4. To avoid repetition, we will only discuss how $\eta$ affects the statistics of the terminal utility and residual process. When $\eta$ increases, the price of the contingent claim increases and so does the terminal wealth of the optimal strategy hedging a short position in the claim. Hence, the expected terminal utility increases with $\eta$ as it shows in the corresponding plot in Fig. 3.3. Also, as $\eta$ increases the position in the traded risky asset increases and so does the variance of the terminal wealth. Hence, the variance of the expected terminal utility increases with $\eta$. Similarly, when $\eta$ increases, the claim is more difficult to hedge and hence the mean, median and variance of the terminal residual risk all increase. We can extend this same reasoning to the other parameters.
Figure 3.3: The mean and variance of the terminal utility with respect to model parameters $\sigma$, $\eta$, $\mu$ and $\mu_0$. 
Figure 3.4: The mean, median and variance of the terminal residual risk process with respect to model parameters $\sigma, \eta, \mu$ and $\mu_0$. 


3.2 Effects of parameter mis-estimation

In this section, we study the effect of different scenarios of parameter mis-estimation on the indifference price, and terminal residual risk process. We assume that the true parameter values are those listed in Table 2.1, and that the agent uses incorrect estimates of those parameters. We present some examples where the parameter mis-estimation is benign and where it can be drastic. Monoyios computed in [4] the SDEs satisfied by the residual errors from both the utility-based strategy and the naive Black-Scholes strategy. Here, we derive the SDE of the residual risk process when parameter mis-estimation is involved. We assume that the correlation coefficient $\rho$ and the volatility $\eta$ of the non-traded asset that the agent uses are the true ones. We assume that the agent uses the erroneous estimates $\bar{\sigma}$, $\bar{\mu}$ and $\bar{\mu}_0$ of $\sigma$, $\mu$ and $\mu_0$, respectively.

**Lemma 3.1** (Wealth process with erroneous parameters) The SDE of the wealth process from using the optimal hedging strategy with the wrong parameters $\bar{\sigma}$, $\bar{\mu}$ and $\bar{\mu}_0$ and starting from an initial wealth $x > 0$ is given by

\[
\begin{align*}
\bar{X}_t & = \begin{cases} 
  r\bar{X}_t - \frac{n \rho \eta Y_t}{\bar{\sigma}}\bar{p}_y^{(n)}(\mu - r) dt - \frac{n \rho \eta Y_t}{\bar{\sigma}}\bar{p}_y^{(n)}(\mu - \bar{\sigma}) \sigma dW_t, \\
  \bar{X}_0 & = x,
\end{cases} \\
\bar{p}_y^{(n)}(t, Y_t) & := \frac{\partial}{\partial y}\bar{p}_y^{(n)}(t, Y_t),
\end{align*}
\]

where $\bar{p}_y^{(n)}(t, Y_t)$ is the indifference price computed with the erroneous parameters.

*Proof.* The $Q^M$-dynamics of the wealth process $X^\pi = (X^\pi_t)_{0 \leq t \leq T}$ generated by
a previsible, admissible trading strategy \( \pi = (\pi_t)_{0 \leq t \leq T} \) is given by

\[
\begin{cases}
\quad dX^\pi_t = [rX^\pi_t + \pi_t(\mu - r)] \, dt + \pi_t \sigma \, dW_t. \\
\quad X^\pi_0 = x,
\end{cases}
\]

\( \bar{X} \) is generated by the hedging strategy

\[
\bar{\pi}^{h,n}_t = -\frac{n \rho \eta Y^\top}{\sigma} \bar{p}^{(n)}_y.
\]

Plugging (3.1) into (3.1), we obtain the desired result.

We can see in Lemma 3.1 that the erroneous parameters in the SDE only come from the hedging strategy.

**Lemma 3.2** (Indifference price with erroneous parameters) The indifference price computed with erroneous parameters satisfies the following PDE

\[
\bar{p}^{(n)}_t + (\bar{\mu}_0 - \rho \eta \bar{\lambda}) y \bar{p}^{(n)}_y + \frac{1}{2} \eta^2 y^2 \bar{p}^{(n)}_{yy} = r \bar{p}^{(n)} + \frac{1}{2} \eta^2 e^{-r(T-t)} \gamma (1 - \rho^2) y^2 \bar{p}^{(n)}_y \bar{p}^{(n)}_y,
\]

where \( \bar{\lambda} := \frac{\bar{\mu} - r}{\bar{\sigma}} \).

**Proof.** For a given \( t, y \), the indifference price \( \bar{p}^{(n)}(t, y) \) is computed using the parameters \( \bar{\sigma}, \bar{\mu} \) and \( \bar{\mu}_0 \). Hence, it satisfies the PDE in (2.11) with the wrong parameters.

Note that in Lemma 3.2, the PDE only depends on the erroneous parameters because the indifference price for a given \( t \) and \( y \) does not depend on the dynamics of \( Y_t \).

**Lemma 3.3** (Indifference price process with erroneous parameters) Applying
Ito’s formula, we obtain the following SDE for the indifference price

$$
\mathrm{d}\bar{p}^{(n)}(t, Y_t) = \left[ \bar{p}^{(n)}_t + \mu_0 Y_t \bar{p}^{(n)}_y + \frac{1}{2} \eta^2 Y_t^2 \bar{p}^{(n)}_{yy} + \eta \rho Y_t \bar{p}^{(n)}_y \right] \mathrm{d}t + \eta \rho Y_t \bar{p}^{(n)}_y \mathrm{d}W_t + \eta \sqrt{1 - \rho^2} Y_t \bar{p}^{(n)}_y \mathrm{d}W_t^\perp,
$$

Proof. Applying Itô’s formula to $\bar{p}^{(n)}$, we get

$$
\mathrm{d}\bar{p}^{(n)}(t, Y_t) = \bar{p}^{(n)}_t \mathrm{d}t + \bar{p}^{(n)}_y \mathrm{d}Y_t + \frac{1}{2} \bar{p}^{(n)}_{yy} \mathrm{d}[Y]_t.
$$

Using the dynamics of $Y$ (the correct value of $\mu_0$), we obtain the above SDE.

Finally, we can obtain the SDE followed by the residual risk process with erroneous parameters $\bar{R}_t = \bar{X}_t + n\bar{p}^{(n)}(t, Y_t)$.

**Lemma 3.4** (Residual risk process with erroneous parameters) The residual risk process with erroneous parameters $\bar{R}_t$ satisfies the SDE

$$
\mathrm{d}\bar{R}_t = \left[ r \bar{R}_t + \frac{1}{2} n^2 \eta^2 e^{-r(T-t)} \gamma (1 - \rho^2) Y_t^2 \bar{p}^{(n)}_{yy} + \left( \mu_0 - \bar{\mu}_0 - \eta \rho \left( \frac{\mu - r}{\sigma} - \lambda \right) \right) n Y_t \bar{p}^{(n)}_y \right] \mathrm{d}t
$$

$$
+ \eta Y_t \bar{p}^{(n)}_y \left( 1 - \frac{\sigma}{\bar{\sigma}} \right) \rho \mathrm{d}W_t + \sqrt{1 - \rho^2} \mathrm{d}W_t^\perp.
$$

Proof. Applying Itô’s formula to the definition of the residual risk process with erroneous parameters and using Lemmas 3.2 and 3.3 give us the above SDE.

As expected, the erroneous parameters come from the hedging strategy and the indifference price, whereas the true parameters come from the dynamics of
the underlying assets. If \( \sigma = \bar{\sigma}, \mu = \bar{\mu} \) and \( \mu_0 = \bar{\mu}_0 \), then the SDE in Lemma 3.4 reduces to the SDE of the residual risk process in (2.12).

### 3.3 Numerical results

In this section, we study how using erroneous parameters \( \bar{\sigma}, \bar{\mu} \) and \( \bar{\mu}_0 \) to compute the indifference price and the hedging strategy affect the terminal utility and residual error. We have studied in the previous section how changes in these parameters affect the indifference pricing, the hedging strategy and its performance. We saw that the volatility \( \sigma \) of the traded asset and the drift \( \mu_0 \) of the non-traded asset causes changes in one direction, whereas the drift \( \mu \) of the traded asset causes changes in the opposite direction. To understand more precisely how misestimating these parameters affect the performance, we assume that the true value of the parameters are the ones we have in Table 2.1 and we set \( \rho = 0.85 \). We then assume that each studied parameter can be either underestimated or overestimated by a fixed amount 0.05 (e.g. the true value of \( \sigma \) in the simulation is 0.25 hence the agent either uses a \( \bar{\sigma} \) of 0.2 or 0.3).

We simulate all the possible scenarios with the misestimated parameter values shown in Table 3.3. As we have seen in the previous section, the terminal util-

<table>
<thead>
<tr>
<th></th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
<th>Scenario 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\sigma} )</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>( \bar{\mu} )</td>
<td>0.05</td>
<td>0.05</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>( \bar{\mu}_0 )</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Scenario 5</th>
<th>Scenario 6</th>
<th>Scenario 7</th>
<th>Scenario 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\sigma} )</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>( \bar{\mu} )</td>
<td>0.05</td>
<td>0.05</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>( \bar{\mu}_0 )</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Table 3.3: Values of the misestimated parameter in all the different scenarios.
ity and residual risk are less sensitive to $\sigma$, despite the high sensitivity of the hedging strategy to the volatility. Hence, we expect the sign of mis-estimation of the drift parameters to have the bigger impact on the performance. When both drifts are underestimated or overestimated, their effects roughly cancel each other and the effect on the terminal utility and residual risk is mainly determined by the sign of the mis-estimation of $\sigma$. What happens, in fact, is that the agent either believes that the non-traded asset is riskier than it actually is but also believes that the traded asset is a more effective hedging instrument or that the non-traded asset is less risky and the traded is less effective in hedging the risk. Either way, he does not become significantly more or less aggressive than he should be. Indeed, the put price and the statistics of the terminal utility and residual risk do not deviate drastically in these scenarios. In this case, the parameter mis-estimation is relatively benign as we can see in Scenarios 1, 2, 7 and 8 in Table 3.4.

<table>
<thead>
<tr>
<th>mis-estimation Scenario</th>
<th>Terminal Utility $\mathbb{E}[U(X_T)]$</th>
<th>sd($U(X_T)$)</th>
<th>Terminal residual risk $\mathbb{E}[R_T]$</th>
<th>sd($R_T$)</th>
<th>med($R_T$)</th>
<th>Put price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 0</td>
<td>-0.8999</td>
<td>0.1041</td>
<td>0.1487</td>
<td>6.8803</td>
<td>1.2387</td>
<td>8.8904</td>
</tr>
<tr>
<td>Scenario 1</td>
<td>-0.9406</td>
<td>0.0961</td>
<td>0.2398</td>
<td>7.4231</td>
<td>0.4719</td>
<td>8.8538</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>-0.9008</td>
<td>0.0845</td>
<td>1.0137</td>
<td>6.8619</td>
<td>2.5300</td>
<td>8.8538</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>-0.8493</td>
<td>0.1231</td>
<td>5.8444</td>
<td>8.3506</td>
<td>5.2477</td>
<td>14.2189</td>
</tr>
<tr>
<td>Scenario 4</td>
<td>-0.8689</td>
<td>0.0815</td>
<td>4.6724</td>
<td>6.8610</td>
<td>5.5597</td>
<td>12.2826</td>
</tr>
<tr>
<td>Scenario 5</td>
<td>-0.9293</td>
<td>0.1347</td>
<td>-3.0836</td>
<td>7.1019</td>
<td>-1.9421</td>
<td>5.6527</td>
</tr>
<tr>
<td>Scenario 6</td>
<td>-0.9316</td>
<td>0.0874</td>
<td>-2.4083</td>
<td>7.0976</td>
<td>-0.1033</td>
<td>5.6527</td>
</tr>
<tr>
<td>Scenario 7</td>
<td>-0.8888</td>
<td>0.1288</td>
<td>1.3234</td>
<td>7.5842</td>
<td>1.3255</td>
<td>9.8944</td>
</tr>
<tr>
<td>Scenario 8</td>
<td>-0.9059</td>
<td>0.0850</td>
<td>0.4385</td>
<td>6.8812</td>
<td>2.0669</td>
<td>8.3149</td>
</tr>
</tbody>
</table>

Table 3.4: Statistics of the terminal utility and residual risk for the different mis-estimation scenarios.

On the other hand, when one of the drifts is underestimated and the other is overestimated, their effects add up and the effect of $\sigma$ fades in comparison. In this case, the effect of the mis-estimation is quite serious. In Scenarios 3 and 4, it seems that more profit is made because of the mis-estimation and that we
obtain more expected terminal utility. However, the situation is not as good as it seems. There is a crucial underlying assumption that the put option is sold at $t = 0$ to finance the hedging portfolio. Unfortunately in this case, the agent believes that the underlying non-traded asset is riskier than it actually is and that the traded asset is less effective in hedging, and hence becomes very cautious and overprices the claim. When the parameter mis-estimation is such as to make the agent more conservative, and thus raise the indifference price above what it would be with correct parameters, then naturally this extra insurance has a benefit on his terminal hedging error. But of course his quote would be less competitive. This illustrates how crucial is the balance between risk and insurance that can be captured by indifference valuation, and how dependent this can be on certain parameter inputs.

Conversely, in Scenarios 5 and 6, the agent believes that the underlying is less risky than it is and that the traded asset is a more effective hedging instrument than it really is. The agent becomes very aggressive and decides to sell the claim at a much lower price incurring serious losses as shown in Table 3.4.
Chapter 4

Conclusion and future work

We have verified and extended some of the results in the literature about the performance of optimal utility-based hedging strategies and we have interpreted the dependence of the results on the correlation coefficient in the basis risk model. We have studied thoroughly the sensitivities of optimal utility-based hedging and valuation to model parameters and we derived general expressions for the partial derivatives of the marginal price and marginal hedging strategies with respect to the model parameters $\sigma, \eta, \mu$ and $\mu_0$. We also specialized these formulae for a put option. Using these sensitivities, we gave useful insights about utility-based strategies and their dependence on different model parameters.

Finally, we examined the impact of the drift parameters mis-estimation on the optimal hedging strategy. We showed that when the signs of the mis-estimation are such that their effects on the strategy add up, the impact is quite serious and can severely compromise the performance of the optimal strategy. Since an accurate estimate of the drift parameters is impossible, it becomes less clear whether the optimal utility-based hedging is an improvement over the Black-Scholes strategy which does not depend on the drift.
Future works may include extending the results to other incomplete market models, such as stochastic volatility or transaction costs models. It would be interesting to examine whether the utility-based hedging strategy remains statistically superior to the Black-Scholes strategy in the other models and whether the effect of drift parameter mis-estimation is similar to what we obtained in the basis risk model.

Another interesting direction would be to study the same problem for different utility functions. While the exponential utility simplifies the problem of utility-based hedging and valuation significantly by removing the dependence on the initial wealth, it is unclear why a major investor (e.g. big bank) would have the same trading strategy as a small investor. In fact, in certain cases constant relative risk aversion utility functions are more realistic. Hence, it would be interesting to study the performance of utility-based hedging and valuation for logarithmic or power utilities.
Bibliography


