# Kummer's Conjecture for Cubic Gauss Sums

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## 1 Introduction

Kummer [5] computed the sums

$$S_p = \sum_{n=1}^{p} \exp(2\pi i n^3/p)$$

for primes $p \equiv 1 \pmod 3$ up to 500, and found that $S_p/(2\sqrt{p})$ lay in the intervals $[-1, -\frac{1}{2}]$, $(-\frac{1}{2}, \frac{1}{2})$, $[\frac{1}{2}, 1]$ with frequencies approximately in the ratio 1 : 2 : 3. He conjectured, somewhat hesitantly, that this might be true asymptotically.

Kummer's conjecture was disproved by Heath-Brown and Patterson [4]. In order to state their result we must introduce a little notation. Let $\omega = \exp(2\pi i/3)$ and let $(*/*)_3$ be the cubic residue symbol for $\mathbb{Z}[\omega]$. For each $c \in \mathbb{Z}[\omega]$ such that $c \equiv 1 \pmod 3$ the cubic Gauss sum is

$$g(c) = \sum_{d \pmod c} (\frac{d}{c})_3 e(\frac{d}{c}),$$

where we have defined

$$e(z) = \exp(2\pi i(z + \overline{z})) \qquad (1)$$

for all complex $z$. One then has

$$g(c)^3 = \mu(c)c^2\overline{c}, \qquad (2)$$

where $\mu(*)$ is the Möbius function for $\mathbb{Z}[\omega]$, see Hasse [2; pp. 443-445] for example. It is therefore natural to normalize $g(c)$ by writing

$$\tilde{g}(c) = \frac{g(c)}{|c|}.$$

One then finds that if $p = N(\pi)$, where $\pi \equiv 1 \pmod 3$ is a prime of $\mathbb{Z}[\omega]$, then

$$\frac{S_p}{2\sqrt{p}} = \Re(\tilde{g}(\pi)).$$

Heath-Brown and Patterson showed that the numbers $\tilde{g}(\pi)$ are uniformly distributed around the unit circle, thereby disproving Kummer's conjecture.

To establish uniform distribution the natural route is to use the Weyl criterion, which requires the sums

$$\sum_{\substack{N(\pi)\le X\\ \pi\equiv 1 (\mathrm{mod}\ 3)}} \tilde{g}(\pi)^k$$

to be $o(X/\log X)$ for each fixed non-zero integer $k$. The formula (2) shows that $\tilde{g}(\pi)^3 = -\pi/|\pi|$. Thus if $k$ is a multiple of 3, the required bound is a standard consequence of the zero-free region for $L$-functions with a Grössencharakter. When $k = 3l + 1$ we need to examine

$$\sum_{\substack{N(\pi)\le X\\ \pi\equiv 1 (\mathrm{mod}\ 3)}} \tilde{g}(\pi)(\frac{\pi}{|\pi|})^l.$$

Similarly, when $k = 3l - 1$ we may restrict attention to sums of the above shape, via the observation that $g(\overline{c}) = \overline{g(c)}$. The principal result of Heath-Brown and Patterson is then the estimate

$$\sum_{\substack{N(c)\le X\\ c\equiv 1 (\mathrm{mod}\ 3)}} \tilde{g}(c)\Lambda(c)(\frac{c}{|c|})^l \ll_\varepsilon X^{30/31+\varepsilon} + |l|X^{29/31+\varepsilon}, \tag{3}$$

valid for any $l \in \mathbb{Z}$ and any $\varepsilon > 0$. Here $\Lambda(c)$ is the von Mangoldt function, defined as $\log N(\pi)$ if $c$ is a power of the prime $\pi$, and 0 otherwise.

This type of bound probably does not express the whole truth, for Heath-Brown and Patterson [4] (following Patterson [8], who presents a heuristic justification) have made the following conjecture.

**Conjecture** *For any $\varepsilon > 0$ we have*

$$\sum_{\substack{N(c)\le X\\ c\equiv 1 (\mathrm{mod}\ 3)}} \tilde{g}(c)\Lambda(c)(\frac{c}{|c|})^l = \begin{cases} bX^{5/6} + O_\varepsilon(X^{1/2+\varepsilon}), & (l = 0),\\ O_\varepsilon(X^{1/2+\varepsilon}), & (l \neq 0), \end{cases} \tag{4}$$

*where*

$$b = \frac{2}{5}(2\pi)^{2/3}\Gamma(\frac{2}{3}).$$

This is supported both by a heuristic argument and by the available numerical evidence. It expresses a bias towards the $\tilde{g}(\pi)$ having positive real part, thereby explaining the non-uniformity found by Kummer.

Unfortunately present methods appear to be inadequate for a resolution of Patterson's conjecture. The goal of the present paper is however to establish an improved version of the result (3) of Heath-Brown and Patterson, which only just fails to achieve the required degree of precision. Specifically we shall establish the following bound.

**Theorem 1** *For any $\varepsilon > 0$ we have*

$$\sum_{\substack{N(c)\le X \\ c\equiv 1(\text{mod } 3)}} \tilde{g}(c)\Lambda(c)(\frac{c}{|c|})^l \ll_\varepsilon X^{5/6+\varepsilon} + |l|X^{3/4+\varepsilon},$$

*for every $l \in \mathbb{Z}$.*

Possible improvements of (3) were investigated by Coleman in his thesis [1; Chapter 2]. Coleman proved unconditionally that

$$\sum_{\substack{N(c)\le X \\ c\equiv 1(\text{mod } 3)}} \tilde{g}(c)\Lambda(c)(\frac{c}{|c|})^l \ll_\varepsilon (X^{29/32} + |l|X^{41/64})(X(|l|+1))^\varepsilon,$$

and, subject to a 'Large Values Conjecture', (which is statement rather stronger than that of our Theorem 2), that

$$\sum_{\substack{N(c)\le X \\ c\equiv 1(\text{mod } 3)}} \tilde{g}(c)\Lambda(c)(\frac{c}{|c|})^l \ll_\varepsilon (X^{5/6} + |l|X^{7/12})(X(|l|+1))^\varepsilon.$$

Coleman explains that his factor $(|l|+1)^\varepsilon$ can probably be dispensed with, so that his second term is then better than that occuring in Theorem 1. This is due to savings in the '$T$-aspect' in Coleman's analysis, related to the formulation of his Large Values Conjecture.

The proof of Theorem 1 follows the line of attack established by Heath-Brown and Patterson, as will be explained in §2, but injects two new ideas into the argument. The first of these replaces a pointwise bound for a Dirichlet series, which Heath-Brown and Patterson used in an application of Perron's formula, by a mean-value bound. This will be described in §3. Coleman's analysis includes a saving of the same type, although his argument is distinctly more complicated. The second innovation is a far more sophisticated estimate for the 'Type II sum', which is $\Sigma_3(X,u)$ in the notation of [4]. In order to do this we shall establish the following.

**Theorem 2** *Let $c_n$ be an arbitrary sequence of complex numbers, where $n$ runs over $\mathbb{Z}[\omega]$. Then*

$$\sum_{N(m)\le M}{}^{*} \,|\sum_{N(n)\le N}{}^{*} c_n(\frac{n}{m})_3|^2 \ll_\varepsilon (M+N+(MN)^{2/3})(MN)^\varepsilon \sum_n{}^{*}|c_n|^2,$$

*for any $\varepsilon > 0$, where $\Sigma^*$ denotes summation over square-free elements of $\mathbb{Z}[\omega]$ congruent to 1 modulo 3.*

It seems possible that the term $(MN)^{2/3}$ could be removed with further effort, and the bound would then be essentially best possible. However the above suffices for our purposes. It should be noted that if the variables are not restricted to be square-free a result as sharp as Theorem 2 would be impossible. The proof of Theorem 2 is modelled on the corresponding argument for sums (over $\mathbb{Z}$) containing the quadratic residue symbol, due to the author [3]. The latter is distinctly unpleasant, but fortunately some of the diffculties may be reduced in our situation by the introduction of the term $(MN)^{2/3}$ in Theorem 2. None the less the proof of this result will form a substantial part of the present paper.

## 2 Preliminary Arguments

We shall begin by introducing a litle more notation, following Heath-Brown and Patterson [4]. We write

$$\tilde{g}_l(c) = \tilde{g}(c)(\frac{\overline{c}}{|c|})^l,$$

$$H_l(X) = \sum_{\substack{N(c)\le X\\ c\equiv 1(\text{mod }3)}} \tilde{g}_l(c)\Lambda(c),$$

and

$$F_l(X,\alpha) = \sum_{\substack{N(c)\le X\\ c\equiv 1(\text{mod }3),\,\alpha|c}} \tilde{g}_l(c),$$

for any $\alpha \in \mathbb{Z}[\omega]$.

As in [4;§3] we apply Vaughan's identity, though now with a minor modification. We write

$$\Sigma_j(X,u) = \sum_{a,b,c} \Lambda(a)\mu(b)\tilde{g}_l(abc),$$

where $a,b,c$ run over square-free elements of $\mathbb{Z}[\omega]$, subject to the conditions $a,b,c \equiv 1 \pmod 3$ and $X < N(abc) \le 2X$ and

$$\begin{array}{cc} N(bc)\le u, & j=0,\\ N(b)\le u, & j=1,\\ N(ab)\le u, & j=2',\\ N(a),N(b)\le u<N(ab), & j=2'',\\ N(b)\le u<N(a),N(bc), & j=3,\\ N(a),N(bc)\le u, & j=4. \end{array}$$

Then

$$\Sigma_0(X,u)+\Sigma_{2'}(X,u)+\Sigma_{2''}(X,u)+\Sigma_3(X,u)=\Sigma_1(X,u)+\Sigma_4(X,u).$$

The reader will note that we have split $\Sigma_2(X,u)$ (as given in [4;§3]) into $\Sigma_{2'}(X,u)$ and $\Sigma_{2''}(X,u)$. We shall suppose that $1\le u\le X^{1/3}$. Then, just as in [4;§3], we have

$$\Sigma_0(X,u)=H_l(2X)-H_l(X),$$

$$|\Sigma_1(X,u)|\le 3\log(2X)\sum_{N(\alpha)\le u}\max_{X<z\le 2X}|F_l(z,\alpha)|, \tag{5}$$

$$|\Sigma_{2'}(X,u)|\le 2\log u\sum_{N(\alpha)\le u}\max_{X<z\le 2X}|F_l(z,\alpha)|, \tag{6}$$

and

$$\Sigma_4(X,u)=0.$$

We shall bound $\Sigma_1(X,u)$ and $\Sigma_{2'}(X,u)$ by means of the following estimate.

**Lemma 1** *For any $\varepsilon>0$ and any $\alpha\in\mathbb{Z}[\omega]$ with $N(\alpha)\le X^{1/3}$ we have*

$$F_l(X,\alpha)\ll_\varepsilon \delta_l X^{5/6}N(\alpha)^{-1}+X^{2/3+\varepsilon}N(\alpha)^{-1/2}+|l|X^{1/2+\varepsilon}N(\alpha)^{-1/4},$$

*where $\delta_l$ is 1 for $l=0$ and is 0 otherwise.*

This may be compared with Theorem 4 of [4] which is the bound

$$F_l(X,\alpha)\ll_\varepsilon \delta_l X^{5/6}N(\alpha)^{-1}+X^{3/4+\varepsilon}N(\alpha)^{-5/8}+|l|X^{1/2+\varepsilon}N(\alpha)^{-1/4}.$$

We take $u=X^{1/3}$, so that (5) and (6) yield

$$\Sigma_1(X,u),\Sigma_{2'}(X,u)\ll_\varepsilon X^{5/6+\varepsilon}+|l|X^{3/4+\varepsilon},$$

on re-defining $\varepsilon$.

For the proof of Theorem 1 it therefore remains to obtain similar bounds for $\Sigma_{2''}(X,u)$ and $\Sigma_3(X,u)$, which will be achieved with the aid of Theorem 2. We begin by recalling that

$$\tilde g_l(vw)=\tilde g_l(v)\tilde g_l(w)\overline{\left(\frac{w}{v}\right)_3},$$

see Hasse [2; pp.443-445], for example. Thus, if we write

$$A(v)=\sum_{\substack{ab=v\\ N(a),N(b)\le u}}\Lambda(a)\mu(b)\tilde g_l(ab)$$

and

$$B(w)=\tilde{g}_l(w),$$

we find that

$$\Sigma_{2''}(X,u)=\sum_{\substack{X<N(vw)\le 2X\\ N(v),N(w)>u}} A(v)B(w)\overline{(\frac{w}{v})_3}.$$

Similarly, if we write

$$C(v)=\Lambda(v)\tilde{g}_l(v)$$

and

$$D(w)=\sum_{\substack{bc=w\\ N(b)\le u}}\mu(b)\tilde{g}_l(bc),$$

we find that

$$\Sigma_3(X,u)=\sum_{\substack{X<N(vw)\le 2X\\ N(v),N(w)>u}} C(v)D(w)\overline{(\frac{w}{v})_3}.$$

We note at once that

$$A(v),B(w),C(v),D(w)\ll_\varepsilon X^\varepsilon \tag{7}$$

for all relevent $v,w$. Moreover the functions $A,B,C$, and $D$ are supported on square-free integers of $\mathbb{Z}[\omega]$, congruent to 1 modulo 3.

We now proceed to estimate $\Sigma_{2''}(X,u)$, the treatment of $\Sigma_3(X,u)$ being identical. Our first task is to remove the condition $X<N(vw)\le 2X$. In order to do this we set

$$A_j(v)=\begin{cases} A(v), & 2^j u<N(v)\le 2^{j+1}u,\\ 0, & \text{otherwise},\end{cases}$$

for each non-negative integer $j$. We define $B_j(w)$ similarly. Values of $j$ for which $u2^j>2X$ are plainly irrelevent, and similarly for $k$. It follows that there must exist some pair $j,k$ for which

$$\Sigma_{2''}(X,u)\ll(\log X)^2|\sum_{X<N(vw)\le 2X} A_j(v)B_k(w)\overline{(\frac{w}{v})_3}|. \tag{8}$$

If we set $V=2^{j+1}u$ and $W=2^{k+1}u$ we deduce that

$$X<VW,\quad VW/4<2X \tag{9}$$

if the sum is to be non-empty. Since $V,W\ge u=X^{1/3}$ we may therefore assume that

$$V,W\ll X^{2/3}. \tag{10}$$

We now define a Dirichlet series

$$F(s) = \sum_{v,w} A_j(v) B_k(w) \overline{(\frac{w}{v})_3} N(vw)^{-s}.$$

It then follows from Perron's formula, as given by Titchmarsh [9; Lemma 3.19], that

$$\sum_{X<N(vw)\le 2X} A_j(v) B_k(w) \overline{(\frac{w}{v})_3} = \frac{1}{2\pi i} \int_{1+\varepsilon-iX}^{1+\varepsilon+iX} F(s) \frac{(2X)^s - X^s}{s} ds + O_\varepsilon(X^{4\varepsilon}),$$

since

$$\sum_{v,w} |A_j(v) B_k(w) \overline{(\frac{w}{v})_3} N(vw)^{-1-\varepsilon}| \ll_\varepsilon X^{3\varepsilon},$$

by (7).

We now see that

$$\sum_{X<N(vw)\le 2X} A_j(v) B_k(w) \overline{(\frac{w}{v})_3} \ll_\varepsilon X^{4\varepsilon} + X^{1+\varepsilon} (\log X) \max_t |F(1+\varepsilon+it)|. \quad (11)$$

However, if we write

$$\tilde{A}(v) = V A_j(v) N(v)^{-1-\varepsilon-it}, \quad \tilde{B}(w) = W B_k(w) N(w)^{-1-\varepsilon-it},$$

we find that

$$\tilde{A}(v), \tilde{B}(w) \ll_\varepsilon X^\varepsilon,$$

and that

$$F(1+\varepsilon+it) = (VW)^{-1} \sum_{N(v)\le V} \tilde{A}(v) \sum_{N(w)\le W} \tilde{B}(w) \overline{(\frac{w}{v})_3}.$$

On comparing the bounds (8) and (11) we deduce that

$$\Sigma_{2''}(X,u) \ll_\varepsilon X^{4\varepsilon} + (\log X)^3 X^\varepsilon | \sum_{N(v)\le V} \tilde{A}(v) \sum_{N(w)\le W} \tilde{B}(w) \overline{(\frac{w}{v})_3}|, \quad (12)$$

for some $t$, in view of the first of the inequalities (9).

Finally we employ Theorem 2 in conjunction with Cauchy's inequality, which shows that

$$\begin{aligned} &| \sum_{N(v)\le V} \tilde{A}(v) \sum_{N(w)\le W} \tilde{B}(w) \overline{(\frac{w}{v})_3}|^2 \\ &\quad \le \{ \sum_{N(v)\le V} |\tilde{A}(v)|^2 \} \{ \sum_{N(v)\le V} |\tilde{B}(w) \overline{(\frac{w}{v})_3}|^2 \} \\ &\quad \ll_\varepsilon V X^{2\varepsilon} (V + W + (VW)^{2/3}) (VW)^\varepsilon \sum_{N(w)\le W} |\tilde{B}(w)|^2 \\ &\quad \ll_\varepsilon V X^{2\varepsilon} (V + W + (VW)^{2/3}) (VW)^\varepsilon W X^{2\varepsilon} \\ &\quad \ll_\varepsilon X^{1+5\varepsilon} (V + W + X^{2/3}), \end{aligned}$$

by the second of the inequalities (9). In view of (10) the above is $O_\varepsilon(X^{5/3+5\varepsilon})$, so that (12) yields the following, after re-defining $\varepsilon$.

**Lemma 2** *For any* $\varepsilon > 0$ *we have*

$$\Sigma_{2''}(X,u) \ll_\varepsilon X^{5/6+\varepsilon},$$

*when* $u = X^{1/3}$, *and similarly for* $\Sigma_3(X,u)$.

This clearly suffices for Theorem 1.

# 3 Proof of Lemma 1

Our proof of Lemma 1 follows the corresponding treatment by Heath-Brown and Patterson [4; §4]. For any $\alpha \in \mathbb{Z}[\omega]$ with $\alpha \equiv 1 \pmod 3$ we write

$$f(s) = \sum_{c\equiv 1 (\mathrm{mod}\ 3)} \tilde{g}_l(\alpha c) N(c)^{-s}.$$

Then, as in [4; p.128] we have

$$F_l(X,\alpha) = \delta_l p(\alpha,1)\frac{6}{5}X^{5/6} + \frac{1}{2\pi i}\{\int_{\sigma_2-iT}^{\sigma_3-iT} + \int_{\sigma_3-iT}^{\sigma_3+iT} + \int_{\sigma_3+iT}^{\sigma_2+iT}\} f(s)(\frac{X}{N(\alpha)})^s \frac{ds}{s}$$
$$+O_\varepsilon(X^{\sigma_2}N(\alpha)^{-1}T^{-1}),$$

where $\sigma_2 = 1+\varepsilon$ and $\sigma_3 = \frac{1}{2}+\varepsilon$. The constant $p(\alpha,1)$ satisfies $p(\alpha,1) \ll N(\alpha)^{-1}$. Moreover, as noted in [4; p.129] the first and third integrals are

$$O_\varepsilon(X^{\sigma_2}N(\alpha)^{-1}T^{-1}) + O_\varepsilon(X^{\sigma_3}(T+|l|)N(\alpha)^{-1/4}T^{-1}),$$

for $T \geq 1$. We shall take

$$T = X^{1/3}N(\alpha)^{-1/2}, \tag{13}$$

so that $T \geq 1$, assuming that $N(\alpha) \leq X^{1/3}$. With this choice we find that

$$\begin{aligned} F_l(X,\alpha) &= \frac{1}{2\pi i}\int_{\sigma_3-iT}^{\sigma_3+iT} f(s)(\frac{X}{N(\alpha)})^s\frac{ds}{s} + O(X^{5/6}N(\alpha)^{-1}) \\ &\quad +O_\varepsilon(X^{2/3+\varepsilon}N(\alpha)^{-1/2}) + O_\varepsilon(X^{1/2+\varepsilon}N(\alpha)^{-1/4}) \\ &\quad +O_\varepsilon(X^{1/6+\varepsilon}N(\alpha)^{1/4}|l|). \end{aligned}$$

The error terms are all satisfactory for Lemma 1, since $N(\alpha) \leq X^{1/3}$.

In [4] the remaining integral was estimated by means of a pointwise bound. We shall use the following mean value result instead.

**Lemma 3** *For any $T \geq 1$ we have*

$$\int_{-T}^{T} |f(\frac{1}{2}+\varepsilon+it)|^2 dt \ll_\varepsilon N(\alpha)^{1/2+4\varepsilon}(T+|l|)^2.$$

Cauchy's inequality then yields

$$\int_{-T}^{T} |f(\frac{1}{2}+\varepsilon+it)| dt \ll_\varepsilon N(\alpha)^{1/4+2\varepsilon}T^{1/2}(T+|l|),$$

so that

$$\int_{-T}^{T} |f(\sigma_3+it)|\frac{dt}{|\sigma_3+it|} \ll_\varepsilon N(\alpha)^{1/4+2\varepsilon}(T^{1/2}+|l|),$$

on integrating by parts. It now follows that

$$\int_{\sigma_3-iT}^{\sigma_3+iT} f(s)(\frac{X}{N(\alpha)})^s\frac{ds}{s} \ll_\varepsilon X^{1/2+\varepsilon}N(\alpha)^{-1/4+\varepsilon}(T^{1/2}+|l|).$$

Since $N(\alpha) \leq X^{1/3}$ this is satisfactory for Lemma 1, on re-defining $\varepsilon$.

The remainder of this section will be devoted to the proof of Lemma 3. We shall work with the function

$$Z(r,s,l) = \zeta(3s-2,l)\psi(r,s,l) \tag{14}$$

where

$$\zeta(s,l) = \sum_{c\equiv 1 (\mathrm{mod}\ 3)} N(c)^{-s}(\frac{\overline{c}}{|c|})^{3l}$$

and

$$\psi(r,s,l) = \sum_{c\equiv 1 (\mathrm{mod}\ 3)} N(c)^{-s}g(r,c)(\frac{\overline{c}}{|c|})^{3l},$$

with

$$g(r,c) = \sum_{d (\mathrm{mod}\ c)} (\frac{d}{c})_3 e(\frac{rd}{c}).$$

It will suffice to consider the case in which $r \in \mathbb{Z}[\omega]$ is square-free. The function $Z(r,s,l)$ satisfies a slightly complicated functional equation, due to Patterson [7; Theorem 6.1]. In order to describe the equation we introduce the functions

$$G_l(s) = (2\pi)^{-s}\Gamma(s+\frac{|l|}{2}-\frac{1}{3})\Gamma(s+\frac{|l|}{2}-\frac{2}{3})$$

and

$$F(r,s,l) = G_l(s)Z(r,s,l).$$

According to [7; Theorem 6.1] this function is entire, except possibly in the case $l=0$, when there may be simple poles at $s=2/3$ and $s=4/3$. (The statement of [7; Theorem 6.1] does not specify that the poles must be simple, but this is evident from the proof.)

The functional equation then expresses $F(r,s,l)$ as a finite linear combination of terms $N(r)^{1-s}F(r\eta,2-s,-l)$ with coefficients bounded in the strip $\frac{3}{4}\le\Re(s)\le\frac{5}{4}$. Here the numbers $\eta$ run over divisors of 9. We note that $F(\lambda^3 r,s,l)=F(r,s,l)$ for any $r$, where $\lambda=1-\omega$, see [7; (5.25)], and we may therefore conclude that if

$$\tilde{Z}(r,s,l)=\sum_{\eta|9}\sum_{\delta=\pm1}|Z(r\eta,s,\delta l)|^2$$

and

$$\tilde{F}(r,s,l)=G_l(s)^2\tilde{Z}(r,s,l),$$

then

$$\tilde{F}(r,s,l)\ll N(r)^{2-2\sigma}|\tilde{F}(r,2-s,l)|,\quad \frac{3}{4}\le\Re(s)\le\frac{5}{4},$$

where $\sigma=\Re(s)$ as usual. Since

$$G_l(2-s)\ll(T+|l|)^{4-4\sigma}|G_l(s)|$$

for $T\le t\le 2T$ and $T\ge 1$, we deduce that

$$\tilde{Z}(r,s,l)\ll(T+|l|)^{8-8\sigma}N(r)^{2-2\sigma}|\tilde{Z}(r,2-s,l)|, \tag{15}$$

for $\frac{3}{4}\le\Re(s)\le\frac{5}{4},\ \ T\le t\le 2T$.

We now take $r\equiv 1\,(\mathrm{mod}\ 3)$ to be square-free, and write

$$Z(r\eta,s,l)=\sum_{n=1}^{\infty}a_n(r\eta,l)n^{-s},$$

when $s$ has sufficiently large real part. By a familiar Mellin transform we have

$$\sum_{n=1}^{\infty}a_n(r\eta,l)n^{-s}e^{-n/X}=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}Z(r\eta,s+w,l)X^w\Gamma(w)dw,$$

where $s=\sigma+it$ and $1<\sigma\le 5/4$. We move the line of integration to $\Re(w)=2-2\sigma$, passing poles at $w=0$ and, possibly, at $w=4/3-s$. The residue at the former is just $Z(r\eta,s,l)$, while that at the latter is

$$X^{4/3-s}\Gamma(4/3-s)\mathrm{res}\{Z(r\eta,w,l);w=4/3\}.$$

According to [7; Theorem 9.1] the residue of the pole of $F(r\eta,s,0)$ at $s=4/3$ is $O(N(r)^{-1/6})$. We may therefore conclude, by a rather crude estimate, that

$$\sum_{n=1}^{\infty}a_n(r\eta,l)n^{-s}e^{-n/X}=Z(r\eta,s,l)+\frac{1}{2\pi i}\int_{2-2\sigma-i\infty}^{2-2\sigma+i\infty}Z(r\eta,s+w,l)X^w\Gamma(w)dw$$

$$+O(Xe^{-|t|}).$$

This enables us to bound

$$I=\int_T^{2T}|Z(r\eta,\sigma+it,l)|^2dt$$

as

$$I\ll I_1+I_2+Xe^{-T}, \tag{16}$$

where

$$I_1=\int_T^{2T}|\sum_{n=1}^{\infty}a_n(r\eta,l)n^{-\sigma-it}e^{-n/X}|^2dt,$$

and

$$I_2=\int_T^{2T}|\int_{2-2\sigma-i\infty}^{2-2\sigma+i\infty}Z(r\eta,\sigma+it+w,l)X^w\Gamma(w)dw|^2dt.$$

The first of these integrals is readily estimated, using the mean-value theorem of Montgomery and Vaughan [6], as

$$\begin{aligned} I_1 &\ll \sum_{n=1}^{\infty}(T+n)|a_n(r\eta,l)|^2n^{-2\sigma}e^{-2n/X} \\ &\ll (T+X)\sum_{n=1}^{\infty}|a_n(r\eta,l)|^2n^{-2\sigma}, \end{aligned} \tag{17}$$

since $\frac{n}{X}e^{-2n/X}\ll 1$. In order to proceed further we shall require information on the size of $a_n(r\eta,l)$. As noted in [4; page 124] we have

$$g(r,c_1c_2)=(\frac{c_1}{c_2})_3(\frac{c_1}{c_2})_3g(r,c_1)g(r,c_2)$$

when $c_1,c_2\equiv 1 \,(\mathrm{mod}\ 3)$ are coprime. Moreover, if $\pi\equiv 1\,(\mathrm{mod}\ 3)$ is a prime of $\mathbb{Z}[\omega]$ then $g(r,\pi^e)=0$ if $\pi\not| r$ and $e\geq 2$, or if $\pi||r$ and either $\pi||c$ or $\pi^3|c$. Finally

$$g(r,\pi)=\overline{(\frac{r}{\pi})_3}g(\pi)$$

when $\pi\not| r$ and

$$g(r,\pi^2)=N(\pi)\overline{g(r/\pi,\pi)}$$

when $\pi||r$. In view of (2) it follows that $|g(r,\pi^e)|\leq N(\pi)^{a(e)}$ for square-free $r$, where $a(1)=\frac{1}{2}$, $a(2)=\frac{3}{2}$ and $a(e)=0$ otherwise. Similar bounds hold when $r$ is square-free apart from powers of $\sqrt{-3}$. Using the definition (14) we now deduce that if $n=\prod p^f$, where each exponent $f\geq 1$ is decomposed as $f=e+3g$, with $e=0,1$ or 2, then

$$|a_n(r\eta,l)|\leq\prod p^{a(e)+2g}\leq\prod p^{f-1/2}.$$

It follows that if $\varepsilon > 0$ then

$$\sum_{n=1}^{\infty} |a_n(r\eta, l)|^2 n^{-2\sigma} \leq \prod_{p\neq 3} \{1 + p^{1-2\sigma} + p^{3-4\sigma} + p^{5-6\sigma} + \ldots\} \ll_\varepsilon 1$$

uniformly for $\sigma \geq 1 + \varepsilon$. We therefore conclude from (17) that

$$I_1 \ll_\varepsilon T + X \tag{18}$$

uniformly for $\sigma \geq 1 + \varepsilon$.

We turn now to the integral $I_2$. Since $\Gamma(2 - 2\sigma + i\tau) \ll_\varepsilon e^{-|\tau|}$ in the range $1 + \varepsilon \leq \sigma \leq 5/4$, we deduce via Cauchy's inequality that

$$|\int_{2-2\sigma-i\infty}^{2-2\sigma+i\infty} Z(r\eta, \sigma + it + w, l) X^w \Gamma(w) dw|^2 \leq X^{4-4\sigma} I' I'',$$

where

$$I' = \int_{2-2\sigma-i\infty}^{2-2\sigma+i\infty} |\Gamma(w)|\ |dw| \ll_\varepsilon 1$$

and

$$\begin{aligned} I'' &= \int_{2-2\sigma-i\infty}^{2-2\sigma+i\infty} |Z(r\eta, \sigma + it + w, l)|^2 |\Gamma(w)|\ |dw| \\ &\ll_\varepsilon \int_{-\infty}^{\infty} |Z(r\eta, 2 - \sigma + iy, l)|^2 e^{-|t-y|} dy. \end{aligned}$$

We insert this in the definition of $I_2$, observing that

$$\int_T^{2T} e^{-|t-y|} dt \ll \begin{cases} 1, & y \in [T, 2T], \\ e^{2T-y}, & y \geq 2T, \\ e^{y-T}, & y \leq T, \end{cases}$$

whence

$$\begin{aligned} I_2 \ll_\varepsilon\ & X^{4-4\sigma} \int_T^{2T} |Z(r\eta, 2 - \sigma + iy, l)|^2 dy \\ & + X^{4-4\sigma} \int_{2T}^{\infty} |Z(r\eta, 2 - \sigma + iy, l)|^2 e^{2T-y} dy \\ & + X^{4-4\sigma} \int_{-\infty}^{T} |Z(r\eta, 2 - \sigma + iy, l)|^2 e^{y-T} dy \end{aligned}$$

To estimate the second and third integrals on the right the pointwise bound

$$Z(r\eta, \alpha + iy, l) \ll_\varepsilon N(r)^{(3/2+\varepsilon-\alpha)/2} (1 + l^2 + y^2)^{3/2+\varepsilon-\alpha}, \tag{19}$$

valid for $1/2 \le \alpha \le 3/2$, suffices (see [4; page 127]). This yields

$$I_2 \ll_\varepsilon X^{4-4\sigma}\{\int_T^{2T} |Z(r\eta, 2-\sigma+iy, l)|^2 dy + N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon}\}.$$

We may now compare this with (16) and (18) to deduce that

$$\begin{aligned} I &= \int_T^{2T} |Z(r\eta, \sigma+it, l)|^2 dt \\ &\ll_\varepsilon X^{4-4\sigma}\int_T^{2T} |Z(r\eta, 2-\sigma+it, l)|^2 dt + T + X \\ &\qquad + X^{4-4\sigma}N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon}. \end{aligned}$$

We proceed to sum over $\eta$ and the alternative signs of $l$. Thus, defining

$$\tilde{I}(\sigma) = \int_T^{2T} \tilde{Z}(r, \sigma+it, l)dt,$$

we obtain

$$\begin{aligned} \tilde{I} &\ll_\varepsilon X^{4-4\sigma}\int_T^{2T} |\tilde{Z}(r, 2-\sigma+it, l)|^2 dt + T + X \\ &\qquad + X^{4-4\sigma}N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon} \\ &= X^{4-4\sigma}\tilde{I}(2-\sigma) + T + X \\ &\qquad + X^{4-4\sigma}N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon}. \end{aligned}$$

We are now ready to apply the estimate (15), whence

$$\tilde{I}(2-\sigma) \ll (T+|l|)^{8\sigma-8}N(r)^{2\sigma-2}\tilde{I}(\sigma).$$

If we insert this into the previous bound we conclude that

$$\begin{aligned} \tilde{I}(\sigma) &\ll_\varepsilon X^{4-4\sigma}(T+|l|)^{8\sigma-8}N(r)^{2\sigma-2}\tilde{I}(\sigma) + T + X \\ &\qquad + X^{4-4\sigma}N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon}. \end{aligned}$$

We write the implied constant as $C_\varepsilon$. Now, if we choose $\sigma \ge 1+\varepsilon$, and

$$X = (\frac{2}{C_\varepsilon})^{1/4\varepsilon}(T+|l|)^2 N(r)^{1/2}$$

then it follows that

$$\tilde{I} \ll_\varepsilon T + X + X^{4-4\sigma}N(r)^{\sigma-1/2+\varepsilon}(T+|l|)^{4\sigma-2+4\varepsilon} \ll_\varepsilon (T+|l|)^2 N(r)^{1/2}.$$

Since $I \ll \tilde{I}$, a similar bound holds for $I$. Indeed, since $|\zeta(3s-2, l)| \gg_\varepsilon 1$ for $\sigma \ge 1+\varepsilon$, we may conclude that

$$\int_T^{2T} |\psi(r, \sigma+it, l)|^2 dt \ll_\varepsilon (T+|l|)^2 N(r)^{1/2}.$$

Moreover, on summing over values of $T \geq 1$ running over powers of 2, we can deduce that

$$\int_{-T}^{T} |\psi(r, \sigma + it, l)|^2 dt \ll_\varepsilon (T + |l|)^2 N(r)^{1/2}, \tag{20}$$

using the pointwise bound (19) on $[-1, 1]$.

It remains to derive the corresponding bound for the function $f(s)$. In the notation of [4; page 124] we have

$$f(s) = N(\alpha)^s \psi_\alpha(1, s + \frac{1}{2}, l),$$

and according to [4; Lemma 3] we have

$$\psi_\alpha(1, z, l) = \Delta \sum_{d|\alpha} \mu(d) N(d)^2 N(d^2\alpha)^{-z} (\frac{\overline{d^2\alpha}}{|d^2\alpha|})^{3l} g(\alpha/d) \psi(\alpha/d, z, l).$$

Here

$$\Delta = \prod_{\pi|\alpha} \{1 - N(\pi)^{2-3z} (\frac{\overline{\pi}}{|\pi|})^{3l}\}^{-1},$$

and $d \in \mathbb{Z}[\omega]$ is restricted to integers $d \equiv 1 \pmod 3$. We may now use (2) to deduce that

$$\psi_\alpha(1, z, l) \ll_\varepsilon N(\alpha)^{-1/2} \sum_{d|\alpha} \mu^2(\alpha/d) N(d)^{-1/2} |\psi(\alpha/d, z, l)|,$$

for $\Re(z) \geq 1 + \varepsilon$, whence

$$\begin{aligned} |\psi_\alpha(1, z, l)|^2 &\ll_\varepsilon N(\alpha)^{-1+\varepsilon} \sum_{d|\alpha} \mu^2(\alpha/d) N(d)^{-1} |\psi(\alpha/d, z, l)|^2 \\ &= N(\alpha)^{-2+\varepsilon} \sum_{r|\alpha} \mu^2(r) N(r) |\psi(r, z, l)|^2. \end{aligned}$$

It follows that

$$|f(\frac{1}{2} + \varepsilon + it)|^2 \ll_\varepsilon N(\alpha)^{-1+3\varepsilon} \sum_{r|\alpha} \mu^2(r) N(r) |\psi(r, \frac{1}{2} + \varepsilon + it, l)|^2,$$

and, since $r$ is restricted to be square-free, (20) yields

$$\begin{aligned} \int_{-T}^{T} |f(\frac{1}{2} + \varepsilon + it)|^2 dt &\ll_\varepsilon N(\alpha)^{-1+3\varepsilon} \sum_{r|\alpha} N(r)(T + |l|)^2 N(r)^{1/2} \\ &\ll_\varepsilon N(\alpha)^{1/2+4\varepsilon} (T + |l|)^2, \end{aligned}$$

as required for Lemma 3.

## 4 Strategy for the proof of Theorem 2

Our proof of Theorem 2 uses many of the ideas of the author [3; §§2-8], although the present argument is rather simpler. We have to investigate

$$\Sigma_1 = \sum_{M<N(m)\le 2M}{}^{*} \ |\sum_{N<N(n)\le 2N}{}^{*} \ c_n(\frac{n}{m})_3|^2,$$

where $\Sigma^*$ denotes summation over square-free elements of $\mathbb{Z}[\omega]$ congruent to 1 modulo 3, as before. It will simplify notation if we suppose that the coefficients $c_n$ are supported on such integers $n \in \mathbb{Z}[\omega]$ lying in the range $N < N(n) \le 2N$.

We begin by defining the norm

$$\mathcal{B}_1(M,N) = \sup\{\Sigma_1 : \sum_n |c_n|^2 = 1\}.$$

Thus our aim is to show that

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^\varepsilon (M + N + (MN)^{2/3}).$$

We observe at once that a non-trivial bound for $\Sigma_1$ can be obtained by dropping the square-free condition on $m$ and including a weight $\exp(-2\pi N(m)/M)$, so that

$$\Sigma_1 \ll \sum_m \exp(-2\pi N(m)/M)|\sum_{N<N(n)\le 2N}{}^{*} \ c_n(\frac{n}{m})_3|^2,$$

the sum being over all $m \in \mathbb{Z}[\omega]$ for which $m \equiv 1 \pmod 3$. If we now expand the above expression we obtain sums

$$\sum_m \exp(-2\pi N(m)/M)(\frac{m}{n_1})_3\overline{(\frac{m}{n_2})_3}.$$

According to Lemma 2 of Heath-Brown and Patterson [3] each of these sums is $O_\varepsilon(N(n_1n_2)^{(1+\varepsilon)/2})$, providing that the character that occurs is non-principal. Since $n_1$ and $n_2$ are square-free, the only remaining case is that in which $n_1 = n_2$. It follows that

$$\begin{aligned}\Sigma_1 &\ll_\varepsilon N^\varepsilon(M\sum_n |c_n|^2 + N\sum_{n_1,n_2}|c_{n_1}c_{n_2}|) \\ &\ll_\varepsilon N^\varepsilon(M+N^2)\sum_n |c_n|^2,\end{aligned}$$

by Cauchy's inequality. We therefore conclude that

$$\mathcal{B}_1(M,N) \ll_\varepsilon N^\varepsilon(M+N^2). \tag{21}$$

This will be the starting point for an iterative bound for $\mathcal{B}_1(M,N)$.

Our first result is the following.

**Lemma 4** *We have*

$$\mathcal{B}_1(M,N)=\mathcal{B}_1(M,N).$$

This is a consequence of 'duality' and the law of cubic reciprocity (see [3; Lemma 1] for the quadratic case). We also note that the cubic residue symbol may be inverted whenever desired.

A second key property of the norm $\mathcal{B}_1(M,N)$ is that it is, essentially, increasing, as the next lemma shows.

**Lemma 5** *There is an absolute constant $C\geq 1$ as follows. Let $M_1,N\gg 1$ and $M_2\geq CM_1\log(2M_1N)$. Then*

$$\mathcal{B}(M_1,N)\ll\mathcal{B}(M_2,N).$$

*Similarly, if $M,N_1\gg 1$ and $N_2\geq CN_1\log(2N_1M)$, then*

$$\mathcal{B}(M,N_1)\ll\mathcal{B}(M,N_2).$$

This is a trivial modification of [3; Lemma 9].

We shall also use the norm

$$\mathcal{B}_2(M,N)=\sup\{\Sigma_2:\sum_n|c_n|^2=1\},$$

where

$$\Sigma_2=\sum_{M<N(m)\leq 2M}|\sum_{N<N(n)\leq 2N}{}^{*}\ c_n(\frac{n}{m})_3|^2, \tag{22}$$

the summation over $m$ running over all integers of $\mathbb{Z}[\omega]$ in the relevent range.

It is then plain that

$$\mathcal{B}_1(M,N)\leq\mathcal{B}_2(M,N). \tag{23}$$

However we also have the following estimation in the reverse direction.

**Lemma 6** *There exist $X,Y\gg 1$ such that $XY^2\ll M$ and*

$$\mathcal{B}_2(M,N)\ll(\log M)^2M^{1/3}X^{-1/3}Y^{-2/3}\min\{Y\mathcal{B}_1(X,N)\,,\,X\mathcal{B}_1(Y,N)\}.$$

We shall prove this in the next section.

As in [3; §2] we now introduce the weight

$$W(x)=\begin{cases}\exp(-\frac{1}{(2x-1)(5-2x)}), & \text{if } \frac12<x<\frac52,\\ 0 & \text{otherwise.}\end{cases}$$

This function is infinitely differentiable for all $x$. It now follows that

$$\Sigma_2\ll\sum_{m\in\mathbb{Z}[\omega]}W(\frac{N(m)}{M})|\sum_n c_n(\frac{m}{n})_3|^2.$$

When the sum on the right is expanded we obtain

$$\sum_{n_1,n_2} c_{n_1}\overline{c}_{n_2} \sum_{m\in\mathbb{Z}[\omega]} W(\frac{N(m)}{M})(\frac{m}{n_1})_3\overline{(\frac{m}{n_2})_3}.$$

It turns out that we may restrict attention to the case in which $n_1$ and $n_2$ are coprime. We therefore set

$$\Sigma_3 = \Sigma_3(M,N) = \sum_{(n_1,n_2)=1} c_{n_1}\overline{c}_{n_2} \sum_{m\in\mathbb{Z}[\omega]} W(\frac{N(m)}{M})(\frac{m}{n_1})_3\overline{(\frac{m}{n_2})_3}, \tag{24}$$

and define

$$\mathcal{B}_3(M,N) = \sup\{\Sigma_3 : \sum_n |c_n|^2 = 1\}. \tag{25}$$

We shall then prove the following, which should be compared with Lemma 2 of [3].

**Lemma 7** *Let $\varepsilon > 0$ be given. Then there exist positive integers $\Delta_2 \geq \Delta_1$ such that*

$$\mathcal{B}_2(M,N) \ll_\varepsilon N^\varepsilon \mathcal{B}_3(\frac{M}{\Delta_1}, \frac{N}{\Delta_2}).$$

We complete the chain of relations amongst the various norms by giving, in §7, an estimate for $\mathcal{B}_3(M,N)$ in terms of $\mathcal{B}_2$.

**Lemma 8** *Let $N \geq 1$. Then for any $\varepsilon > 0$ we have*

$$\begin{aligned}\mathcal{B}_3(M,N) \quad \ll_\varepsilon \quad & MN^{4\varepsilon-1}\max\{\mathcal{B}_2(K,N) :\ K \leq N^2/M\} \\ & + M^{-1}N^{3+4\varepsilon}\sum_{K>N^2/M} K^{-2-\varepsilon}\mathcal{B}_2(K,N),\end{aligned}$$

*where $K$ runs over powers of* 2.

This bound, which depends on the Poisson summation formula, is the key result in the proof of Theorem 2. It is important to note that it does not cover the case in which $N = \frac{1}{2}$, say, for which we have the trivial bound

$$\mathcal{B}_3(M,N) \ll M, \quad (N \leq 1). \tag{26}$$

Lemmas 6, 7 and 8 allow us to estimate $\mathcal{B}_1(M,N)$ recursively, as follows.

**Lemma 9** *Suppose that $\frac{4}{3} < \xi \leq 2$, and that*

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^\varepsilon(M + N^\xi + (MN)^{2/3}) \tag{27}$$

*for any $\varepsilon > 0$. Then*

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^\varepsilon(M + N^{(6\xi-4)/(3\xi-1)} + (MN)^{2/3})$$

*for any $\varepsilon > 0$.*

This will be proved in §8. We note that $\xi = 2$ is admissable in (27), by (21). Since

$$\frac{6\xi - 4}{3\xi - 1} < \xi$$

for $\xi > \frac{4}{3}$, the infimum of the possible values for $\xi$ must be $\frac{4}{3}$. We therefore conclude that

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^\varepsilon (M + N^{4/3} + (MN)^{2/3})$$

for any $\varepsilon > 0$. In view of Lemma 4 we then have

$$\begin{aligned}
\mathcal{B}_1(M,N) &\ll_\varepsilon (MN)^\varepsilon \min\{M + N^{4/3} + (MN)^{2/3}\,,\, N + M^{4/3} + (MN)^{2/3}\} \\
&\ll_\varepsilon (MN)^\varepsilon (M + N + (MN)^{2/3}),
\end{aligned}$$

as required for Theorem 2.

# 5 Proof of Lemmas 6 and 7

To handle $\Sigma_2$ we write each of the integers $m$ occuring in the outer summation of (22) in the form $m = ab^2c$, where $a, b \equiv 1 \pmod 3$ are square-free, and $c$ is a product of a unit, a power of $\sqrt{-3}$, and a cube. We split the available ranges for $a, b$ and $c$ into sets $X < N(a) \le 2X$, $Y < N(b) \le 2Y$ and $Z < N(b) \le 2Z$, where $X, Y$ and $Z$ are powers of 2. There will therefore be $O(\log^2 M)$ possible triples $X, Y, Z$. We may now write

$$\Sigma_2 \le \sum_{X,Y,Z} \Sigma_2(X,Y,Z)$$

accordingly, so that

$$\Sigma_2 \ll (\log M)^2 \Sigma_2(X,Y,Z)$$

for some triple $X, Y, Z$. However

$$\Sigma_2(X,Y,Z) = \sum_{b,c} \sideset{}{^*}\sum_{X' < N(a) \le 2X'} |\sideset{}{^*}\sum_{N < N(n) \le 2N} c_n \Big(\frac{b^2c}{n}\Big)_3 \Big(\frac{a}{n}\Big)_3|^2,$$

where $X' = X'(b,c) = M/N(b^2c)$. It follows that $X \ll X' \ll X$, and hence

$$\begin{aligned}
\Sigma_2(X,Y,Z) &\le \sum_{b,c} \mathcal{B}_1(X',N) \sum_n |c_n|^2 \\
&\ll YZ^{1/3} \max\{\mathcal{B}_1(X',N) :\; X \ll X' \ll X\} \sum_n |c_n|^2,
\end{aligned}$$

since there are $O(Z^{1/3})$ possible integers $c$.

In the same way we have

$$\begin{aligned}\Sigma_2(X,Y,Z) &= \sum_{a,c} \ \sideset{}{^*}\sum_{Y'<N(b)\le 2Y'} \ | \sideset{}{^*}\sum_{N<N(n)\le 2N} c_n\Big(\frac{ac}{n}\Big)_3\Big(\frac{b^2}{n}\Big)_3|^2 \\ &= \sum_{a,c} \ \sideset{}{^*}\sum_{Y'<N(b)\le 2Y'} \ | \sideset{}{^*}\sum_{N<N(n)\le 2N} \overline{c_n}\overline{\Big(\frac{ac}{n}\Big)_3}\Big(\frac{b}{n}\Big)_3|^2 \\ &\le \sum_{a,c}\mathcal{B}_1(Y',N)\sum_n |c_n|^2 \\ &\ll XZ^{1/3}\max\{\mathcal{B}_1(Y',N):\ Y\ll Y'\ll Y\}\sum_n |c_n|^2.\end{aligned}$$

Lemma 6 then follows, since $Z^{1/3}\ll M^{1/3}X^{-1/3}Y^{-2/3}$.

We turn now to Lemma 7. We begin by expanding the sum

$$\sum_{m\in\mathbb{Z}[\omega]} W\Big(\frac{N(m)}{M}\Big)|\sum_n c_n\Big(\frac{m}{n}\Big)_3|^2$$

and sorting the resulting terms to produce

$$\sum_\delta \sum_{m\in\mathbb{Z}[\omega]} W\Big(\frac{N(m)}{M}\Big)\sum_{(n_1,n_2)=\delta} c_{n_1}\overline{c}_{n_2}\Big(\frac{m}{n_1}\Big)_3\overline{\Big(\frac{m}{n_2}\Big)_3}.$$

Clearly we may assume that $\delta\equiv 1\,(\mathrm{mod}\ 3)$. On setting $n_i=\delta r_i$ and

$$\chi(m)=\Big(\frac{m}{r_1}\Big)_3\overline{\Big(\frac{m}{r_2}\Big)_3},$$

we may write the contribution from each $\delta$ as

$$\sum_{(m,\delta)=1} W\Big(\frac{N(m)}{M}\Big)\sum_{(r_1,r_2)=1} c_{r_1\delta}\overline{c}_{r_2\delta}\chi(m)$$

$$\begin{aligned} &= \sum_{d|\delta}\mu(d)\sum_{d|m} W\Big(\frac{N(m)}{M}\Big)\sum_{(r_1,r_2)=1} c_{r_1\delta}\overline{c}_{r_2\delta}\chi(m) \\ &= \sum_{d|\delta}\mu(d)\sum_{s\in\mathbb{Z}[\omega]} W\Big(\frac{N(s)}{M/N(d)}\Big)\sum_{(r_1,r_2)=1} c^*_{r_1}\overline{c^*_{r_2}}\chi(s),\end{aligned}$$

where $d$ runs over non-associated divisors of $\delta$ and

$$c^*_r = c_{r\delta}\Big(\frac{d}{r}\Big)_3.$$

These coefficients are supported on square-free integers $r \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$. In view of (24) and (25) the above expression has modulus at most

$$\sum_{d|\delta} \mathcal{B}_3(\frac{M}{N(d)}, \frac{N}{N(\delta)}) \sum_r |c_{r\delta}|^2.$$

If we now write, temporarily,

$$\tilde{B}_3(M,N) = \max\{\mathcal{B}_3(\frac{M}{\Delta_1}, \frac{N}{\Delta_2}) : \quad 1 \le \Delta_1 \le \Delta_2\},$$

we find that

$$\begin{aligned} \Sigma_3 \quad &\le \quad \tilde{B}_3(M,N) \sum_\delta \sum_{d|\delta} \sum_r |c_{r\delta}|^2 \\ &\le \quad \tilde{B}_3(M,N) \sum_n d(n)^2 |c_n|^2 \\ &\ll_\varepsilon \quad N^\varepsilon \tilde{B}_3(M,N) \sum_n |c_n|^2, \end{aligned}$$

where $d(n)$ is the divisor function for $\mathbb{Z}[\omega]$. Lemma 7 now follows.

# 6 Preliminaries to the Proof of Lemma 8

Our treatment of Lemma 8 requires the following application of the Poisson summation formula, which corresponds to Lemma 11 of [3]. We shall write

$$\chi(m) = (\frac{m}{n_1})_3 \overline{(\frac{m}{n_2})_3},$$

which is a primitive character to modulus $q = n_1 n_2$, providing that the $n_i$ are coprime to each other, and to 3, and are square-free.

**Lemma 10** *With the above notations we have*

$$\sum_{m \in \mathbb{Z}[\omega]} W(\frac{N(m)}{M}) \chi(m) = \frac{\chi(\sqrt{-3}) g(n_1) \overline{g(n_2)} M}{N(q)} \sum_{k \in \mathbb{Z}[\omega]} \widetilde{W}(\sqrt{\frac{N(k)}{N(q)} M}) \overline{\chi}(k),$$

*where*

$$\widetilde{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x+y\omega)) e(t(x+y\omega)/\sqrt{-3}) dx dy$$

*for non-negative real $t$.*

The reader should recall the definition (1).

To establish Lemma 10 we start from the Poisson summation formula for $\mathbb{Z}[\omega]$, which takes the form

$$\sum_{j\in\mathbb{Z}[\omega]} f(j) = \sum_{k\in\mathbb{Z}[\omega]} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+y\omega)e(k(x+y\omega)/\sqrt{-3})dxdy.$$

This is itself an easy consequence of the classical Poisson summation formula in 2 dimensions. Since $\chi$ is a character to modulus $q$ we then have

$$\sum_{m\in\mathbb{Z}[\omega]} W(\frac{N(m)}{M})\chi(m) = \sum_{r(\text{mod } q)} \chi(r) \sum_{j\in\mathbb{Z}[\omega]} W(\frac{N(r+jq)}{M})$$

$$= \sum_{r(\text{mod } q)} \chi(r) \sum_{k\in\mathbb{Z}[\omega]} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\frac{N(r+(x+y\omega)q)}{M})e(\frac{k(x+y\omega)}{\sqrt{-3}})dxdy.$$

We change variables in the integral, writing

$$\sqrt{N(\frac{q}{k})}\frac{k(r+(x+y\omega)q)}{q\sqrt{M}} = u+v\omega,$$

with $u, v \in \mathbb{R}$. (If $k=0$ we omit the factors involving $k/q$.) The Jacobian of this transformation being $N(q)/M$ we find that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\frac{N(r+(x+y\omega)q)}{M})e(\frac{k(x+y\omega)}{\sqrt{-3}})dxdy =$$

$$\frac{M}{N(q)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(N(u+v\omega))e(\frac{(u+v\omega)\sqrt{N(k/q)M}}{\sqrt{-3}})dudv,$$

whence

$$\sum_{m\in\mathbb{Z}[\omega]} W(\frac{N(m)}{M})\chi(m) =$$

$$\frac{M}{N(q)}e(-kr/q\sqrt{-3}) \sum_{k\in\mathbb{Z}[\omega]} \widetilde{W}(\sqrt{\frac{N(k)}{N(q)}M}) \sum_{r(\text{mod } q)} \chi(r)e(-kr/q\sqrt{-3}).$$

We observe at this point that the additive character $\psi(v) = e(v/q\sqrt{-3})$ has minimal period $q$, rather than $q\sqrt{-3}$. Since $(q,3)=1$ we may then substitute $r \equiv -s\sqrt{-3}\,(\text{mod } q)$ to obtain

$$\sum_{r(\text{mod } q)} \chi(r)\psi(-kr) = \sum_{s(\text{mod } q)} \chi(-s\sqrt{-3})\psi(ks\sqrt{-3}).$$

Since $n_1$ and $n_2$ are coprime we now have

$$\sum_{s(\mathrm{mod}\ q)} \chi(s)e(ks/q) =$$

$$(\frac{n_2}{n_1})_3 \overline{(\frac{n_1}{n_2})_3} \sum_{s_1(\mathrm{mod}\ n_1)} (\frac{s_1}{n_1})_3 e(\frac{ks_1}{n_1}) \sum_{s_2(\mathrm{mod}\ n_2)} \overline{(\frac{s_2}{n_2})_3} e(\frac{ks_2}{n_2}),$$

by the usual argument for multiplicativity of exponential sums. The law of cubic reciprocity gives

$$(\frac{n_2}{n_1})_3 \overline{(\frac{n_1}{n_2})_3} = 1.$$

Moreover

$$\sum_{s_1(\mathrm{mod}\ n_1)} (\frac{s_1}{n_1})_3 e(\frac{ks_1}{n_1}) = \overline{(\frac{k}{n_1})_3} g(n_1)$$

and

$$\sum_{s_2(\mathrm{mod}\ n_2)} \overline{(\frac{s_2}{n_2})_3} e(\frac{ks_2}{n_2}) = (\frac{k}{n_2})_3 \overline{g(n_2)},$$

since the cubic characters involved are primitive. This completes the proof of Lemma 10.

Our next result will be used to 'separate the variables' in a function of a product.

**Lemma 11** *Let $\rho : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function whose derivatives satisfy $\rho^{(k)}(x) \ll_{k,A} |x|^{-A}$ for $|x| \geq 1$, for any positive constant $A$. Let*

$$\rho_+(s) = \int_0^\infty \rho(x) x^{s-1} dx.$$

*Then $\rho_+(s)$ is holomorphic in $\Re(s) = \sigma > 0$, and satisfies*

$$\rho_+(s) \ll_{A,\sigma} |s|^{-A}$$

*there, for any positive constant $A$. Moreover if $\sigma > 0$ we have*

$$\rho(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \rho_+(s) x^{-s} ds$$

*for any positive $x$.*

This is merely Lemma 12 of [3].

# 7 Proof of Lemma 8

In the notation of Lemma 10 we have

$$\Sigma_3(M,N) = \sum_{(n_1,n_2)=1} c_{n_1}\overline{c}_{n_2} \sum_{m\in\mathbb{Z}[\omega]} W(\frac{N(m)}{M})\chi(m).$$

We proceed to evaluate the inner sum via Lemma 10, whence

$$\Sigma_3(M,N) = M\sum_{k\in\mathbb{Z}[\omega]}\sum_{(n_1,n_2)=1} a_{n_1}\overline{a}_{n_2}\widetilde{W}\left(\sqrt{\frac{N(k)M}{N(n_1n_2)}}\right)\overline{\chi}(k),$$

where

$$a_n = c_n(\frac{\sqrt{-3}}{n})_3\frac{g(n)}{N(n)}.$$

Note that $k=0$ may be omitted if $N\geq 1$, since then $N(n_1n_2)>1$ and $\chi(0)=0$, the character being non-trivial. We may now apply Lemma 11 to the function $\rho(x)=\widetilde{W}(x)$, which does indeed satisfy the necessary hypothesis, as one sees by repeated integration by parts. We decompose the available $k$ into sets for which $K<N(k)\leq 2K$, where $K$ runs over powers of 2, and use $\sigma=\varepsilon$ for $K\leq N^2/M$, and $\sigma=4+\varepsilon$ otherwise. This yields

$$\Sigma_3(M,N) \ll_\varepsilon M\sum_K (KM)^{-\sigma/2}\int_{-\infty}^{\infty}|\rho_+(\sigma+it)|S(\sigma+it)|dt,$$

where

$$S(s) = \sum_{K<N(k)\leq 2K} |\sum_{(n_1,n_2)=1} b_{n_1}b'_{n_2}\overline{\chi}(k)|,$$

with

$$b_n = a_nN(n)^{s/2},\quad b'_n = \overline{a_n}N(n)^{s/2}.$$

To bound $S(s)$ we use the Möbius function to pick out the coprimality condition in the usual way, giving

$$\begin{aligned} S(s) &\ll \sum_d \sum_{K<N(k)\leq 2K} |\sum_{d|n_1,n_2} b_{n_1}b'_{n_2}\overline{\chi}(k)| \\ &= \sum_d \sum_{K<N(k)\leq 2K} |\sum_{d|n} b_n\overline{(\frac{k}{n})_3}|\cdot|\sum_{d|n} b'_n(\frac{k}{n})_3| \\ &\leq S_1^{1/2}S_2^{1/2} \end{aligned}$$

by Cauchy's inequality, where

$$S_1 = \sum_d \sum_{K<N(k)\leq 2K} |\sum_{d|n} b_n\overline{(\frac{k}{n})_3}|^2$$

$$
\begin{aligned}
&\leq \sum_d \mathcal{B}_2(K,N)\sum_{d|n}|b_n|^2 \\
&\leq \mathcal{B}_2(K,N)\sum_n d(n)|c_n|^2 N(n)^{\sigma-1} \\
&\ll_\varepsilon N^{\varepsilon+\sigma-1}\mathcal{B}_2(K,N),
\end{aligned}
$$

and similarly for $S_2$. It follows that

$$S(s) \ll_\varepsilon N^{\varepsilon+\sigma-1}\mathcal{B}_2(K,N),$$

and since

$$\int_{-\infty}^{\infty} |\rho_+(\sigma+it)|dt \ll_\varepsilon 1,$$

we deduce that

$$\Sigma_3(M,N) \ll_\varepsilon M\sum_K (KM)^{-\sigma/2}N^{\varepsilon+\sigma-1}\mathcal{B}_2(K,N).$$

Then Lemma 8 follows, on re-defining $\varepsilon$.

# 8 The Recursive Estimate

By the symmetry expressed in Lemma 4 the hypothesis (27) yields

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^\varepsilon(M^\xi + N + (MN)^{2/3}).$$

We feed this into Lemma 6, whence

$$\mathcal{B}_2(M,N) \ll_\varepsilon (MN)^{2\varepsilon}M^{1/3}X^{-1/3}Y^{-2/3}\min\{Yf(X,N)\,,\,Xf(Y,N)\},$$

where

$$f(Z,N) = Z^\xi + N + (ZN)^{2/3}.$$

If $X \geq Y$ we bound the minimum by $Yf(X,N)$, whence

$$\mathcal{B}_2(M,N) \ll_\varepsilon (MN)^{2\varepsilon}M^{1/3}X^{-1/3}Y^{-2/3}\{YX^\xi + YN + Y(XN)^{2/3}\}.$$

Here we have

$$M^{1/3}X^{-1/3}Y^{-2/3}YX^\xi \ll M^\xi Y^{1-2\xi}$$

since $X \ll MY^{-2}$. On recalling that $\xi \geq 4/3 > 1/2$ we and $Y \gg 1$ we see that this is $O(M^\xi)$. Moreover

$$M^{1/3}X^{-1/3}Y^{-2/3}YN \leq M^{1/3}N,$$

since we are supposing $X \geq Y$. Finally

$$\begin{aligned} M^{1/3}X^{-1/3}Y^{-2/3}Y(XN)^{2/3} &= M^{1/3}X^{1/3}Y^{1/3}N^{2/3} \\ &\ll M^{2/3}N^{2/3} \\ &\ll M^{1/3}N + M^{4/3} \\ &\ll M^{1/3}N + M^{\xi}, \end{aligned}$$

since $XY \ll M$ and $\xi \geq 4/3$. It follows that

$$\mathcal{B}_2(M,N) \ll_\varepsilon (MN)^{2\varepsilon}(M^{1/3}N + M^{\xi}) \tag{28}$$

when $X \geq Y$.

In the alternative case we bound the minimum by $Xf(Y,N)$, whence

$$\mathcal{B}_2(M,N) \ll_\varepsilon (MN)^{2\varepsilon}M^{1/3}X^{-1/3}Y^{-2/3}\{XY^{\xi} + XN + X(YN)^{2/3}\}.$$

Here

$$M^{1/3}X^{-1/3}Y^{-2/3}XY^{\xi} \ll M^{1/3}X^{2/3}Y^{4/3} \ll M \ll M^{\xi},$$

since $\xi \leq 2$ and $XY^2 \ll M$. Moreover

$$M^{1/3}X^{-1/3}Y^{-2/3}XN \leq M^{1/3}N,$$

since we are now supposing that $Y \geq X$. Finally, for $Y \geq X$ and $XY^2 \ll M$ we have $X \ll M^{1/2}$, whence

$$M^{1/3}X^{-1/3}Y^{-2/3}X(YN)^{2/3} = M^{1/3}X^{2/3}N^{2/3} \ll M^{2/3}N2/3 \ll M^{1/3}N + M^{\xi},$$

as before. It follows that (28) holds when $Y \leq X$ too. It will be convenient to observe that (28) still holds when $M < \frac{1}{2}$, since then $\mathcal{B}_2(M,N) = 0$.

We are now ready to use (28) (with a new value for $\varepsilon$) in Lemma 8, to obtain a bound for $\mathcal{B}_3(M,N)$. We readily see that

$$\max\{\mathcal{B}_2(K,N) :\ K \leq N^2/M\} \ll_\varepsilon N^{\varepsilon}(M^{-1/3}N^{5/3} + M^{-\xi}N^{2\xi})$$

and

$$\sum_{K > N^2/M} K^{-2-\varepsilon}\mathcal{B}_2(K,N) \ll_\varepsilon N^{\varepsilon}(M^{5/3}N^{-7/3} + M^{2-\xi}N^{2\xi-4}).$$

Thus, if $N \geq 1$, we will have

$$\mathcal{B}_3(M,N) \ll_\varepsilon N^{5\varepsilon}((MN)^{2/3} + M^{1-\xi}N^{2\xi-1}).$$

When this is used in Lemma 7 we find that

$$\begin{aligned} \mathcal{B}_3(\frac{M}{\Delta_1},\frac{N}{\Delta_2}) &\ll_\varepsilon N^{5\varepsilon}((MN)^{2/3} + M^{1-\xi}N^{2\xi-1}\Delta_1^{\xi-1}\Delta_2^{1-2\xi}) \\ &\leq N^{5\varepsilon}((MN)^{2/3} + M^{1-\xi}N^{2\xi-1}\Delta_2^{-\xi}) \\ &\leq N^{5\varepsilon}((MN)^{2/3} + M^{1-\xi}N^{2\xi-1}), \end{aligned}$$

providing that $N/\Delta_2 \geq 1$. In the alternative case (26) applies, whence

$$\mathcal{B}_2(M,N) \ll_\varepsilon (MN)^{5\varepsilon}(M + (MN)^{2/3} + M^{1-\xi}N^{2\xi-1}).$$

In view of Lemma 5 and (23) we may now deduce that

$$\begin{array}{rcl} \mathcal{B}_1(M,N) & \ll & \mathcal{B}_1(M',N) \\ & \leq & \mathcal{B}_2(M',N) \\ & \ll_\varepsilon & (M'N)^{5\varepsilon}(M' + (M'N)^{2/3} + M'^{1-\xi}N^{2\xi-1}) \end{array}$$

for any $M' \geq CM\log(2MN)$. We shall choose

$$M' = C\max\{M\,,\,N^{(6\xi-5)/(3\xi-1)}\}\log(2MN),$$

whence

$$\mathcal{B}_1(M,N) \ll_\varepsilon (MN)^{21\varepsilon}\{M + (MN)^{2/3} + N^{(6\xi-5)/(3\xi-1)} + N^{(6\xi-4)/(3\xi-1)}\}.$$

Lemma 9 now follows.

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