Odd Perfect numbers

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1 Introduction

It is not known whether or not odd perfect numbers can exist. However it is known that there is no such number below 10^{300}, (see Brent [1]). Moreover it has been proved by Hagis [4] and Chein [2] independently that an odd perfect number must have at least 8 prime factors. In fact results of this latter type can in principle be obtained solely by calculation, in view of the result of Pomerance [6] who showed that if N is an odd perfect number with at most k prime factors, then

$$N \leq (4k)^{(4k)^{2^{k^2}}}. \tag{1}$$

Pomerance's work was preceded by a theorem of Dickson [3] showing that there can be only a finite number of such N. Clearly however the above bound is vastly too large to be of any practical use. The principal object of the present paper is to sharpen the estimate (1). Indeed we shall handle odd 'multiply perfect' numbers in general, as did Kanold [5], who extended Dickson's work, and Pomerance. Our result is the following.

Theorem *Let $\alpha > 1$ be a rational number. Let N be an odd number with at most k prime factors, and suppose that*

$$\sigma(N) = \alpha N.$$

Then

$$N < (4d)^{4^k},$$

where d is the denominator of α. In particular, if $\alpha = 2$, so that N is an odd perfect number, then

$$N < 4^{4^k}.$$

Our estimate shows that $\omega(N) \gg \log\log N$ for any odd perfect number, so that such integers cannot have a smaller than average number of prime factors. While the bound for N is clearly much less than that given by Pomerance, it is, unfortunately, still too large for practical use.

2 A preparatory Lemma

We begin our argument with the following lemma.

Lemma 1 *Let r be a positive integer and let $n_1, \ldots, n_r$ be integers such that $1 < n_1 < \ldots < n_r$. Suppose that $\frac{a}{b}$ is a rational number in the range*

$$\prod_{i=1}^{r}(1-\frac{1}{n_i}) \leq \frac{a}{b} < \prod_{i=1}^{r-1}(1-\frac{1}{n_i}). \tag{2}$$

Then

$$\prod_{i=1}^{r} n_i \leq (4a)^{2^r-1}.$$

The proof is by induction on r. When $r=1$ we see from (2) that

$$1-\frac{1}{n_1} \leq \frac{a}{b} < 1.$$

It follows that $b > a$, whence $b \geq a+1$, and hence

$$1-\frac{1}{n_1} \leq \frac{a}{a+1}.$$

We therefore deduce that

$$n_1 \leq a+1 \leq 4a = (4a)^{2^1-1},$$

as required.

For the induction step we begin by showing that some integer n_i must satisfy

$$n_i \leq 2^{i+1}a. \tag{3}$$

We observe that

$$1 \geq \prod_{i=1}^{r-1}(1-\frac{1}{n_i}) > \frac{a}{b},$$

whence $b \geq a+1$ as before. Then, if every n_i satisfies $n_i > 2^{i+1}a$, we will have

$$\begin{aligned} \frac{a}{a+1} &\geq \frac{a}{b} \\ &\geq \prod_{i=1}^{r}(1-\frac{1}{n_i}) \\ &> 1-\sum_{i=1}^{r}\frac{1}{n_i} \\ &> 1-a^{-1}\sum_{i=1}^{\infty}2^{-i-1} \\ &= 1-\frac{1}{2a}. \end{aligned}$$

Since $2a \geq a+1$ this is a contradiction.

We now take k to be the smallest integer i for which (3) holds. Thus

$$n_1 \dots n_k \leq n_k^k \leq (2^{k+1}a)^k. \tag{4}$$

and so, in the case $k = r$, we have

$$\prod_{i=1}^{r} n_i \leq (2^{r+1}a)^r. \tag{5}$$

When $1 \leq k \leq r-1$ we will have

$$\prod_{i=k+1}^{r} (1 - \frac{1}{n_i}) \leq \frac{a'}{b'} < \prod_{i=k+1}^{r-1} (1 - \frac{1}{n_i}),$$

where

$$a' = a \prod_{i=1}^{k} n_i, \quad b' = b \prod_{i=1}^{k} (n_i - 1).$$

It therefore follows from the induction assumption that

$$\prod_{i=k+1}^{r-1} n_i \leq (4a')^{2^{r-k}-1} = (4a)^{2^{r-k}-1} (\prod_{i=1}^{k} n_i)^{2^{r-k}-1},$$

whence

$$\prod_{i=1}^{r} n_i \leq (4a)^{2^{r-k}-1} (\prod_{i=1}^{k} n_i)^{2^{r-k}}.$$

In view of (4) we therefore have

$$\prod_{i=1}^{r} n_i \leq (4a)^{2^{r-k}-1} (2^{k+1}a)^{k2^{r-k}}.$$

This estimate also holds when $k = r$, as one sees from (5). It remains to check that

$$4^{2^{r-k}-1} 2^{k(k+1)2^{r-k}} \leq 4^{2^r - 1}$$

and

$$a^{2^{r-k}-1} a^{k2^{r-k}} \leq a^{2^r - 1}$$

for $1 \leq k \leq r$. These bounds however follow from the estimates

$$1 + \frac{k(k+1)}{2} \leq 2^k, \quad \text{and} \quad 1 + k \leq 2^k.$$

This completes the proof of the lemma.

3 The Induction Lemma

Our next result provides the induction step for the proof of our theorem. In order to state the lemma we introduce the notation

$$\Pi(S) = \prod_{p \in S} p$$

for any finite set S of primes p.

Lemma 2 *Let N be an odd number divisible by a set S of primes, and suppose that*

$$\frac{\sigma(N)}{N} = \frac{n}{d} > 1.$$

Then N is the product of two coprime factors U and V with the following properties.

(i) $\omega(V)$ $(= v,$ *say) is at least 1.*

(ii) *U is divisible by a set T of primes, where $v + \#T - \#S$ $(= w,$ say) is non-negative.*

(iii) $\frac{\sigma(U)}{U} = \frac{\nu}{\delta}$, *with* $d\sigma(V)|\delta$.

(iv) $4\delta\Pi(T) \le (4d\Pi(S))^{2^{v+w}}$.

We begin the proof of the lemma by observing that

$$\prod_{p \in S}(1 - \frac{1}{p})$$

is either equal to 1 (if S is empty) or has even numerator when written as a fraction in lowest terms. On the other hand $\frac{d}{n} = \frac{N}{\sigma(N)}$ is less than 1 and has odd denominator when written as a fraction in lowest terms. It follows that

$$\prod_{p \in S}(1 - \frac{1}{p}) \neq \frac{d}{n}.$$

This is the crucial point at which we use the fact that N is odd.

We first examine the case in which

$$\prod_{p \in S}(1 - \frac{1}{p}) > \frac{d}{n}. \tag{6}$$

Since

$$\prod_{p|N}(1 - \frac{1}{p}) < \frac{N}{\sigma(N)} = \frac{d}{n},$$

it follows that

$$\prod_{p|N, p\notin S}(1-\frac{1}{p}) < \frac{d'}{n'} < 1,$$

where

$$d' = d\prod_{p\in S} p = d\Pi(S), \quad n' = n\prod_{p\in S}(p-1). \tag{7}$$

We therefore see that there is a non-empty set $S' = \{p_1, \ldots, p_w\}$, say, consisting of prime factors of N, which is disjoint from S, and for which

$$\prod_{i=1}^{w}(1-\frac{1}{p_i}) \le \frac{d'}{n'} < \prod_{i=1}^{w-1}(1-\frac{1}{p_i}). \tag{8}$$

Lemma 1 therefore applies and shows that

$$\Pi(S') \le (4d')^{2^w-1}$$

and hence

$$\Pi(S\cup S') \le (4d)^{2^w-1}\Pi(S)^{2^w}. \tag{9}$$

Moreover the first inequality of (8), together with the definitions (7), shows that

$$\prod_{p\in S\cup S'}(1-\frac{1}{p}) \le \frac{d}{n}.$$

As before we cannot have equality here, so that

$$\prod_{p\in S\cup S'}(1-\frac{1}{p}) < \frac{d}{n}. \tag{10}$$

When (6) does not hold, so that

$$\prod_{p\in S}(1-\frac{1}{p}) < \frac{d}{n},$$

we see that in fact (9) and (10) hold with $w = 0$ and $S' = \emptyset$.

We now observe that

$$\prod_{p\in S\cup S'}\frac{1-p^{-e(p)-1}}{1-p^{-1}} \le \frac{\sigma(N)}{N} = \frac{n}{d},$$

where for every prime p we take $e(p)$ to be the exponent of p in N. We therefore have

$$\prod_{p\in S\cup S'}(1-p^{-e(p)-1}) \le \frac{n''}{d''},$$

where

$$d'' = d\prod_{p\in S\cup S'} p = d\Pi(S\cup S'),\ \ n'' = n\prod_{p\in S\cup S'}(p-1). \tag{11}$$

Moreover (10) shows that $n''/d'' < 1$. We can therefore apply Lemma 1 in the same way as before to show that $S\cup S'$ contains a non-empty subset S'', say, such that

$$\prod_{p\in S''}(1-p^{-e(p)-1}) \le \frac{n''}{d''}$$

and

$$\prod_{p\in S''} p^{e(p)+1} \le (4d'')^{2^v-1}, \tag{12}$$

where $v = \#S''$. If we now put

$$V = \prod_{p\in S''} p^{e(p)},$$

$U = N/V$ and $T = (S\cup S')\setminus S''$, we see that U and V certainly have properties (i) and (ii) of Lemma 2. Moreover

$$\frac{\sigma(U)}{U} = \frac{n}{d}\frac{V}{\sigma(V)} = \frac{\nu}{\delta},$$

where

$$\delta = d\prod_{p\in S''}(p^{e(p)+1}-1).$$

Thus $d\sigma(V)|\delta$ as required for property (iii). Finally

$$\begin{aligned}\delta &\le d(4d'')^{2^v-1}\\ &\le d(4d\Pi(S))^{2^w(2^v-1)},\end{aligned}$$

by (12), (11) and (9). It therefore follows from (9) that

$$\begin{aligned}4\delta\Pi(T) &\le 4\delta\Pi(S\cup S')\\ &\le 4d(4d\Pi(S))^{2^w(2^v-1)}(4d)^{2^w-1}\Pi(S)^{2^w}\\ &= (4d\Pi(S))^{2^{v+w}},\end{aligned}$$

as required for property (iv) of Lemma 2.

4 Completion of the Proof

We can now finish the proof of the theorem. We shall take N, α and k as in the statement of the theorem and apply Lemma 2 repeatedly, starting with $N =$

N_0, $S = S_0 = \emptyset$, and $\alpha = n_0/d_0$. In general we shall apply Lemma 2 to N_i, S_i, n_i and d_i, and produce the next set of values as $N_{i+1} = U$, $S_{i+1} = T$, $n_{i+1} = \nu$, and $d_{i+1} = \delta$. The numbers v and w which occur will be denoted $v(i+1)$ and $w(i+1)$ respectively Moreover we shall label the values of V that arise by taking $N_i = UV = N_{i+1}V_{i+1}$. From property (i) of the lemma we see that the process will terminate in $s \le k$ steps, by producing a value $N_s = 1$. We will then have $N = V_1 \ldots V_s$ and

$$N \le \sigma(N) = \sigma(V_1)\ldots\sigma(V_s) | d_s,$$

by repeated application of property (iii) of the lemma. Moreover, part (iv) of Lemma 2 shows that

$$4d_{i+1}\Pi(S_{i+1}) \le (4d_i\Pi(S_i))^{2^{v(i+1)+w(i+1)}},$$

so that a trivial induction argument yields

$$4d_s\Pi(S_s) \le (4d_0\Pi(S_0))^{2^{\Sigma}},$$

where

$$\Sigma = \sum_{i=1}^{s}(v(i) + w(i)).$$

It therefore follows that

$$N \le (4d)^{2^{\Sigma}}.$$

However, as $v(i) = \omega(V_i)$, we have $\sum v(i) = \omega(N) \le k$. Moreover, since

$$w(i) = v(i) + \#S_i - \#S_{i-1}, \quad (1 \le i \le s)$$

and $S_0 = S_s = \emptyset$, it follows that

$$\sum_{i=1}^{s} w(i) = \sum_{i=1}^{s} v(i) + \sum_{i=1}^{s-1} \#S_i - \sum_{i=1}^{s-1} \#S_i = \sum_{i=1}^{s} v(i) = \omega(N) \le k.$$

Thus $\Sigma \le 2k$, and the theorem follows.

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