Square-Full Numbers in Short Intervals

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A positive integer \( n \) is called square-full if \( p^2 | n \) for every prime factor \( p \) of \( n \). Let \( Q(x) \) denote the number of square-full integers up to \( x \). It was shown by Bateman and Grosswald [1] that

\[
Q(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{3/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/2} + o(x^{1/2}).
\]  

Bateman and Grosswald also remarked that any improvement in the exponent \( 1/6 \) would imply a “quasi Riemann Hypothesis” of the type \( \zeta(s) \neq 0 \) for \( \Re(s) \geq 1 - \delta \). Thus (1) is essentially as sharp as one can hope for at present. From (1) it follows that, for the number of square-full integers in a short interval, we have

\[
Q(x + x^{1/2}y) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)} y
\]  

when \( y \geq x^{1/3} \) and \( y = o(x^{1/6}) \). (It seems more suggestive to write the interval as \( [x, x + x^{3/2}y] \) than \( (x, x+y] \), since only intervals of length \( x^{3/2} \) or more can be of relevance here.) It was shown by Shiu [5] that (2) in fact holds on the longer range \( y \geq x^{0.1526} \) (and \( y = o(x^{1/6}) \)). The exponent of \( x \) was later lowered to 0.1490342 by Jia Chao-hua [2], and to 0.14254 by Liu Hongquan [3]. These last two authors used more delicate exponential sum estimates than were employed by Shiu.

The purpose of the present note is to indicate how a simple observation during the preliminary part of the argument leads immediately to a further improvement in these results. Before stating our theorem it is convenient to introduce the notations

\[ y' = (x + x^{3/2}y)^{1/2} - x^{1/2} \]

and

\[ S(x) = \sum_{a^2b^3 \leq x} 1. \]

It follows of course that \( y' \sim y/2 \) for \( y = o(x^{1/6}) \). As we shall see later we have

\[ S(x) \sim \zeta(3/2)x^{3/4}. \]

We can now state our results.
\textbf{Theorem} \ Let $\theta_0$ be a positive constant with the property that for any $\theta > \theta_0$ there exists a $\delta = \delta(\theta) > 0$ such that
\[ S(x + \frac{1}{2}y) - S(x) = \zeta(\frac{3}{2})y'(1 + O(x^{-\delta})) \] (3)
uniformly for $x^\theta \leq y \leq x^{\frac{1}{2}}$. Then for any $\theta > \theta_0$ there exists $\eta = \eta(\theta) > 0$ such that
\[ Q(x + \frac{1}{2}y) - Q(x) = \frac{\zeta(3/2)}{\zeta(3)}y'(1 + O(x^{-\eta})) \] (4)
uniformly for $x^\theta \leq y \leq x^{\frac{1}{2}}$.

\textbf{Corollary} \ The asymptotic formula (2) holds for $y \geq x^{0.1315162}$ if $y = o(x^{\frac{1}{2}})$.

For the proof of the theorem we begin by giving ourselves a $\theta > \theta_0$. As in Shiu’s work we start with the fact that
\[ Q(x + \frac{1}{2}y) - Q(x) = \sum_{m \leq M} \mu(m). \] (5)
We first consider the terms $m \leq M$, where
\[ M = x^{(\theta - \theta_0)/3}. \]
This choice ensures that
\[ \left( \frac{x}{m^3} \right)^{\theta'} \leq \frac{y}{m^3} \leq \left( \frac{x}{m^3} \right)^{\frac{1}{2}} \]
for $m \leq M$, with
\[ \theta' = \frac{\theta_0}{1 - 2(\theta - \theta_0)} > \theta_0. \]
According to our hypothesis (3), applied with $x$ replaced by $x/m^3$ and $y$ by $y/m^3$, the values $m \leq M$ contribute to (5) a total
\[ \sum_{m \leq M} \mu(m) \left[ S\left( \frac{x + \frac{1}{2}y}{m^3} \right) - S\left( \frac{x}{m^3} \right) \right] = y' \zeta(3/2) \sum_{m \leq M} \frac{\mu(m)}{m^3} \]
\[ + O(y' x^{-\delta} \sum_{m \leq M} m^{-3 + 6\delta}) \]
\[ = y' \left( \frac{\zeta(3/2)}{\zeta(3)} + O(M^{-2}) \right) + O(y' x^{-\delta}) \]
for a suitably small constant $\delta > 0$. This gives us the main term of (4), together with acceptable error terms.
On the other hand, the contribution to (5) arising from numbers $m \geq M$ is at most

\[
| \sum_{x < a^2b^3m^6 \leq x + x^{\frac{1}{2}}y \atop m \geq M} \mu(m) | \leq \sum_{x < a^2b^3m^6 \leq x + x^{\frac{1}{2}}y \atop m \geq M} 1 \\
\leq \sum_{x < a^2c^3 \leq x + x^{\frac{1}{2}}y \atop c \geq M^2} d(c), \quad (6)
\]

where $d(c)$ is the usual divisor function. We define

\[
D = \max_{c \leq x} d(c).
\]

Then (6) is

\[
\leq D \sum_{x < a^2c^3 \leq x + x^{\frac{1}{2}}y \atop c \geq M^2} 1 \\
= D\{S(x + x^{\frac{1}{2}}y) - S(x) - \sum_{x < a^2c^3 \leq x + x^{\frac{1}{2}}y \atop c \leq M^2} 1\}. \quad (7)
\]

Our hypothesis (3) yields

\[
S(x + x^{\frac{1}{2}}y) - S(x) = \zeta(3/2)y'(1 + O(x^{-\delta})),
\]

while

\[
\sum_{x < a^2c^3 \leq x + x^{\frac{1}{2}}y \atop c \leq M^2} 1 = \sum_{c \leq M^2} \left\lfloor (\frac{x + x^{\frac{1}{2}}y}{c^3})^{\frac{1}{2}} \right\rfloor - \left\lfloor (\frac{x}{c^3})^{\frac{1}{2}} \right\rfloor \\
= \sum_{c \leq M^2} \{c^{-3/2}y' + O(1)\} \\
= y' \left\{\zeta(\frac{3}{2}) + O(M^{-1/2})\right\} + O(M^2).
\]

If $\theta$ has been chosen sufficiently close to $\theta_0$ we can deduce that (7) is

\[
\ll Dy'x^{-\delta},
\]

with a new value of $\delta$. Since $D \ll x^\varepsilon$ for any $\varepsilon > 0$, the theorem follows.
Despite the “short interval” form of our hypothesis (3), there seems to be no advantage over a direct estimation of the error term $\Delta(x)$ in the asymptotic formula

$$S(x) = \zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}} + \zeta\left(\frac{2}{3}\right)x^{\frac{2}{3}} + \Delta(x).$$

It was shown by Richert [4] that

$$\Delta(x) \ll x^{2/15},$$

by a simple exponential sum method, and Shiu [5] improved this slightly to

$$\Delta(x) \ll x^{0.1318161\ldots},$$

by using two dimensional sums. This estimate provides our corollary. However it is apparent that further small reductions in the exponent are possible by more complicated exponential sum techniques.

References