On the Fractal Characteristics of a Stabilised Newton Method

M. Drexler    I. J. Sobey
Oxford University
Numerical Analysis Group

C. Bracher
Technical University at Munich
Department of Theoretical Physics

In this report, we present a complete theory for the fractal that is obtained when applying Newton’s Method to find the roots of a complex cubic. We show that a modified Newton’s Method improves convergence and does not yield a fractal, but basins of attraction with smooth borders. Extensions to higher-order polynomials and the numerical relevance of this fractal analysis are discussed.

Key words and phrases: Newton’s Method, Fractals, Cayley’s Problem, Iterative Mappings, Polynomials

M. Drexler would like to acknowledge the financial support of the German Academic Exchange Service (DAAD) through the programme HSP II/AUFE.

Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England    OX1 3QD
E-mail: namd@comlab.oxford.ac.uk

December, 1995
Contents

1 Introduction 5

2 The Complex Cubic as a Test Case 7
   2.1 Classical Analysis ........................................ 7
   2.2 Evaluation of the Gradient Field ............................ 9
   2.3 Generalisation to Higher-order Polynomials .................. 11

3 Fractal Characteristics of the Orthodox Newton Method 14
   3.1 Definitions .................................................. 14
   3.2 The One-dimensional Restriction ............................... 17
   3.3 The Two-dimensional Mapping and its Properties ............ 19
      3.3.1 The General Inverse Mapping ............................ 19
      3.3.2 The Image of the Negative Real Axis ................... 21
      3.3.3 Formation of the Fractal ............................... 25
   3.4 Self-similarity and Fractal Dimension .......................... 33
      3.4.1 Symmetries ........................................ 34
      3.4.2 Local Scale Factors .................................. 35
      3.4.3 Interpretation and Fractal Dimension ................... 41

4 Fractal Characteristics of a Modified Newton Method 44
   4.1 Definitions .................................................. 44
   4.2 The Julia Set of the Modified Method ......................... 46
      4.2.1 One-dimensional Restriction ............................... 46
      4.2.2 Two-dimensional Case .................................. 47
   4.3 Influence of the Stabilisations ............................... 49
      4.3.1 Shift Scaling ........................................ 50
      4.3.2 Line Searches ........................................ 54
      4.3.3 Norm Dependency ....................................... 55
   4.4 A Modified Newton Method with Empty Julia Set ............. 57

5 Concluding Remarks 59
   5.1 Generalisation to Higher-order Polynomials .................... 59
   5.2 Numerical Relevance ....................................... 64
   5.3 Beyond Polynomials ....................................... 70
List of Figures

1. Zero contours for real and imaginary part of $z^3 - 1$ ........................................ 8
2. Contour plot of $L_1$ positive definite region ..................................................... 9
3. Gradient field for $f_1$ ....................................................................................... 10
4. Gradient field for $f_2$ ....................................................................................... 11
5. The Newton fractal in the interval $[-2,2] \times [-2,2]$ ............................................ 15
6. First-quadrant image of the negative real axis ....................................................... 22
7. Images of the negative real axis and its rotations after up to three inverse Newton iterations ................................................................. 23
8. Images of the negative real axis and its rotations after $k$ inverse Newton iterations ................................................................. 24
9. Typical arrangement of the images under the inverse Newton mapping ............................... 27
10. Relation between the dominant vector and $t$ ......................................................... 28
11. Plot of the Julia set for large $|z_0|$ ................................................................. 29
12. Effect of the inverse Newton mapping on a basic set of vectors .............................. 31
13. Basins of attraction for $z^3 = 1$, ..................................................................... 38
14. Basins of attraction for modified Newton method, $s_u = 10, s_d = 2$ ............... 38
15. Basins of attraction for $z^3 = 1$, ..................................................................... 39
16. Basins of attraction for $\sin z = \frac{1}{2}$ ............................................................ 39
17. Blob of order 1 on the negative real axis, bordered by blobs of higher order ......................... 42
18. Julia sets for $z^3 = 1$ with various shift limits. ................................................. 52
19. Basins of attraction of root $(1,0)$ for $z^3 = 1$ with various shift limits. ......... 53
20. Basins of attraction for Newton’s method with line searches .............................. 56
21. Basins of attraction for the method with empty Julia set ................................. 58
22. Convergence history for starting point $z_A$ ...................................................... 65
23. Convergence history for three starting points close to a Julia point .................... 65
24. Convergence history for starting points inside the attractive circle .................... 67
25. Basins of attraction of non-zero roots for $\sin z = \frac{1}{2}$ with various shift limits. ................................. 72

List of Tables

1. Starting Points for Numerical Experiments .................................................... 64
2. Comparison of Convergence for various Newton methods, $z^3 = 1$. ............ 68
3. Comparison of Convergence for various Newton methods, $z^4 = 1$. .......... 69
4. Convergence on a Sun SparcClassic, $z^4 = 1$ .............................................. 70
5. Comparison of Convergence for various Newton methods, $\sin z = \frac{1}{2}$ .......... 71
Description of the Colour Plates

**Fig. 13** Basins of attraction for \( z^3 = 1 \), using the orthodox Newton method. Array of \( 300 \times 300 \) equidistant points cast over \([-2, 2] \times [-2, 2] \). An offset of \( k \cdot 85 \) has been added to the actual iteration number according to the converged solution. Colouring according to converged solution

- **blue** for root \((1, 0)\), iteration range \(1 \ldots 80\),
- **yellow** for root \(e^{\frac{4\pi}{3}}\), iteration range \(85 \ldots 160\),
- **green/red** for root \(e^{\frac{2\pi}{3}}\), iteration range \(170 \ldots 250\).

**Fig. 14** Basins of attraction for \( z^3 = 1 \), using the modified Newton method as defined in 4.1. Array of \( 300 \times 300 \) equidistant points cast over \([-2, 2] \times [-2, 2] \). An offset of \( k \cdot 85 \) has been added to the actual iteration number according to the converged solution. Colouring according to converged solution as in Fig. 13.

**Fig. 15** Basins of attraction for \( z^5 = 1 \), using the orthodox Newton method. Array of \( 300 \times 300 \) equidistant points cast over \([-2, 2] \times [-2, 2] \). An offset of \( k \cdot 50 \) has been added to the actual iteration number according to the converged solution. Colouring according to converged solution

- **blue** for root \((1, 0)\), iteration range \(1 \ldots 45\),
- **green** for root \(e^{\frac{2\pi}{5}}\), iteration range \(50 \ldots 95\),
- **orange** for root \(e^{\frac{4\pi}{5}}\), iteration range \(100 \ldots 145\),
- **yellow** for root \(e^{\frac{6\pi}{5}}\), iteration range \(150 \ldots 195\),
- **red-brown** for root \(e^{\frac{8\pi}{5}}\), iteration range \(200 \ldots 245\).

**Fig. 16** Basins of attraction for \( \sin z = \frac{z}{7} \), using the orthodox Newton method. Array of \( 300 \times 300 \) equidistant points cast over \([-1.2, -0.78] \times [-0.21, 0.21] \). An offset of \( k \cdot 85 \) has been added to the actual iteration number according to the converged solution. Colouring according to converged solution

- **blue** for positive real root \( \sin x = \frac{x}{7} \) and other roots with \( x > 0 \), iteration range \(1 \ldots 80\),
- **green/red** for root \((0, 0)\), iteration range \(85 \ldots 160\),
- **yellow** for negative real root \( \sin x = \frac{x}{7} \) and other roots with \( x < 0 \), iteration range \(170 \ldots 250\).
1 Introduction

Newton's method is a widely established algorithm to solve non-linear systems. Its appeal lies in a great simplicity (see [10] for a geometric interpretation), easy generalisation to multiple dimensions and a quadratic local convergence rate [14]. Despite these features, little is known about its global behaviour. Many practitioners using it for applied problems assume from the classical Newton-Kantorovich theorem that the use of Newton's method is safe if only it can be started close enough to a solution. So, in practice, a local residual is often used to switch between a global solution strategy for problems and a local Newton solver. If the local Newton solver does not decrease the residual in a suitable fashion, it is assumed to 'diverge' and another trial started from a different approximate. For a survey of strategies of this kind, see for example [13]. It is often implicitly assumed that globally the residual behaves in an 'unpredictable' way with increases and decreases alike, until the iterates somehow get close to the root. In this work, we will show that for a certain class of functions, the residual actually behaves in an explicable way and the iterates' path from a given starting point to the root can be described in terms of simple rotation and prolongation mappings.

It is well known that the basins of attraction of different roots have fractal boundaries when Newton's method is used to determine the complex roots of polynomials; the simplest example being the cubic $z^3 = 1$. The problem of describing the basins of attraction of the roots is known as Cayley's problem and arose as early as 1879. In a substantial paper, Gaston Julia [7] used this problem as an example of describing sets that later came to bear his name. The Julia set may be described as the union of all points that are eventually mapped onto a singular point. He also derived some properties of the set solving Cayley's problem, namely a reflective symmetry with regard to the real axis, an invariance under rotations by multiples of $\frac{2\pi}{3}$ and the inclusion of $z = 0$ and $z = \infty$ as images of each other. Also, the fragmented character of the set was mentioned. Almost seven decades later, interest in iterated polynomial mappings and their properties re-emerged, but much attention was devoted to the Mandelbrot set ([9], [11], [2]) and Newton's method was treated in a historical context and as an accessible example to introduce the concept of Julia sets [12].

Our research was motivated by the application of a modified Newton method, which had been devised to solve applications in turbulence modelling [4], to a complex cubic. To our initial surprise, the fractal character of the structure seemed to have vanished completely. Of course, the question was, whether the modified method had changed the characteristic length scale so that we couldn't see the fractal structure or whether it had disappeared completely - and if so, why. So, in contrast to the research mentioned above that focused on the properties of Julia sets, we examine the mechanism by which Newton's method generates a fractal structure and how modifications affect that mechanism.
This work is structured in four general parts. The first part will give a
discussion of the complex cubic in a classical sense and state the properties and
quantities which are necessary for the subsequent analysis.

The second part examines the way in which the fractal structure emerges for
the boundaries of the basins of attraction and its properties, using an orthodox
Newton method. We are able to arrive at analytic values for all the scale factors
which describe the fractal, its local and global symmetries, give an explicit map
for determining the Julia set, and present an estimate for the fractal dimension
of the structure. From the explicit map for the Julia set, we also arrive at
an explanation how the fractal structure emerges which relies only on simple
geometric operations involving vectors.

In the third part, we discuss the modified Newton method in a fractal context.
We can show that suitable modifications remove the fractal character completely.
We also present a way of modifying Newton’s method which gradually removes
the fractal structure and yet in one limit is equivalent to the orthodox Newton
method. Finally, we state an algorithm for a modified Newton method which
has an empty Julia set, i.e. no fractal character.

The final part forms both a conclusion and an extension of the previous anal-
ysis. It generalises the results of the cubic to general polynomials of the form
$z^n = 1$ and gives analytic values for most of the scale factors and the symmetries.
The theoretical results are then verified and explained using numerical experi-
ments and the modified method is discussed in a numerical context. It emerges
from that discussion that the fractal concepts are clearly visible in the numeri-
cal context and actually explain the convergence paths completely. Finally, the
stabilised numerical method is applied to a problem involving a transcendental
function and it is observed that the same principles still hold.

In writing this paper, we are aware that some of the observations which we
state about fractals and Newton’s method are known, some are known but not
well documented, and some are new. We however feel that it is useful to present
a comprehensive survey of results for fractals associated with Newton’s method
in a common framework, and the new results are best introduced in such a
framework.
2 The Complex Cubic as a Test Case

As we have mentioned above, it is well established that the boundaries of the basins of attraction for different roots of polynomials of the form

\[ z^\nu = 1, \quad z \in \mathbb{C}, \quad \nu \in \mathbb{N}, \nu > 2 \]  

are fractals. We choose \( \nu = 3 \) and concentrate on the complex cubic

\[ z^3 = 1, \quad z \in \mathbb{C}. \]  

With this choice, we hope to study all relevant aspects when trying to find a numerical solution to equations of the above form. The low degree of a cubic polynomial allows a closed-form solution to most of the problems arising in the analysis. Yet, as we shall show, the cubic test case does not incur loss of generality.

This section will firstly present a classical analysis of the test function in order to summarize important facts that will form the background for the following fractal analysis. Secondly, we shall discuss the gradient field that is of great importance for understanding the orthodox Newton method and its modifications. Finally, we generalise the results for polynomials of the form (2.1).

2.1 Classical Analysis

Analytical solutions to (2.2) can easily be obtained as

\[ z_k = e^{2(k-1)\pi i}, \quad k = 1, 2, 3 \]  

or

\[ z_1 = 1, \quad z_{2,3} = -\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}. \]  

We note that the set of roots is invariant under a rotation of \( \frac{2\pi}{3} \) and its multiples. For any further analysis it is useful to split the equation (2.2) into its real and imaginary part by setting \( z = x + iy \). We therefore obtain a system in two dimensions

\[ f(x, y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x^3 - 3xy^2 - 1 \\ 3x^2y - y^3 \end{pmatrix} = 0. \]  

Any function of a complex variable can be stated in such fashion, splitting the real and imaginary part. In this work, we consider complex functions, for which powerful analytical tools are available, as an example of two-dimensional systems. However, it is worth noting that fractal properties are not necessarily linked with complex functions and any two-dimensional system could exhibit fractal behaviour.
For the zero contours of the system \((2.5)\), we have
\[
\begin{align*}
  f_1 : y &= \pm \frac{1}{\sqrt{3}} \sqrt{x^2 - \frac{1}{x}} \\
  f_2 : y &= \pm \sqrt{3} x \\
  y &= 0.
\end{align*}
\]  
(2.6)

These contours, the intersection of which defines the roots, are depicted in Fig. 1. The solid lines mark \(f_1 = 0\), the dashed lines \(f_2 = 0\). The horizontal axis denotes \(x\) and the vertical axis \(y\). This convention will be used throughout this work for all plots of the complex plane unless stated otherwise.

![Figure 1: Zero contours for real and imaginary part of \(z^3 - 1\)](image)

From \((2.5)\), we can immediately form the Jacobian
\[
J = \begin{bmatrix}
  3x^2 - 3y^2 & -6xy \\
  6xy & 3x^2 - 3y^2
\end{bmatrix}
\]  
(2.7)

that will be used by a multidimensional Newton method to find the next iterate. Determining the eigenvalues of \(J\), we get
\[
\lambda_{1,2} = 3 \left( x^2 - y^2 \right) \pm i 6xy = 3 \left( x \pm iy \right)^2.
\]  
(2.8)

We particularly note the special cases

- \(\lambda_1 = \lambda_2\) for \(\{(x = 0) \lor (y = 0)\}\) and
- singularity for \(\{(x = 0) \land (y = 0)\}\) only.

In this case of singularity, the Jacobian degenerates to the zero matrix and the inverse
\[
J^{-1} = \frac{1}{9 (x^2 + y^2)^2} \begin{bmatrix}
  3x^2 - 3y^2 & 6xy \\
  -6xy & 3x^2 - 3y^2
\end{bmatrix}
\]  
(2.9)
is not defined. Trying to determine a region where the classical Newton-Kantorovich stability is fulfilled, we furthermore form the Hessian for each of the scalar subfunctions in (2.5) and obtain for the eigenvalues in each case

$$\lambda_{f_1,2} = \pm 6\sqrt{x^2 + y^2}. \quad (2.10)$$

Therefore, the Hessian is indefinite for any point other than the singular origin. We conclude that there is no region of convergence that is guaranteed by a positive definite Hessian.

Transforming the function to an $L_2$ norm space by trying to minimize $f^T f$ yields for the function to the minimized

$$F = f^T f = 3x^2y^2(x^2 + y^2) + 2x(4y^2 - x^2) + x^6 + y^6 + 1. \quad (2.11)$$

The condition on positive definiteness of the Hessian for this function is

$$3(x^2 + y^2)^2 > 2\sqrt{(x^4 - y^4 - x)^2 + (2xy^3 + 2x^3y + y^2)^2}, \quad (2.12)$$

yielding a region that exhibits threefold rotational symmetry with respect to the origin, but no hint towards a fractal nature of the basin of attraction. Fig. 2 shows that region, with the innermost contour being the zero contour on the outside of which (2.12) just holds. On the outer contours, the condition is satisfied.

![Figure 2: Contour plot of $L_2$ positive definite region](image)

### 2.2 Evaluation of the Gradient Field

Newton’s method works by locally approximating any scalar subfunction of the system (2.5) by its tangential plane. The intersection of these two planes for the two-dimensional system determines a line in the non-degenerate case. The point
where this line intersects the zero plane is chosen to be the next Newton point (cf. [10]).

For an understanding of what step Newton’s method determines at a certain point, the local gradient therefore is very important. For the complex cubic, we obtain two gradients, one for the real part

\[
\nabla f_1 = \begin{pmatrix} 3(x^2 - y^2) \\ -6xy \end{pmatrix}
\]

(2.13)

and one for the imaginary part

\[
\nabla f_2 = \begin{pmatrix} 6xy \\ 3(x^2 - y^2) \end{pmatrix}
\]

(2.14)

The gradient field is depicted in Fig. 3 and Fig. 4. The vectors are scaled by a factor of 27.0 to keep the proportions in order. It is clearly visible that the two gradient fields are orthogonal, a fact which will be further discussed in the following section.

Recalling that the plane tangent to \( f \) at \((x_0, y_0)\) is

\[
z = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \mathbf{x} = f(x_0, y_0) + f_x(x_0, y_0) x + f_y(x_0, y_0) y,
\]

(2.15)

we can determine the general Newton direction for any starting point.

For example, it is quite straightforward to deduce from the figures that any starting point on the real axis will always be mapped onto the real axis. The plane for the real part will be sloped only towards the right, and start either...
above or below the zero plane. In the same fashion, the plane for the imaginary part will slope strictly upwards. Therefore, its intersection with the zero plane has to be parallel to the real axis. As $f_2(x, 0) = 0$, it will coincide with the real axis and therefore the next Newton point has to lie on the real axis. If the plane started below the zero plane, i.e. for $x < 1$, the next iterate will lie to the right of the starting point, otherwise to the left.

### 2.3 Generalisation to Higher-order Polynomials

Extending the above analysis to polynomials of the form (2.1), we shall briefly state a few results concerning the general case. Separating real and imaginary part and writing $z = x + iy$, (2.1) can be written as the two-dimensional system

\[
\begin{align*}
\delta_1 &= \frac{\partial}{\partial x} f_1 = \sum_{k=0}^{[\frac{\nu}{2}]} (-1)^k \left( \frac{\nu}{2k} \right) x^{\nu - 2k} y^{2k} - 1 = 0, \\
\delta_2 &= \frac{\partial}{\partial x} f_2 = \sum_{k=0}^{[\frac{\nu}{2}]} (-1)^k \left( \frac{\nu}{2k + 1} \right) x^{\nu - 2k - 1} y^{2k+1} = 0,
\end{align*}
\]

with $|x|$ denoting the integer part of $x$ obtained by truncation. We take partial derivatives to obtain

\[
\begin{align*}
\delta_1 &= \frac{\partial}{\partial x} f_1 = \sum_{k=0}^{[\frac{\nu}{2}]} (-1)^k (\nu - 2k) \left( \frac{\nu}{2k} \right) x^{\nu - 2k - 1} y^{2k} \\
\delta_2 &= \frac{\partial}{\partial x} f_2 = \sum_{k=0}^{[\frac{\nu}{2}]} (-1)^k (\nu - 2k - 1) \left( \frac{\nu}{2k + 1} \right) x^{\nu - 2k - 2} y^{2k+1}.
\end{align*}
\]
For any \( n > m \), we can write
\[
(n - m) \binom{n}{m} - (m + 1) \binom{n}{m+1} = n \binom{n}{m} - \binom{n}{m+1} - m \binom{n+1}{m+1} = 0.
\] (2.18)

Therefore, using (2.17), the gradients can be written as
\[
\nabla f_1 = \begin{pmatrix} \delta_1 \\ -\delta_2 \end{pmatrix}, \quad \nabla f_2 = \begin{pmatrix} \delta_2 \\ \delta_1 \end{pmatrix}.
\] (2.19)

It is immediately obvious that the gradients are orthogonal. Therefore, the remarks on the gradient field stated above also hold for the general case.

The Jacobian
\[
J = \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{pmatrix}.
\] (2.20)

is orthogonal up to a scale factor \( \sqrt{\delta_1^2 + \delta_2^2} \) - therefore the mappings associated with Newton’s method are locally angle-preserving. We note that the Cauchy-Riemann equations
\[
\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \wedge \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}
\] (2.21)

are fulfilled - \( f(x, y) = f_1 + if_2 \) is an analytical function. The inverse of the Jacobian is
\[
J^{-1} = \frac{1}{\delta_1^2 + \delta_2^2} \begin{pmatrix} \delta_1 & \delta_2 \\ -\delta_2 & \delta_1 \end{pmatrix}.
\] (2.22)

Therefore, the eigenvalues of \( J \) both have modulus \( \sqrt{\delta_1^2 + \delta_2^2} \), and are conjugate complex, as \( (\delta_1^2 + \delta_2^2) \) is real and positive (cf. (2.8)). The inverse of the Jacobian exists unless \( \delta_1 = \delta_2 = 0 \), and hence \( \frac{\partial f}{\partial z} = 0 \).

Regarding the Hessian, we obtain from (2.21)
\[
\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial x \partial y}, \quad \frac{\partial^2 f_1}{\partial y^2} = -\frac{\partial^2 f_2}{\partial x \partial y},
\] (2.23)

\[
\frac{\partial^2 f_1}{\partial x \partial y} = \frac{\partial^2 f_2}{\partial x^2} = -\frac{\partial^2 f_2}{\partial y^2}
\]

As
\[
\Delta f_1 = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} = 0,
\]
\[
\Delta f_2 = \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} = 0,
\] (2.24)
we can see that the trace of both Hessians is zero,

$$\text{tr}(\mathbf{H}(f_1)) = \text{tr}(\mathbf{H}(f_2)) = 0.$$  \hspace{1cm} (2.25)

Therefore, the sum of eigenvalues of each Hessian is zero. As the Hessians are symmetric, their eigenvalues must be real and hence of opposite sign (cf. (2.10)). Therefore, the functions \(f_1\) and \(f_2\) have no local extrema for \(z \neq 0\). Considering the case \(\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial y^2} = 0\), i.e. \(z = 0\), we can state that this point cannot be a local extremum either. With \(f_1\) and \(f_2\) being harmonic functions (2.24), and the arithmetic mean of all function values of a harmonic function assembled on a circle equaling the function value in the centrepoint of the circle (proof via Cauchy integral for a loop), there must be points in the vicinity of the origin with function values larger than \(f(0)\). Therefore, the origin is a saddle point and no region of convergence is guaranteed by a positive definite Hessian.
3 Fractal Characteristics of the Orthodox Newton Method

In this section, we present an analysis of the orthodox Newton Method that leads to an explanation of the emerging fractal when applied to a complex cubic. We give the necessary definitions and start with a restriction of the method to one dimension. The insights gained in this analysis will provide guidance for the analysis of the general two-dimensional problem. After finishing the general analysis, we will be able to give a qualitative and quantitative description of the fractal. This description will be used to determine characteristic features of the fractal. Among those are scale factors that describe the self-similar structure and symmetries that mark the invariants of the fractal. These characteristics are discussed both on a global and a local scale. Combining the knowledge about the formation of the fractal and its self-similar characteristics, we give an estimate for its fractal dimension.

3.1 Definitions

Throughout this chapter, we will be concerned with Newton’s method defined in the standard way.

Definition 3.1 The orthodox Newton method on \( f(z) \) is defined by the iteration

\[
z^{(k+1)} = z^{(k)} - \frac{f(z^{(k)})}{f_z(z^{(k)})} = g(z^{(k)})
\]

(3.1)

with \( z^{(k)} \) denoting the \( k \)th iterate, and \( f_z \) denoting \( \frac{df}{dz} \).

The view we shall take of the Newton method is slightly different to that implied by (3.1). Rather than finding the next iterate given a starting point, we shall ask which points \( z^{(k)} = z \) are mapped into a given point \( z^{(k+1)} = z_0 \) by (3.1). We therefore define

Definition 3.2 The complex Newton polynomial of order \( \nu \) is defined by

\[
(\nu - 1)z^\nu - \nu z_0 z^{\nu-1} + 1 = 0, \quad z, z_0 \in \mathbb{C}
\]

(3.2)

Despite the existence of a 'common-sense concept' of a fractal, it can be difficult to give a general definition for such a structure. Following Mandelbrot [9], a strict definition of a fractal is

Definition 3.3 A fractal is a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension.
This definition, however precise, is hard to apply in practice if the Hausdorff-Besicovitch dimension is difficult to determine or even unknown - which in fact is the case for many well-known fractals. Therefore, alternative and more intuitive definitions of a fractal are in use. For the purposes of this study, we use 'fractal' according to the definition by Falconer [5].

**Definition 3.4** We refer to a set $S$ as a fractal with the following in mind.

- $S$ has a fine structure, i.e. detail on arbitrary small scales.
- $S$ is too irregular to be described in traditional geometrical language, both locally and globally.
- Often $S$ has some form of self-similarity, perhaps approximate or statistical.
- Usually, the 'fractal dimension' of $S$ (defined in some way) is greater than its topological dimension.
- In most cases of interest $S$ is defined in a very simple way, perhaps recursively.

A suitable working definition for the purposes of this study is

**Definition 3.5** The Newton fractal of order $v$ is defined by the union of all points that are mapped into the singular origin by the Newton mapping (3.2).

![Newton Fractal](image)

**Figure 5**: The Newton fractal in the interval $[-2, 2] \times [-2, 2]$

It is obvious from this definition that with $z$ belonging to the Newton fractal, $z_0$ also belongs to the fractal. A picture of the Newton fractal in the vicinity of
the origin can be seen in Fig. 5. It is worth noting that the fractal is the union of points that lie on the boundary of the basins of attraction, and does not consist of the basins themselves. Fig. 5 also provides a good illustration for most of the points in definition 3.4.

As this study is concerned with the iteration of a function of a complex variable, we will refer to the underlying framework of Julia set theory. Following Falconer [5], we define the necessary sets as follows.

**Definition 3.6** The Julia set $\mathcal{J}(g)$ of a complex-variable function $g$ is the closure of the set of repelling periodic points of $g$.

**Definition 3.7** The Fatou set $\mathcal{F}(g)$ of a complex-variable function $g$ is the complement of the Julia set $\mathcal{J}(g)$. $\mathcal{F}(g)$ is also known as the stable set of $g$.

Using this, we could equivalently define the Newton fractal as the union of all Julia points of the Newton mapping. An important notion for identifying Julia points on the fractal structure will be their order.

**Definition 3.8** The order of a specific Julia point on a Newton fractal is defined as the number of Newton iterations it takes to reach the origin from that point.

An excellent overview of the Julia set theory concerned with polynomials of a complex variable can be found in Falconer’s book [5]. In this work, we want to highlight one particular lemma that is very instructive for understanding the character of fractals associated with polynomials and Newton’s method. We first have to define the important concept of a basin of attraction.

**Definition 3.9** The basin of attraction $\mathcal{A}(w)$ of an attractive fixed point $w$ of a function $g$ is defined by $\mathcal{A}(w) = \{z \in \mathbb{C} : g^k(z) \to w \text{ as } k \to \infty \}$.

With this definition, we are able to state the lemma.

**Lemma 3.10** Let $w$ be an attractive fixed point of $g$. Then, denoting the boundary of the basin of attraction $\partial \mathcal{A}(w)$ by $\partial \mathcal{A}(w)$, we get: $\mathcal{J}(g) = \partial \mathcal{A}(w)$. The same is true if $w = \infty$.

For a proof and further background on Julia set theory, see [5]. One implication of the lemma is that any point of the Julia set must lie on the boundary of all basins of attraction for all attractive fixed points of $g$. Thus, an approximate close to a Julia point with only a small perturbation might converge to any of the roots. Further aspects of this lemma will be discussed in later sections. A colour plot of the basins of attraction illustrating the fractal character and lemma 3.10 is given in Fig. 13.
3.2 The One-dimensional Restriction

We begin by considering the restriction of the orthodox method (3.1) to one dimension, the real axis. As pointed out in section 2.2, the method will not leave the real axis for any starting point with $y = 0$. Therefore, it suffices to only regard the first component of the two-dimensional system (2.5). In this case, substituting $z^{(n)} = x$, $z^{(n+1)} = a$ for simplicity, the orthodox Newton method can be written as

$$a = x - \frac{x^3 - 1}{3x^2} = \frac{2x^3 + 1}{3x^2} \quad (3.3)$$

From (3.3), we clearly see that $x = 0$ will be the only point for which the method is not defined - in the nomenclature of iterated mappings this is a repelling point and therefore the first member of the Julia set. The idea in determining the Julia set is now to look for the points that are mapped into the origin by Newton’s method at some stage, as their union constitutes the Julia set. We therefore look at the inverse Newton mapping, i.e. solve (3.3) for $x$, assuming we have $a$ given (cf. [2]).

The image of $a = 0$ can be determined from (3.3) by inspection, yielding

$$\chi_1 = -\frac{1}{\sqrt{2}} \quad (3.4)$$

We can restate (3.3) as a cubic equation,

$$2x^3 - 3x^2a + 1 = 0. \quad (3.5)$$

This can be solved using Cardani’s formula [1], yielding one real solution for $-\infty < a < 1$

$$x = \frac{F(a)^2 + a^2}{2F(a)} + \frac{a}{2} \quad (3.6)$$

$$F(a) = \sqrt[3]{a^3 - 2} + 2\sqrt{1 - a^3}.$$

We note that there exist two real solutions for $a = 1$, and three real solutions for $a > 1$. As the singular origin lies to the left of $a = 1$ and, by the argument in 2.2, any point that will be mapped into the origin has to lie to its left as well, we will be concerned with $a < 0$.

Letting $\chi_0 = 0$, the first non-zero part of the Julia set $\{\chi_j\}$ on the real axis therefore is $\chi_1 = -\frac{1}{\sqrt{2}}$, and subsequently,

$$\chi_{j+1} = \frac{F(\chi_j)^2 + \chi_j^2}{2F(\chi_j)} + \frac{\chi_j}{2} \quad (3.7)$$

with $F(\chi_j)$ defined as in (3.6).
Clearly, by the recursive nature of the equation, the Julia set consists of an infinite, but countable number of points. The rest of this section will be concerned with the specific properties of this one-dimensional Julia set.

First, we try to establish the fixed points under the recursion (3.6), for which the following must hold:

\[
\frac{a}{2} = \frac{F(a)^2 + a^2}{2F(a)}
\] (3.8)

Solving for \(F(a)\), and ruling out the case \(a = 0\), we obtain

\[
F(a) = \frac{1}{2} (1 \pm \sqrt{3})a
\] (3.9)

and substituting \(k = \frac{1}{2} (1 \pm \sqrt{3})\), this rewrites as

\[
a^3 = \frac{2}{(k^3 - 1)^2} \left( -k^3 \pm \sqrt{k^6 - 3(k^3 - 1)^2} \right).
\] (3.10)

As we require the solution \(a\) to be real, and the term under the square root is by inspection negative, this states that the one-dimensional mapping (3.6) has no fixed points for \(a < 0\).

With this result, we look at the properties of the mapping as \(a\) becomes larger and quickly establish that

\[
x - a = \frac{F(a)^2 + a^2 - F(a)a}{2F(a)} < 0
\] (3.11)

as

\[
\text{sgn}(a) = \text{sgn}(F(a)) \quad \text{for} \quad a < 0,
\]

\[
\text{and} \quad c^2 < cd < d^2 \quad \text{for} \quad \{(c < d) \land (\text{sgn}(c) = \text{sgn}(d))\}.
\] (3.12)

Therefore, each point of the iteration (3.6) lies to the left of its predecessor.

To establish the properties for very large \(|a|\), we first note that

\[
\lim_{a \to -\infty} \frac{F(a)}{a} = \lim_{a \to -\infty} \sqrt[3]{\frac{1 - \frac{2}{a^3}}{\frac{1}{a^6} + 2}} = 1.
\] (3.13)

We now determine the ratio between two consecutive points in the recursive process,

\[
\frac{x}{a} = \frac{F(a)^2 + a^2 + F(a)a}{2F(a)a} = \frac{1}{2} + \frac{1}{2} (\Omega + \frac{1}{\Omega})
\]

\[
\Omega = \frac{F(a)}{a}
\] (3.14)
From a variational viewpoint, we know that
\[ \min \{ \Omega + \frac{1}{\Omega} \} = 2 \]  
and therefore
\[ \frac{x}{a} \geq \frac{3}{2}. \]  
As the minimum in (3.15) is attained for \( \Omega = 1 \), we know that for large \( j \),
\[ \lambda_{3,\infty} = \lim_{j \to \infty} \frac{\chi_{j+1}}{\chi_{j}} = \frac{3}{2}. \]  
These results state that the points of the one-dimensional Julia set start from \( \chi_{1} = -\frac{1}{\sqrt{2}} \) and approach \(-\infty\) like a geometric progression with factor \( \lambda_{3,\infty} \). The Julia set \( \{ \chi_{j} \} \) consists of the union of these points and the origin. All other points on the real axis are in the Fatou set and will converge to the root \( z = 1 \).

### 3.3 The Two-dimensional Mapping and its Properties

With the one-dimensional case understood, we can consider the solution of the general two-dimensional problem. In principle, we will adhere to the line of analysis presented in the previous section, although there obviously will be differences in detail. The general strategy however, will again be to determine the Julia set via the inverse mapping and successive images of the singular origin. After giving the general solution for the inverse Newton mapping, we will consider the image of the real axis to quantify some of its properties. With that knowledge, we will be able to finally explain the formation of the fractal.

#### 3.3.1 The General Inverse Mapping

In multiple dimensions, the orthodox Newton iteration for a vector-valued function \( f(x) \) can be stated as
\[ x^{(n+1)} = x^{(n)} - J^{-1} f(x^{(n)}). \]  
Using the inverse Jacobian (2.9) and the complex cubic (2.5), we obtain the system of equations to be solved for the inverse Newton iteration, where \( x^{(n)} = (x, y) \) and \( x^{(n+1)} = (a, b) \):
\[ \begin{align*}
2x^3 - 3ax^2 + 6y(b - y)x + 3y^2a + 1 &= 0 \\
2y^3 - 3by^2 + 6x(a - x)y + 3x^2b &= 0.
\end{align*} \]  
Writing
\[ z = x + iy, \quad z_0 = a + ib \]
we see that (3.19) is equivalent to (3.2) for $\nu = 3$

$$2z^3 - 3z_0z^2 + 1 = 0. \quad (3.21)$$

In this form, the system can be solved for $z$. With $z^*$ denoting the conjugate of $z$, the desired $(x, y)$ can be obtained from

$$x = \frac{1}{2}(z + z^*), \quad y = \frac{1}{2i}(z - z^*). \quad (3.22)$$

As (3.21) has generally three distinct solutions, there are three $(x, y)$ to be expected as the solutions to the inverse Newton mapping. We also note the similarity of (3.21) to the one-dimensional case (3.5). The solution of the system, however, cannot be expected to be as simple as it was in one dimension, as now all three roots of the cubic are of interest.

Solving (3.21), we get for the solution

$$z_1 = \frac{1}{2} \left[ \sqrt[3]{\varphi} + z_0 + \frac{(z_0)^2}{\sqrt[3]{\varphi}} \right], \quad (3.23a)$$

$$z_2 = -\frac{1}{4} \sqrt[3]{\varphi} \left\{ \left[ z_0 - \sqrt[3]{\varphi} \right]^2 - i\sqrt{3} \left[ \sqrt[3]{\varphi^2} - (z_0)^2 \right] \right\}, \quad (3.23b)$$

$$z_3 = -\frac{1}{4} \sqrt[3]{\varphi} \left\{ \left[ z_0 - \sqrt[3]{\varphi} \right]^2 + i\sqrt{3} \left[ \sqrt[3]{\varphi^2} - (z_0)^2 \right] \right\}, \quad (3.23c)$$

with the nonlinearity

$$\varphi = (z_0)^3 - 2 + 2\sqrt{1 - (z_0)^3}. \quad (3.24)$$

Denoting the real and imaginary part of the cube root of the nonlinearity by

$$\xi = \text{Re}(\sqrt[3]{\varphi}), \quad \eta = \text{Im}(\sqrt[3]{\varphi}), \quad (3.25)$$

and

$$\alpha = \xi^2 + \eta^2, \quad \beta = b^2 - a^2 \quad (3.26)$$

we can evaluate (3.22) to obtain the solution in $(x, y)$:

$$x_1 = \frac{1}{2} \left[ \xi + \frac{2ab\eta - \beta\xi}{\alpha} + a \right] \quad (3.27a)$$

$$y_1 = \frac{1}{2} \left[ \eta + \frac{\beta\eta + 2ab\xi}{\alpha} + b \right],$$

$$x_2 = \frac{1}{4\alpha} \left[ \alpha(2a - \xi - \sqrt{3} \eta) + \beta(\xi + \sqrt{3} \eta) + 2ab(\sqrt{3} \xi - \eta) \right] \quad (3.27b)$$

$$y_2 = \frac{1}{4\alpha} \left[ \alpha(2b - \eta + \sqrt{3} \xi) + \beta(\sqrt{3} \xi - \eta) - 2ab(\sqrt{3} \eta + \xi) \right],$$
\[
x_3 = \frac{1}{4\alpha} \left[ \alpha (2a - \xi + \sqrt{3} \eta) + \beta (\xi - \sqrt{3} \eta) - 2ab(\sqrt{3} \xi + \eta) \right]
\]
\[
y_3 = \frac{1}{4\alpha} \left[ \alpha (2b - \eta - \sqrt{3} \xi) - \beta (\sqrt{3} \xi + \eta) + 2ab(\sqrt{3} \eta - \xi) \right],
\] (3.27c)

Setting
\[
c = 1 + 3ab^2 - a^3, \quad d = b(b^2 - 3a^2),
\] (3.28)
the expressions for the real and imaginary part of \( \varphi \) are
\[
\text{Re}(\varphi) = \sqrt{2} \sqrt{c^2 + d^2 + 2c - c - 1}
\]
\[
\text{Im}(\varphi) = \text{sgn}(d) \sqrt{2} \sqrt{c^2 + d^2 - 2c - d},
\] (3.29)

so that \((\xi, \eta)\) are determined from (3.25).

### 3.3.2 The Image of the Negative Real Axis

For further analysis of the two-dimensional mapping, we initially restrict our view to the image of the negative real axis, i.e. \((a < 0, b = 0)\). From the general solution, it is obvious that
\[
\xi = \sqrt{a^3 - 2 + 2\sqrt{1 - a^2}}, \quad \eta = 0.
\] (3.30)

With this, the general solution becomes
\[
x_1 = \frac{1}{2} \left[ \frac{\xi^2 + a^2}{\xi} + a \right]
\]
\[
y_1 = 0
\] (3.31a)

\[
x_{2,3} = -\frac{(a - \xi)^2}{4 \xi}
\]
\[
y_{2,3} = \pm \sqrt{3} \frac{\xi^2 - a^2}{4 \xi}
\] (3.31b)

As the solution \((x_1, y_1)\) is the one-dimensional restriction already discussed, we shall be concerned with \((x_{2,3}, y_{2,3})\), considering without loss of generality the positive branch with \(y_1 > 0\). A parametric plot of this branch is given in Fig. 6. The crosses mark the images of the Julia points \(\{x_j\}\) on the negative real axis.

We note that, eliminating the parameter, (3.31b) can be written as
\[
y_{2,3} = \pm \sqrt{x_{2,3} - (x_{2,3})^2}.
\] (3.32)
Using (3.13) for $\xi$, it follows from (3.31b) that

$$\lim_{a \to -\infty} x_2 = 0$$
$$\lim_{a \to -\infty} y_2 = 0.$$  \hspace{1cm} (3.33)

Therefore, the origin is the image of $\infty$ under the inverse Newton mapping.

By Taylor expansion of $\xi$ for large $|a|$, we obtain

$$\xi = a + \frac{2}{3|a|} + O(a^{-2}).$$  \hspace{1cm} (3.34)

Substituting the first terms of this expansion into (3.31b), the image of the negative real axis becomes

$$x_2 \approx \frac{1}{9a^2}$$
$$y_2 \approx \frac{1}{\sqrt{3|a|}}.$$  \hspace{1cm} (3.35)

Abandoning the parametric description, the image of the axis can therefore be approximated by

$$y \approx \frac{4}{\sqrt{x}}.$$  \hspace{1cm} (3.36)
Numerical experiment shows that this approximation is valid to an error of 1% for \( a < -3.1 \). In the following sections, we will use that result to further quantify characteristic properties of the fractal.

As the whole negative real axis is mapped on the arc given by (3.31b), the infinite number of Julia points in the interval \( (-\infty, 0] \) is mapped on the arc between the points \((0, 0)\) and \((\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}})\). The latter point is the image of the origin under the inverse Newton mapping and can also be obtained from the first Julia point \( \chi_1 \) on the real axis by a \( \frac{2\pi}{3} \) rotation.

To show how the self-similar structure is emerging from images of the negative real axis, we consider Fig. 7. It shows the locus of points that are mapped onto the negative real axis (and its rotations by \( \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \)) in up to three Newton iterations. The arcs mapping onto the rotations are dashed, the ones mapping onto the real axis solid.

![Figure 7: Images of the negative real axis and its rotations after up to three inverse Newton iterations](image)

For a closer look on the different mappings, we consider the sequence Fig. 8. The labels refer to the closest solid arc new in the picture compared with the picture before and are labeled in the form \( l_1 - l_2 - l_3 \), where \( l_k \) denotes the arc obtained using mapping \( l_k \) in (3.27). \( k \) stands for the number of inverse Newton iterations. We can see that with increasing \( k \), a better approximation to the
Figure 8: Images of the negative real axis and its rotations after \( k \) inverse Newton iterations.
fractal structure is obtained. In the following section, we will explain how this happens and also give an explanation of the way in which the three images of an arc are scaled and rotated.

### 3.3.3 Formation of the Fractal

In order to understand the formation of the fractal, it is helpful to rewrite (3.27) in slightly different notation. Knowing the general form of a two-dimensional rotation matrix about an angle $\theta$ to be

$$
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix},
$$

we can restate (3.27) as

$$
x_1 = b_1 + b_2 + b_3 + t \quad (3.38a)
$$

$$
x_2 = R_{120}b_1 + R_{240}(b_2 + b_3) + t \quad (3.38b)
$$

$$
x_3 = R_{240}b_1 + R_{120}(b_2 + b_3) + t \quad (3.38c)
$$

where

$$
x_k = \begin{pmatrix}
x_k \\
y_k
\end{pmatrix} \quad (3.39a)
$$

$$
b_1 = \frac{1}{2} \begin{pmatrix}
\xi \\
\eta
\end{pmatrix} \quad (3.39b)
$$

$$
b_2 = \frac{ab}{\xi^2 + \eta^2} \begin{pmatrix}
\eta \\
\xi
\end{pmatrix} \quad (3.39c)
$$

$$
b_3 = \frac{b^2 - a^2}{2(\xi^2 + \eta^2)} \begin{pmatrix}
-\xi \\
\eta
\end{pmatrix} \quad (3.39d)
$$

$$
t = \frac{1}{2} \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} \quad (3.39e)
$$

The definitions of $\xi$ and $\eta$ are retained from (3.25). $R_{120}$ and $R_{240}$ denote the rotation matrices about an angle of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively.
It is remarked that \( \mathbf{b}_2 \) and \( \mathbf{b}_3 \) are orthogonal. We denote

\[
\mathbf{b}_{23} = \mathbf{b}_2 + \mathbf{b}_3 = \frac{1}{2(\xi^2 + \eta^2)} \left( 2ab\eta - (b^2 - a^2)\xi, 2ab\xi + (b^2 - a^2)\eta \right).
\]

(3.40)

Using the orthogonality property, it can easily be stated in the \( L_2 \) norm

\[
\|\mathbf{b}_{23}\| = \sqrt{\|\mathbf{b}_2\|^2 + \|\mathbf{b}_3\|^2} = \frac{1}{2} \sqrt{\xi^2 + \eta^2} \frac{a^2 + b^2}{\sqrt{\xi^2 + \eta^2}}
\]

(3.41)

Furthermore, employing the \( L_2 \) norm, we can establish that

\[
\begin{align*}
\|\mathbf{b}_{23}\| &= \frac{\sqrt{a^2 + b^2}}{\sqrt{\xi^2 + \eta^2}} \\
\|\mathbf{t}\| &= \frac{\sqrt{\xi^2 + \eta^2}}{\sqrt{a^2 + b^2}} \\
\|\mathbf{b}_1\| &= \sqrt{\frac{\xi^2 + \eta^2}{a^2 + b^2}} \\
\|\mathbf{t}\| &= \sqrt{\frac{\xi^2 + \eta^2}{a^2 + b^2}}
\end{align*}
\]

(3.42a)

(3.42b)

Combining these equalities, we arrive at

\[
\|\mathbf{b}_{23}\| \cdot \|\mathbf{b}_1\| = \|\mathbf{t}\|^2.
\]

(3.43)

This equation yields the important result that \( \mathbf{t} \) can never be the dominant vector of the triple. Its length can at the most be equal to that of either \( \mathbf{b}_1 \) or \( \mathbf{b}_{23} \).

From the definitions in 3.3.1, we can see that this case is approached in the limit for large \((a, b)\).

From (3.42), we can infer that \( \mathbf{b}_{23} \) will be the dominant vector, if

\[
\frac{\xi^2 + \eta^2}{a^2 + b^2} < \frac{\xi^2 + \eta^2}{a^2 + b^2}
\]

or

\[
|\xi^2 + \eta^2| < |a^2 + b^2|.
\]

(3.44)

(3.45)

Due to the complicated form of \( \xi \) and \( \eta \), it is hard to draw general conclusions for \( a \) and \( b \) from this condition. However, we will shortly see that the assumption of \( \mathbf{b}_{23} \) being the dominant vector is justified at least for small \((a, b)\). The notion that either \( \mathbf{b}_1 \) or \( \mathbf{b}_{23} \) is dominant also confirms the point that the rotations in (3.38) will have some visible influence on the location of the three images of \((a, b)\).

The case where the points are all lying close to the tip of some huge \( \mathbf{t} \), rotated by minute quantities, is ruled out.

We postulate that the three images \( x_k \) of \((a, b)\) will be arranged in the following fashion. We take \((a, b)\) to be a Julia point of order \( n \), lying on a specific side of the directed line \( L \), drawn between the origin and the Julia point of order 1 in the same quadrant.
- One image lies in the same direction as \((a, b)\), but outside a circle of radius \(\frac{1}{\sqrt{2}}\) centered at the origin. It is also further away from the origin than \((a, b)\). This is denoted by *prolongation*.

- Two images are situated inside a circle of radius \(\frac{1}{\sqrt{2}}\) centered at the origin, but rotated onto the two other main branches of the fractal, i.e. approximately by \(\frac{2\pi}{3}\) and \(\frac{4\pi}{3}\), respectively. One of the images (which are of order \((n+1)\)) will stay on the same side of \(\mathcal{L}_s\) in the respective quadrant, the other one will be reflected (and scaled) through \(\mathcal{L}_s\). These images are denoted by *rotation* and *reflected rotation*.

![Figure 9: Typical arrangement of the images under the inverse Newton mapping](image)

Fig. 9 depicts the arrangement of the images for \((a, b)\) lying close to the negative real axis. The bold lines are the symmetry axes \(\mathcal{L}_s\). The dotted circle is the circle with radius \(\frac{1}{\sqrt{2}}\), which shall be referred to as the *attractive circle* for the rest of this chapter. Point 1 is a result of prolongation and points 2 and 3 are the results of reflected rotation and rotation. The fractal structure is greyshaded in the background.

In order to relate these propositions to the equations (3.38), we will first write down a few scalar products of the vectors involved. They can easily be obtained from the definitions in (3.39) after some arithmetic.

\[
\begin{align*}
\mathbf{b}_{23} \mathbf{t} &= \frac{a^2 + b^2}{4(\xi^2 + \eta^2)} (a\xi + b\eta), \\
\mathbf{b}_1 \mathbf{t} &= \frac{1}{4} (a\xi + b\eta).
\end{align*}
\]
Recalling that for the angle between two vectors $a$ and $b$,

$$\gamma_{ab} = \arccos \frac{ab}{\|a\| \cdot \|b\|},$$

(3.47)

we get with the definitions in (3.39)

$$\gamma_{b_{23}t} = \gamma_{b_{1}t} = \arccos \frac{a\xi + b\eta}{\sqrt{(\xi^2 + \eta^2)(a^2 + b^2)}}.$$  

(3.48)

Therefore, the vector $t$ is the bisector of the angle between $b_{23}$ and $b_{1}$. Furthermore, we obtain for the angle between $b_{23}$ and $b_{1}$

$$\gamma_{b_{1}b_{23}} = \arccos \frac{(a\xi + b\eta)^2 - (a\eta - b\xi)^2}{(a^2 + b^2)(\xi^2 + \eta^2)}.$$  

(3.49)

We note that prolongation occurs when $t$ points in the same direction as the dominant vector (or one of its rotated images). Therefore, we have to consider a situation as depicted in Fig. 10. Clearly, prolongation through addition of the vectors will in this example only occur in the case $b(240)$, where $b$ has been rotated by $\frac{4\pi}{3}$.

![Figure 10: Relation between the dominant vector and t](image)

**Large Julia points** To explain the properties of the mapping (3.38), we at first consider the case of large $|z_0|$, i.e. $\|(a, b)\| \gg 1$. We can then approximate $\varphi$ in (3.24) by

$$\varphi \approx (z_0)^3$$  

(3.50)

and therefore

$$\xi = a, \quad \eta = b.$$  

(3.51)
We can verify that this approximation is valid with an error of less than 2% for $|z_0| > 10$ and less than 1% for $|z_0| > 15$. With this, we immediately obtain for the angles between the vectors

$$\gamma_{b_1t} = \gamma_{b_23t} = 0, \quad \gamma_{b_1b_2} = 0 \quad (3.52)$$

and for their lengths

$$\|b_{23}\| = \|b_1\| = \|t\| = \frac{1}{2} \sqrt{a^2 + b^2}. \quad (3.53)$$

It is easy to see that in this case, prolongation will occur for the straightforward addition of the three vectors and the scale factor of $\lambda_{3,\infty} = \frac{3}{2}$ that was derived in 3.2 is attained. Fig. 11 confirms this scale factor. The other two points will lie close to the origin, as the rotations in (3.38) arrange the vectors almost in the fashion of a triangle with equal sides. The distance of the endpoint from the origin will be determined only by the small differences in angle and length that prevail between the vectors and are neglected in the approximation. Although an exact measure of that displacement would need further expansion of (3.50), it is convincing by inspection that its modulus will be much smaller than $\frac{1}{\sqrt{2}}$.

![Figure 11: Plot of the Julia set for large $|z_0|$](image)

**Small Julia points** For Julia points with small $|z_0|$, i.e. $\|(a, b)\|^3 \ll 1$, we can approximate the square root in (3.24) by binomial expansion,

$$\sqrt{1 - (z_0)^3} = 1 - \frac{(z_0)^3}{2} - \frac{(z_0)^6}{8} - \ldots \quad (3.54)$$

and obtain for $\varphi$

$$\varphi \approx \frac{(z_0)^6}{4}. \quad (3.55)$$
Thus, the real and imaginary part of the cube root become

\[ \xi = -\frac{a^2 - b^2}{\sqrt[4]{4}}, \quad \eta = -\sqrt[4]{2}ab. \] (3.56)

Verifying the validity of the approximation for \( |z_0| = \frac{1}{\sqrt{2}} \), we find that the error is 1.6%. The approximation therefore holds with good cause within the whole attractive circle. Substituting the results for \( \xi \) and \( \eta \) into the expressions for the angles, we obtain

\[ \gamma_{6,1t} = \gamma_{6,23t} = \arccos \left( -\frac{a}{\sqrt{a^2 + b^2}} \right), \quad \gamma_{6,1b_{23}} = \arccos \left( \frac{a^2 - b^2}{a^2 + b^2} \right). \] (3.57)

and for the lengths

\[ \|b_1\| = \frac{2 + b^2}{2 \sqrt{2}}, \]
\[ \|b_{23}\| = \frac{1}{\sqrt{2}}, \]
\[ \|t\| = \frac{1}{2} \sqrt{a^2 + b^2}. \] (3.58)

Knowing that in the attractive circle, \( \|(a, b)\| \leq \frac{1}{\sqrt{2}} \), we can bound the length of the vectors by

\[ \|b_1\| \leq \frac{1}{4 \sqrt{2}}, \]
\[ \|t\| \leq \frac{1}{2 \sqrt{2}}. \] (3.59)

As \( \gamma_{6,1t} \) can now range from 0 to \( \pi \) for general \((a, b)\), we have to find some range that contains the Julia points to be more specific about the mapping properties. This is supported by the fact that we know the shape and dimensions of the image of the real axis analytically. The image of the real axis determines the general appearance of the first lobe and might be regarded as the ‘first-order’ approximation to the fractal. Using this, we derive an estimate for the sectors of the attractive circle that contains the Julia points. To do so, we first consider the angle of the vectors mentioned above with respect to the positive \( x \)-axis. Most importantly, we obtain for the angle between \( b_{23} \) and the vector \((1, 0)\)

\[ \angle_{b_{23}} = \pi \pm \frac{2\pi}{3} k, \quad k = 0, 1. \] (3.60)

The rotational part holds because of the symmetry of the cube root operator when applied to \((3.55)\). Of the three solutions, we choose the one that is closest to the direction of \( t \). This is without loss of generality, as \( b_{23} \) will be mapped onto this vector by one of the rotations in \((3.38)\) if it didn’t coincide with it firstly. From the form of the image of the negative real axis \((3.31b)\), we know
that the image is in the first quadrant, so we choose $\angle b_{23} = \frac{\pi}{6}$. It is also clear from (3.31b), that
\[
\gamma_{b_{1}t} = \gamma_{b_{23}t} \leq \frac{\pi}{6}.
\] (3.61)

We will now examine how this angle changes under consecutive mappings (3.38) and from this try to establish a sector of the attractive circle which the Julia points cannot leave. Fig. 12 shows the effect of the three mappings (3.38) on the vectors. We note that mapping 1 is the prolongation.

![Diagram showing the effect of the inverse Newton mapping on a basic set of vectors](image)

Figure 12: Effect of the inverse Newton mapping on a basic set of vectors

Clearly, the only mapping that actually can increase the angle (i.e. the image point deviates further from the direction of $b_{23}$ than $t$ did) in this case is mapping 2. We see that in this case we can write for the coordinates of the image point $(x_2, y_2)$
\[
x_2 = \| b_{23} \| - \| t \| \cdot \cos \left( \gamma_{b_{23}t} + \frac{\pi}{3} \right) - \| b_1 \| \cdot \cos \left( 2 \gamma_{b_{23}t} - \frac{\pi}{3} \right),
\]
\[
y_2 = \| t \| \cdot \sin \left( \gamma_{b_{23}t} + \frac{\pi}{3} \right) + \| b_1 \| \cdot \sin \left( 2 \gamma_{b_{23}t} - \frac{\pi}{3} \right),
\] (3.62)
to obtain the new angle from
\[
\gamma_{b_{23}t} = \arctan \frac{y_2}{x_2}.
\] (3.63)

With this new angle, $(x_2, y_2)$ can be determined giving rise to another $\gamma_{b_{23}t}$. The angle to which this sequence is converging will be the limit angle for the sector containing the Julia points. Solving the above recurrence numerically, and accounting for a $2\%$ approximation error within the small Julia point approximation, we obtain a limit angle of
\[
\gamma_{b_{23}t}^\infty \approx 35.56^\circ.
\] (3.64)
The angle can be confirmed graphically in an image of the fractal. This result means that for any Julia point, the angle between $b_{23}$ and $b_1$ will be bounded by $2 \gamma_{b_{23}t}$. Therefore, there will always be one prolongation step (similar to mapping 1 in Fig. 12) and two steps that rotate with an angle close to $\pm \frac{2\pi}{3}$. We note that for any $(a, b)$ in the attractive circle, we can bound

$$\frac{\|b_1\|}{\|t\|} < \frac{1}{2}.$$ (3.65)

Furthermore, we know that in the case of mapping 3 in Fig. 12, $t$ can be regarded as the base of an incomplete polygon with $b_1$ and $b_{23}$ emerging on each side with an angle of $\frac{2\pi}{3} - \gamma_{b_{23}t}$. As $b_1$ is shorter than half the base, however, it can never intersect $b_{23}$ and therefore, the endpoint of the sum of vectors always lies on the same side of $b_{23}$, namely the one determined by $t$. Due to the rotation by $\frac{2\pi}{3}$, $t$ prescribes a different side of $b_{23}$ in the two non-prolongation cases. Therefore, the existence of a rotation and a reflected rotation as postulated is confirmed.

We note that if $b_{23}$ becomes any of the other rotated solutions in (3.60), the above argument still holds, only the numbering of the different mappings is permuted. It is also noted from (3.57) that $\gamma_{b_{23}t}$ approaches 0 as $a$ becomes bigger than $b$. This is exactly the case when the validity range of the approximation for small Julia points is left on the negative real axis. For large $(a, b)$, however, $\gamma_{b_{23}t} \approx 0$ again. Therefore, the two approximations for both small and large Julia points seem to describe the fractal across the whole range of points satisfactorily, taking the symmetry of the cube root operator in (3.51) into account.

From these considerations, the following results can be summarised which explain the formation of the fractal.

- The original Julia point $(0, 0)$ has infinitely many images under the inverse Newton mapping on the negative real axis and its rotated images by $\pm \frac{2\pi}{3}$.

- Due to local conformity, any point between two Julia points of order $n$ lies between the two image Julia points of order $(n + 1)$ after one inverse Newton mapping.

- The negative real axis and its rotated images are mapped by one inverse Newton mapping onto arcs described by (3.31b) inside the attractive circle. These arcs form 'lobes' which approximate the general shape of the structure.

- Julia points within the attractive circle are confined to sectors determined by approximate angles of $\pm 35.6^\circ$ around the main symmetry axes $L$, with angles $\frac{\pi}{3}, \pi, \frac{4\pi}{3}$ regarding the positive real axis.

- Under each further inverse Newton iteration, any arc containing Julia points within the attractive circle has three images:
One arc outside the attractive circle in the same general direction (on
the same branch) as the old arc.

Two arcs inside the attractive circle on branches approximately forming
an angle of $\pm \frac{2\pi}{3}$ with the original branch.

The arcs rotated within the attractive circle lie on different sides of
the main axis $L_s$ of their respective branches.

The images inside the attractive circle are scaled down in size.

- The same three images exist for any arc containing Julia points outside the
attractive circle.

- A useful notion is to consider images within the attractive circle as 'nested'
and images outside the attractive circle as 'copied'.

- As any structure outside the attractive circle is an image of the structure
inside the unit circle, the Julia points outside the unit circle stay close to
the main axes of their branches.

- A change of branch is only possible inside the attractive circle, any iteration
outside the attractive circle consists only of a lateral movement along that
branch.

- Due to local conformity, any lateral movement of a point between two Julia
points of order $(n-1)$ and $n$ outside the unit circle cannot extend further
than the neighbouring interval of Julia points of order $n$ and $(n+1)$.

- Outside the attractive circle, the structure consists of locally conformal
images of the lobe on the same main axis $L_s$ within the attractive circle.
These images are repeated again and again towards infinity.

\section{Self-similarity and Fractal Dimension}

In this section, we establish the local properties of the fractal determining self-
similarity, namely symmetry and scaling. From this, we suggest a way of calcul-
ating the dimension of the third-order Newton fractal.

It is an interesting property of the fractal that the global symmetries have
their counterpart in discrete local symmetries, which are of different order. To
examine those, we note that the mapping $z_0 \mapsto z$ (3.2) is conformal for $z \neq 0$, i.e.
in particular locally angle-preserving and preserving length ratios. On the other
hand, by definition of the Newton fractal, every element of the Julia set is an
(iterated) image of the origin $z_0 = 0$. Hence, the neighbourhood of every point of
the fractal has the same structure as the neighbourhood of the origin. The local
appearance and properties of the fractal are therefore (using sufficiently large
magnification) similar at each of its points. It follows from this argument, that it
is sufficient to examine the properties of the origin to reach general conclusions about the local properties at any point of the fractal.

### 3.4.1 Symmetries

From the equation governing Newton’s iteration (3.2), we immediately establish by inspection an invariance under the coordinate transformation

$$z \mapsto ze^{\frac{2\pi i}{3}}, \quad z_0 \mapsto z_0e^{\frac{2\pi i}{3}}, \quad m \in \mathbb{N}.$$  \hspace{1cm} (3.66)

This transformation corresponds to a rotation around the origin by a multiple of \(\frac{2\pi}{3}\). The fractal therefore has a three-fold global symmetry.

After this immediate, global observation we will move on to examine the local symmetry properties of the fractal. According to the above discussion, we will consider the local behaviour at the origin, which then will hold for the other Julia points. To do this, we firstly define the global symmetry axes. They are defined by the origin and its images under one inverse Newton mapping (3.27),

$$|z_s| = \frac{1}{\sqrt{2}}, \quad \arg z_s = \frac{\pi}{3} + \frac{2\pi s}{3}, \quad s = 0, 1, 2.$$  \hspace{1cm} (3.67)

The axes \(L_s\) are determined by the lines through the origin and any of these \(z_s\). By substituting into (3.27), it can be shown that one image of the axis is the axis itself- the necessary condition for symmetry to hold. The Julia points on it define a geometric progression as determined in 3.2. The axes can be parametrised by

$$z_0 = \zeta e^{i\theta_s}, \quad \theta_s = \frac{\pi}{3} + \frac{2\pi s}{3}, \quad 0 \leq \zeta < \infty.$$  \hspace{1cm} (3.68)

For large \(|z_0|\), it is possible to approximate one solution to the Newton polynomial (3.2) with \(\nu = 3\) as

$$z = \sqrt{\frac{1}{3|z_0|}} + \mathcal{O}\left(z_0^{-\frac{1}{3}}\right).$$  \hspace{1cm} (3.69)

Substituting the parametrised global \(s^{th}\) symmetry axis \(L_s\) into (3.69) and noting the reflective symmetry of the \(L_s\), we obtain for \(\zeta \gg 1\):

$$z_{sm}(\zeta) \approx \sqrt{\frac{1}{3\zeta}} \cdot e^{\frac{\pi i}{3} m} \cdot e^{im\pi}, \quad m = 0, 1.$$  \hspace{1cm} (3.70)

For large \(\zeta\), the \(z_{sm}\) asymptotically approach the origin on straight lines. The succession of Julia points on these lines will be discussed in relation with local scale factors. Considering the angle \(\omega_{sm}\), from which the \(z_{sm}\) approach the origin, we obtain immediately

$$\omega_{sm} = \frac{\pi}{6} [1 + 2(3m + s)].$$  \hspace{1cm} (3.71)
By inspection, it can be verified that $3m + s$ ranges from 0 to 5, and therefore the angles can be equivalently stated as

$$\omega_k = \frac{\pi}{6} + \frac{k\pi}{3}, \quad k = 0, 1, \ldots, 5.$$ (3.72)

This result shows that there are 6 locally identical branches of the fractal approaching the origin, separated by an angle of $\frac{\pi}{3}$. Therefore, we can establish a six-fold local rotational symmetry at the origin. As every point of the fractal is a locally conformal image of the origin, this local six-fold invariance under rotations by $\frac{\pi}{3}$ holds for every point of the fractal.

We note that the symmetry is higher locally than globally, and that the global symmetry is contained within the local symmetry. This will be further elaborated for general polynomials in the concluding section.

It is also noted that the fractal exhibits the same local rotational symmetry at every point, which is different from the global symmetry. In this, it is different from all non-pathological connected sets, where local and global symmetry are either identical or do not stay the same for all points.

### 3.4.2 Local Scale Factors

After having established the global scale factor of the fractal to be $\lambda_{3,\infty} = \frac{3}{2}$ (see 3.2 and 3.3.3), we now discuss local scale factors.

The first obvious local scale factor is concerned with the scaling along one branch of the fractal structure. An example of that would be the image of the negative real axis as described in 3.3.2. We are concerned with the scale factor that emerges as the Julia points get closer and closer to the origin. To determine the factor, we simply compare the ratio of the distances between three consecutive Julia points $\{(x_{n+1}, y_{n+1}), (x_n, y_n), (x_{n-1}, y_{n-1})\}$ which are eventually mapped onto the negative real axis. This can in a Euclidean norm be written as

$$\rho_n = \sqrt{\frac{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}}.$$ (3.73)

Using the facts about points close to the origin that have been established in 3.3.2, particularly $|x| \ll |y|$, this simplifies to

$$\rho_n \approx \frac{y_{n+1} - y_n}{y_n - y_{n-1}}.$$ (3.74)

To further simplify this expression, we consider (3.35) and obtain after some arithmetic

$$\rho_n \approx \frac{\sqrt{\lambda_{n-1}}}{\sqrt{\lambda_{n+1}}} \frac{\sqrt{\lambda_n - \sqrt{\lambda_{n+1}}}}{\sqrt{\lambda_n - \sqrt{\lambda_{n-1}}}}.$$ (3.75)
with \( \chi_k \) denoting the point on the negative real axis that was mapped into the corresponding \((x_k, y_k)\) by (3.31b). We now recall the global scale factor and therefore can state for large \( \chi_n \)

\[
\chi_{n+1} \approx \frac{3}{2} \chi_n \quad \chi_{n-1} \approx \frac{2}{3} \chi_n.
\] (3.76)

Substituting this into (3.75) and after some basic arithmetic, we finally obtain for the local scale factor

\[
\lambda_3 = \lim_{n \to \infty} \rho_n = \sqrt[3]{\frac{2}{3}}.
\] (3.77)

This scale factor is valid for consecutive elements on one "fractal chain" running into a Julia point, which is a locally conformal image of the origin. Therefore, this scale factor prevails throughout the fractal. In particular, it is visible on the local symmetry axes discussed in the previous section. This can of course be verified in numerical experiments. The Julia points constituting a fractal chain of order \( n \) (i.e. one that is obtained from the negative real axis after \( n \) inverse Newton mappings) approach the Julia point of order \( (n - 1) \) in a geometrical progression with factor \( \sqrt[3]{\frac{2}{3}} \).

However, there exist scale factors associated with the images of the inverse mapping that do not stay on the same branch of the fractal structure. To determine these, we examine the fixed points of the third-order Newton polynomial

\[
2z^3 - 3z_0 z^2 + 1 = 0,
\] (3.78)

and keep in mind that the solution \( z \) of this mapping depends on the parameter \( z_0 \). We consider a local approximation of this mapping

\[
z(z_0 + \epsilon) = z(z_0) + \epsilon \cdot \frac{dz}{dz_0} \bigg|_{z=z_0} + \ldots
\] (3.79)

and cut off after the linear term. The image of an infinitesimal \( \epsilon \) is scaled by

\[
A_3(z_0) = \lim_{\epsilon \to 0} \left| \frac{z(z_0 + \epsilon) - z(z_0)}{\epsilon} \right| = \left| z'(z_0) \right|.
\] (3.80)

with \( z' \) denoting \( \frac{dz}{dz_0} \). To determine this factor, the governing equation (3.78) may be differentiated implicitly

\[
6z^2 \frac{dz}{dz_0} - 3z_0 z^2 - 6z z_0 \frac{dz}{dz_0} = 0
\] (3.81)

to obtain

\[
A_3(z_0) = \frac{1}{2} \left| \frac{z}{z - z_0} \right|.
\] (3.82)
The trivial fixed points with cycle 1, i.e. $z(\zeta) = \zeta$, are obviously obtained from (3.78) by

$$\zeta^3 = 1.$$  \hfill (3.83)

This is just the solution of the original equation (2.5) and therefore of little further interest.

The fixed points of cycle 2 can be described by $z(\zeta_1) = \zeta_1$ and $z(\zeta_2) = \zeta_2$. Using (3.78), this can be written as

$$2\zeta_1^3 - 3\zeta_2\zeta_1^2 + 1 = 0 \quad \text{and} \quad 2\zeta_2^3 - 3\zeta_1\zeta_2^2 + 1 = 0. \hfill (3.84)$$

Knowing from (3.82) that the scale factors may be written as

$$\Lambda_3(\zeta_1) = \frac{1}{2} \left| \frac{\zeta_2}{\zeta_2 - \zeta_1} \right|, \quad \Lambda_3(\zeta_2) = \frac{1}{2} \left| \frac{\zeta_1}{\zeta_1 - \zeta_2} \right|, \hfill (3.85)$$

we may rearrange (3.84) as

$$\frac{\zeta_2 - \zeta_1}{\zeta_2} = \frac{1}{3\zeta_2^3} (\zeta_2^3 - 1), \quad \frac{\zeta_1 - \zeta_2}{\zeta_1} = \frac{1}{3\zeta_1^3} (\zeta_1^3 - 1). \hfill (3.86)$$

It is therefore sufficient to determine the third power of $\zeta_1$ and $\zeta_2$. The governing equation for these is obtained by using (3.84) and the condition for periodic cycle 2, $z(z(\zeta_2)) = \zeta_1$.

$$20\zeta_1^6 - 15\zeta_2^6 - 3\zeta_3^3 - 2 = 0. \hfill (3.87)$$

Substituting $u = \zeta_3^3$, we can factorize (3.87) as

$$20(u - 1)(u^2 + \frac{u}{4} + \frac{1}{10}) = 0. \hfill (3.88)$$

Knowing that $u = 1$ is a trivial fixed point of cycle 1, we can discard this solution and concentrate on the quadratic part. It follows from root conditions that

$$u_1u_2 = u_1u_1^* = u_2^*u_2 = \frac{1}{10} \hfill (3.89)$$

and therefore

$$|\zeta_1| = |\zeta_2| = \frac{1}{\sqrt[3]{10}}. \hfill (3.90)$$

We can further rewrite the quadratic part as

$$u^2 + \frac{u}{4} + \frac{1}{10} = (u - 1)^2 + \frac{7}{4}(u - 1) + \frac{27}{20} = 0 \hfill (3.91)$$
Figure 13: Basins of attraction for $z^3 = 1$.

Figure 14: Basins of attraction for modified Newton method, $s_w = 10$, $s_d = 2$. 
Figure 15: Basins of attraction for $z^5 = 1$.

Figure 16: Basins of attraction for $\sin z = \frac{z}{2}$. 
to obtain in the same fashion
\[ |\zeta_1^3 - 1| = |\zeta_2^3 - 1| = \sqrt{\frac{27}{20}} \] (3.92)

Substituting these quantities into the equations (3.85) governing the scale factor via (3.86), we finally obtain
\[ \Lambda_{3,1} = \frac{1}{\sqrt{6}}. \] (3.93)

As expected, this scale factor is equal for both \( \zeta_1 \) and \( \zeta_2 \). To determine the fixed points \( \zeta \), we simply solve the quadratic part (3.91) to obtain
\[ u_{1,2} = \frac{1}{8} \left( -1 \pm 3 \sqrt{\frac{3}{5}} \right) = \frac{1}{\sqrt{10}} \exp \left[ \pi i \pm i \arctan \left( 3 \sqrt{\frac{3}{5}} \right) \right]. \] (3.94)

Resubstituting with \( \zeta \), this finally yields six fixed points with cycle 2, namely
\[ \zeta_{k,1} = \frac{1}{\sqrt{10}} \exp \left[ \frac{\pi}{3} i \pm \frac{1}{3} i \arctan \left( 3 \sqrt{\frac{3}{5}} \right) + \frac{2\pi k}{3} i \right], \quad k = 0, 1, 2. \] (3.95)

The \( \zeta_{k,1} \) are located on either side of the main symmetry axes \( \mathcal{L}_s \). The set of points formed by their union is invariant with regard to reflections through \( \mathcal{L}_s \).

We finally establish the fixed points with cycle 3, noting that Julia points are mapped closer to each other with each inverse mapping within the circle with radius \( \frac{1}{\sqrt{3}} \). Therefore, after suitably many iterations, the sequence of points will converge to a cyclic fixed point of the mapping
\[ z(\zeta) = \zeta \cdot e^{2\pi k i/3}, \quad k = 1, 2 \] (3.96)
which is of cycle 3. With this mapping for the fixed point, we immediately can write the scale factor from (3.82) as
\[ \Lambda_{3,2} = \frac{1}{2} \left| \frac{\exp \left( \frac{2\pi k i}{3} \right)}{\exp \left( \frac{2\pi k i}{3} \right) - 1} \right|, \quad k = 1, 2. \] (3.97)

After some arithmetic, we obtain the scale factor as
\[ \Lambda_{3,2} = \frac{1}{4 \sin \frac{\pi}{5}} = \frac{1}{\sqrt{12}}. \] (3.98)

Substituting the mapping (3.96) into (3.2), we obtain
\[ 2\zeta^3 - 3\zeta^3 e^{-\frac{2\pi k i}{3}} + 1 = 0, \quad k = 1, 2 \] (3.99)
and solving for $\zeta$, six fixed points of cycle 3 are defined by

$$\zeta_{l,2} = \frac{1}{\sqrt{3}} \cdot \exp \left[ \left( \frac{\pi}{3} \pm \frac{1}{3} \arctan \frac{3\sqrt{3}}{7} + \frac{2\pi l}{3} \right) i \right], \quad l = 0, 1, 2. \quad (3.100)$$

The $\zeta_{l,2}$ are located on either side of the main symmetry axes $L_s$. The set of points formed by their union is invariant with regard to reflections through $L_s$.

Having established these results, we note that there exist three different local scale factors for the fractal. $\lambda_3$ describes the scaling on a chain of Julia points of order $n$ leading into a Julia point of order $(n - 1)$. $\lambda_{3,1}$ and $\lambda_{3,2}$ describe the scaling between fractal chains of different order. It is notable that the two cross-chain factors are different from $\lambda_3$ and different among themselves. The following section will focus on how to relate the different scale factors to the fractal structure and from there present an attempt at establishing the fractal dimension of the third-order Newton fractal.

### 3.4.3 Interpretation and Fractal Dimension

We denote the area that is bordered by two chains of Julia points which are themselves images of the ones on the negative real axis (or its rotations by $\pm \frac{\pi}{3}$) after $n$ inverse Newton mappings as a **blob of order** $n$. For example, the area in the first quadrant bordered by the arc defined by (3.31b) and its reflection through the main axis of this fractal branch is a blob of order 1. We note that each chain of Julia points on the border of a blob consists of infinitely many higher-order blobs itself, giving rise to the fractal structure that is apparent to the eye. This is illustrated in Fig. 17 for the first-order blob on the negative real axis. The dotted line computed by (3.31b) depicts the line on which the Julia points for the first-order blob are situated. As we have established earlier, it suffices to consider the structure inside the attractive circle, as all structures outside this circle are just obtained by prolongation steps. We also note that a blob of order $(n + 1)$ is obtained from a blob of order $n$ by one inverse Newton mapping.

From the previous sections, we know that the blobs on each chain of Julia points are scaled like a geometric progression with factor $\lambda_3$. Due to the curvature of the chain, the starting blobs of order $n$ of each chain consist of a long side and a short side. Both sides are images of another arc that is situated on a different branch of the fractal. Therefore, the cross-chain scale factors $\lambda_{3,1}$ and $\lambda_{3,2}$ are associated with their scaling. We associate the previous arc with the blob of order $(n - 1)$, so these factors describe the scaling of the new blob with respect to the previous blob. We also note that the fixed points associated with these scale factors have characteristic positions within the blobs that are either on the short or the long chain bordering a blob. Empirically, we assign $\lambda_{3,1}$ to the longer side of the blob and $\lambda_{3,2}$ to the shorter side. We note that this relation is fulfilled only at the start of the chain, and that the long and short side of the
blob approach the same value as the chain evolves into a geometric progression. In the asymptotic part of the chain, the ratio between the arcs of the blobs of order \((n - 1)\) and \(n\) is numerically determined to be

\[
\frac{1}{\Lambda_3} = 2.30276...
\]  

(3.101)

Lacking a proper explanation for this number, we will omit it from the following approach to determine the fractal dimension - although there might be use for it to improve the estimate given.

We further assume that the size ratio of two sets containing \(N_1\) and \(N_2\) points is dependent on their geometrical sizes \(s_1\) and \(s_2\) and the fractal dimension \(d\) in the following fashion.

\[
\frac{N_2}{N_1} = \left(\frac{s_2}{s_1}\right)^d
\]  

(3.102)

This is coherent with the scaling property of a Hausdorff set according to the Hölder condition. This condition can be used to determine the fractal dimension of the middle third Cantor set (see [5]), and also to calculate the fractal dimension of Sierpinski triangles. It seems sensible that even if \(N_1\) and \(N_2\) are not countable, the scaling property should still hold. In the context of integer dimensions, the scaling property has immediate meaning: A line segment scaled by a factor \(s\) is \(s\) units longer, a rectangle with all sides scaled by \(s\) grows in area by \(s^2\) and a cube with all sides scaled by \(s\) grows in volume by \(s^3\).

Now we recall that the number of Julia points in each blob is determined by all the parts that constitute it, and that the blob is a self-similar structure obeying (3.102) with asymptotic scale factors \(\lambda_3, \Lambda_{3,1}\) and \(\Lambda_{3,2}\). We denote the
number of Julia points in the original blob by \( N \) and conjecture that

\[
N = N \frac{\left[ (A_{3,1})^d + (A_{3,1}, A_{3})^d + (A_{3,1}, A_{3}, A_{3})^d + \ldots \right] + 
\left[ (A_{3,2})^d + (A_{3,2}, A_{3})^d + (A_{3,2}, A_{3}, A_{3})^d + \ldots \right]}{1 - A_{3}^d}.
\]

(3.103)

Summing the geometric progressions, we get from this

\[
\frac{(A_{3,1})^d + (A_{3,2})^d}{1 - A_{3}^d} = 1
\]

(3.104)
or, rearranged

\[
(A_{3,1})^d + (A_{3,2})^d + A_{3}^d = 1.
\]

(3.105)

Substituting the values for the scale factors into this equation and solving numerically for \( d \), we finally get as an approximation to the fractal dimension

\[
d \approx 1.801.
\]

(3.106)

This value is lying in the interval \((0, 2)\) for the Hausdorff dimension of iterated rational functions given by Douady [3]. However, the simplifying assumptions made above should be stressed again. Most importantly, we assumed that the character of a geometric progression holds throughout the fractal chain. In reality, this is only true for the end of the chain. Also, we did not take the deformation of the blobs on the chain into account that changes throughout the chain and finally forces the two sides of the blobs on the end of the chain to be equal. To get a more realistic approximation, the first terms of the progressions in (3.103) could be determined numerically and be used to correct the simplified fractal dimension given here. From the form of (3.105), it seems quite likely that the dimension will get lower with the more accurate estimates. Another way of getting a more accurate estimate would be to incorporate the value \( A_{3} \) given above in a suitable fashion.
4 Fractal Characteristics of a Modified Newton Method

In this section, we will introduce modifications that stabilise the orthodox Newton method for the complex cubic and improve its convergence speed. The modifications will be defined and we present an analysis of the Julia set of that modified method. It can be shown that the modified method does not exhibit fractal character any more. After establishing that result, the stabilisations will be examined individually and from that analysis, a method will be presented that has an empty Julia set.

4.1 Definitions

For the fractal analysis, we refer to the definitions given in the previous chapter for the orthodox Newton method.

Looking at the orthodox Newton iteration for a vector-valued function \((\mathbf{x})\), we define

**Definition 4.1** The shift vector \(\mathbf{x}_s\) is determined by the equation

\[
\mathbf{x}_s = \begin{pmatrix} x_s \\ y_s \end{pmatrix} = -J^{-1} \mathbf{f}(\mathbf{x}^{(n)}).
\] (4.1)

Throughout the chapter, we will use the notation (3.18) for the orthodox Newton iteration and denote for simplicity

\[
\mathbf{x}^{(n)} = \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}
\] (4.2)

To specify the modified method, we introduce further definitions:

**Definition 4.2** Dynamic Shift Scaling is determined by the following algorithm:

- Define \(s_d, s_u > 1\) as the limits on the upward and downward shift, s.t.

\[
s_{f1} = \begin{cases} \min \left( 1, \frac{r_{s1}}{r_{s1}} \right) & \text{if } r_{s1} < 0 \\ \min \left( 1, \frac{r_{s1}}{r_{s1}} \right) & \text{if } r_{s1} > 0 \end{cases}
\] (4.3a)

\[
s_{f2} = \begin{cases} \min \left( 1, \frac{r_{s2}}{r_{s2}} \right) & \text{if } r_{s2} < 0 \\ \min \left( 1, \frac{r_{s2}}{r_{s2}} \right) & \text{if } r_{s2} > 0 \end{cases}
\] (4.3b)

with

\[
r_s = \begin{pmatrix} r_{s1} \\ r_{s2} \end{pmatrix} = \begin{pmatrix} \frac{x}{y} \\ \frac{y}{-x} \end{pmatrix}.
\] (4.3c)

Empirically, the shift limits are set to \(s_d = 2\) and \(s_u = 10\).
• Determine the scale factor $s_f$ by

$$s_f = \min(s_{f1}, s_{f2}).$$  \hspace{1cm} (4.3d)

• Scale the shift vector

$$x_s \mapsto s_f x_s$$  \hspace{1cm} (4.3e)

Shift scaling of that kind is established in the literature as a way of stabilising Newton’s method for applied problems [8], [6].

**Definition 4.3** A Line Search shall be specified by

• By stepping $\alpha \in (0, 1]$ in $\frac{1}{10}$ increments, determine the $x_t$ with

$$x_t = x^{(n)} + \alpha x_s$$  \hspace{1cm} (4.4a)

• If there is no residual improvement over $\|f(x^{(n)} + x_s)\|$, then by stepping $\alpha \in (1, 10]$ in $\frac{1}{10}$ increments determine

$$x_t = x^{(n)} + \alpha x_s$$  \hspace{1cm} (4.4b)

This algorithmic step is denoted by overstepping.

• Determine

$$\alpha_{\text{min}} = \{ \alpha \|f(x_t)\| = \min \}$$  \hspace{1cm} (4.4c)

• Update the shift vector

$$x_s \mapsto \alpha_{\text{min}} x_s$$  \hspace{1cm} (4.4d)

The $L_2$ norm shall be used unless stated otherwise. The stepping increments are an empirical choice considering algorithmic simplicity, runtime and resolution.

In order to analyse the line searches, it is important to recall the $L_2$ norm of the cubic system (2.5) to be

$$\|f\|_2 = \sqrt{(x^2 + y^2)^3 + 2x(3y^2 - x^2) + 1}$$  \hspace{1cm} (4.5)

With these definitions, we can state the modified Newton Method as

**Definition 4.4** The modified Newton Method is defined by the following algorithm:
• Determine the shift vector $x_s$.
• If the residual is above a threshold, apply dynamic shift scaling. The threshold is empirically chosen to be $10^{-5}$.
• Execute the line search
• Update the Newton iteration by

$$x^{(n+1)} = x^{(n)} + x_s.$$  \hspace{1cm} (4.6)

This formulation of this method has arisen from the practical solution of other problems, such as the $k-\epsilon$ equations for turbulent fluid flow where a variant of it was successfully employed as a global solution procedure [4].

With the modified method being defined in this way, the fractal analysis will basically follow the steps outlined in the section on the orthodox Newton method, using the added steps in the algorithm to determine changes in the Julia set.

### 4.2 The Julia Set of the Modified Method

As in the previous section, we shall first be concerned with the one-dimensional restriction of the method and then analyse the general case. Analysis will show that although there is no change in the Julia set of the restriction, the chain of mappings into two dimensions is broken due to bordering maxima and minima in the residual norm used for the line searches and therefore the fractal structure does not emerge. For reference, a colour plot of the basins of attraction is given in Fig. 14.

#### 4.2.1 One-dimensional Restriction

As pointed out before, the orthodox Newton method will not leave the real axis once $y = 0$ for any iterate. Looking at the definitions in the previous section, it is important to note that the stabilising modifications only change the scale of the shift $x_s$ and never its direction. Therefore, the modified method is also guaranteed to stay on the real axis once started there.

In the one-dimensional restriction $y = 0$, the $L_2$ norm (4.5) becomes

$$||f||_2 = |x^3 - 1|$$  \hspace{1cm} (4.7)

and, by inspection of the first derivative, it is obvious that $||f||_2$ has a global minimum at $x = 1$ and is monotonically decreasing towards that minimum. Therefore, the line searches will always show best residual improvement at the full Newton shift, and overstepping will not occur due to the way the line search is defined. The line searches therefore will not cause any deviation from the orthodox Newton method.
As the line searches will not come into effect on the real axis, it suffices to consider the dynamic shift scaling in order to establish differences between the modified and the orthodox Newton method. We also know that the shift scaling as defined in the previous section will always have a scale factor $0 < s_f \leq 1$ and therefore never reverse or prolong an orthodox Newton shift. By looking at the gradient, we know that on the negative real axis (which is by the aforementioned facts the only region of interest for the Julia set), only upward shifts will occur. Shift limiting will therefore come into effect if

$$\left| \frac{s_u}{r_{s1}} \right| = \left| \frac{10x}{x_1} \right| < 1. \tag{4.8}$$

Using the definition of the shift vector and noting $x < 0$, this yields the condition

$$-10x \cdot 3x^2 < 1 - x^3 \Leftrightarrow |x| < \frac{1}{3\sqrt{29}} \tag{4.9}$$

As the first point of the Julia set to the left of the origin $\chi_1 = -\frac{1}{\sqrt{2}}$ is mapped into the origin by one Newton shift, and the condition for activating the shift scaling is not effective for this point and any point to its left, dynamic shift scaling does also not alter the modified method as far as the Julia points are concerned.

In a strict sense, the modified method does not allow shifts of infinite lengths and the origin is therefore no longer a member of the Julia set. However, we will continue to regard it as such, as the direction of the shift from the origin is still random. The scaling and line searches only limit its length and therefore assure convergence from the origin. In graphical terms, the visible properties of the Julia points, namely their local similarity to the origin and their bordering on each of the three basins of attraction are retained.

We can therefore conclude that for the one-dimensional restriction, the modified method is essentially equivalent to the orthodox Newton method and therefore no change in the Julia set on the real axis $y = 0$ is expected. There is still a countable, infinite number of Julia points on the real axis and its images under rotation by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ with the properties established in section 3.2. However, as seen in the analysis of the orthodox method, the formation of a fractal structure only occurs through the properties of the two-dimensional inverse Newton mapping and this will therefore be the next step of the analysis.

4.2.2 Two-dimensional Case

As in the analysis of the orthodox Newton method, we now try to establish the set of points which are mapped onto the negative real axis by one Newton iteration. The existence of such a set is crucial to the fractal nature of the method as there are only three images of the origin under one inverse Newton mapping, namely $\chi_1 = -\frac{1}{\sqrt{2}}$ and its two rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Therefore, all the other points of
the fractal Julia set have to be images of $\chi_1$ and the Julia points to its left under further inverse Newton mappings.

From an analysis of the $L_2$ norm (4.5), we see that

- $y = 0$ is a local maximum of the norm for $-\sqrt{2} < x < 0$,
- $y = 0$ is a local minimum of the norm for $-\infty < x < -\sqrt{2}$.

Clearly, $\chi_1$ lies in the first region and all the Julia points to its left on the real axis - obtained by evaluating the recurrence (3.7) - are located in the second region.

We will start with an analysis of the case $-\sqrt{2} < x < 0$. As the line searches determine a scale factor with a local minimization property, none of them will by definition stop on the real axis. Therefore, there is no image of $\chi_1$ under the inverse Newton mapping apart from the one to its left on the negative real axis. By the same argument, the whole interval $(-\sqrt{2}, 0)$ has no image with $y \neq 0$ under the inverse mapping.

We now discuss the case $x < -\sqrt{2}$. We know from the formation of the fractal that the only region mapping onto this interval under the orthodox Newton iteration is the arc described by (3.31b) with $0 < x < \sqrt{2}$ and $|y| < \frac{\sqrt{2}}{2\sqrt{2}}$. As the direction of the Newton shift vector is not changed by any of the modifications, this is the only region that could be mapped onto the real axis also by the modified method. However, the minimum downward shift in $x$ that would be necessary to get to the appropriate point on the real axis is

$$x_s = -\left(\sqrt{2} + x\right).$$  \hfill (4.10)

Inserting the upper limit of $x$ and comparing with the downward shift limit condition, we find

$$\left|\frac{x_s}{x}\right| = 1 + 2\sqrt{4} > s_d = 2.$$  \hfill (4.11)

Therefore, no point in this region ever reaches the part of the real axis with $x < -\sqrt{2}$. The image of the negative real axis found for the inverse orthodox Newton mapping does not exist any more for the modified Newton method!

We also note that, for any point to reach the real axis left of $-\sqrt{2}$ by a downward shift from $x > 0$, its $x$ coordinate must satisfy

$$x > \sqrt{2}.$$  \hfill (4.12)

To complete the argument, it remains to be shown that there are no other points that stop on the negative real axis due to the local minimization property for $x < -\sqrt{2}$. We just ruled out the case where the orthodox shift would stop on the axis, and due to the properties of the shift scaling the points in question would be characterised by orthodox Newton shifts that cross the real axis, i.e.

$$|y_s| > |y| \quad \land \quad \text{sgn}(y_s) \neq \text{sgn}(y).$$  \hfill (4.13)
We therefore examine the equation for $y_s$,

$$y_s = -\frac{y}{3} \left[ 1 + \frac{2x}{(x^2 + y^2)^2} \right]. \tag{4.14}$$

We see that the second condition in (4.13) is fulfilled for any positive $x$. However, for positive $x$, the first condition becomes

$$\frac{x}{(x^2 + y^2)^{3/2}} > 1, \tag{4.15}$$

being impossible to satisfy for $x > 1$ - which would be needed according to the downward shift scaling argument (4.12). So, there exists no image of the negative real axis with $x > 0$. For negative $x$, the condition also cannot be satisfied for $x < -1$, hence the interval $(-\infty, -\sqrt[3]{2})$ can only be reached via a downward shift in $x$.

By a similar argument for the downward shift limit as with positive $x$ (4.12), we can establish that $x < -\sqrt[3]{2}$. However, as $x$ is now negative and $|x| < 1$, the sign of $y_s$ in (4.14) equals the sign of $y$ if the first condition in (4.13) is to be fulfilled, and therefore the second condition in (4.13) is violated. We note that, when $y_s$ and $y$ are of opposite sign, $|y_s| \leq \frac{2}{3}|y|$ holds in this case. Therefore, there exist no orthodox Newton shifts that cross the negative real axis with $x < -\sqrt[3]{2}$.

To summarize, we have shown that the arc which is mapped onto the negative real axis under the orthodox Newton iteration does not exist any more. It was also shown that there exists no other image of the negative real axis (or part of it) that would arise from shifts crossing the axis and being halted there by line searches. Therefore, there exists no point with $y \neq 0$ that is mapped onto the negative real axis within one modified Newton iteration. The Julia set of the modified method is therefore confined to the negative real axis and its images under rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Due to the local conformity of the Newton mapping and lemma 3.10, there still exist slightly distorted copies of the origin at those points with the three basins of attraction coming together. However, between those points, the basins of attraction are bordered by Jordan curves, and the method exhibits no fractal characteristics.

### 4.3 Influence of the Stabilisations

After we have shown a way of removing the fractal character from Newton’s method, this section will be devoted to establishing the quantitative influence of the two stabilisation methods on the fractal character. From these considerations, it will be possible to design a method that has no Julia set even with the relaxed concept of Julia points used with the modified Newton method.
4.3.1 Shift Scaling

The idea to introduce limits on the Newton shifts emerges from a model-trust region approach where a root is suspected in some vicinity to an initial guess. It is then desired that the Newton iterates stay in that region and are not sent off by small (e.g. oscillatory) perturbations. From the analysis in the previous chapter, however, we saw that in this context, shift limits also played a role in eliminating the fractal properties. We will focus on this property and establish to what extent the fractal characteristics can be influenced by the actual value of the shift limits.

It is worthwhile noting that different upward and downward shift limits break the symmetry of the method, as the treatment of points will vary according to the sign of the resulting shift. Thus, it cannot be expected that the rotational symmetry of the orthodox method is to be exactly maintained.

**Upward Shift Limits**

In the previous analysis, the upward shift limit was connected with the formation of the one-dimensional chain of Julia points that runs on the negative real axis into the origin. In this one-dimensional restriction, it is easy to see from the definition that the first Newton step that would be affected by a more restrictive upward shift limit is the one mapping \( \chi_1 = -\frac{1}{\sqrt{5}} \) into the origin. Obviously, the condition for this mapping not to occur any more, and therefore the fractal chain on the negative real axis to break down is

\[
\underline{s_u} < 1. \tag{4.16}
\]

In fact, a restriction of that kind allows no Newton iterate \( x^{(n)} \) to change any of its coordinate signs from negative to positive. The threshold condition in the modified Newton algorithm is therefore critical to allow convergence to the root \( (x = 1, y = 0) \).

As \( s_u \) is gradually decreased towards 1, the one-dimensional pattern of Julia points stays unaffected. However, there are shifts in the two-dimensional mapping that involve substantial upward shifts in \( y \) (e.g. those mapping onto the negative real axis rotated into the first quadrant by \( \frac{2\pi}{3} \)). These are controlled by the upward limit, which leads to a partial "depletion" of the Julia set as certain points do not have three images under the inverse Newton mapping any more. This phenomenon will also occur regarding the negative real axis with downward shifts and will be discussed in greater detail there. Using a relaxation parameter for the Newton shift, Peitgen et al. conjectured in [12] that the fractal dimension decreases with a decreasing relaxation factor. This conjecture is strongly supported by the depletion of the fractal chain and the discussion in 3.4, particularly equation (3.103). Fig. 18 shows this phenomenon.

Decreasing \( s_u \) further below the critical threshold \( s_u = 1 \) does not change the fractal characteristics any more, it only decreases the convergence rate. Each
step has to stay inside a disc of smaller and smaller radius around the current iterate, even if there is no reason for this - apart from the extremely cautious application of the model-trust region approach.

**Downward Shift Limits**

While the upward shift limit was affecting the actual building of the Julia set on the negative real axis, the establishment of the fractal structure was due to mappings from the first and fourth quadrant onto the negative real axis. Those were obviously connected with downward shifts and therefore, restrictive downward shifts are able to control the formation of the fractal. It is again obvious that for

\[ s_d < 1, \]  

no fractal formation can occur as all iterate coordinates are confined to their original half-plane. In addition, the rotated image of \( \chi_1 = -\frac{1}{\sqrt{2}} \) in the first quadrant cannot reach the origin any longer and the chain of Julia points in this quadrant will break down. However, with such a scaling, the points in the first and fourth quadrant that would converge to roots with \( x = -\frac{1}{2} \) cannot do so any longer. The Newton shifts will however point persistently towards these quadrants and the algorithm will get stuck. We are therefore looking for a less restrictive limit on \( s_d \).

For the \( x \)-direction, we can write

\[ x_s = -(a + x) \]  

if \(-a\) is the image of \( x \) on the real axis after one Newton step. Putting this into the shift restriction condition and approximating \( x \approx \frac{1}{9a^2} \) according to (3.35) for \( x \) close to the origin, we obtain

\[ s_d < 1 + 9a^3. \]  

This condition gives an upper limit on \( s_d \) if we only want to allow points in \((-a, 0)\) to have images under an inverse Newton mapping. Conversely, for a given shift limit \( s_d \), points with

\[ a > \sqrt[3]{\frac{1}{9}} (s_d - 1) \]  

have no image in the first or fourth quadrant under the inverse Newton mapping.

These conditions show that the closer \( s_d \) gets to 1, the more points do not have an image under the inverse mapping any longer. The sequence of nodes (i.e. Julia points) approaching the origin on the arc described by (3.31b) with \( 0 < x < \frac{1}{2\sqrt{2}} \) and \( |y| < \frac{\sqrt{5}}{2\sqrt{2}} \) can be thought of as a chain. The closer \( s_d \) gets...
Figure 18: Julia sets for $z^3 = 1$ with various shift limits.
Figure 19: Basins of attraction of root \((1, 0)\) for \(z^3 = 1\) with various shift limits.
to 1, the shorter this chain becomes until, for some value $s_d = 1 + \epsilon$, the first Julia point $\chi_1$ on the negative real axis has no image any more and therefore the fractal breaks down completely.

This behaviour is depicted in Fig. 18, where Julia points were computed on a $240 \times 240$ array. The first picture of a scaled method with $s_u, s_d = 80.0$ is still very close to the orthodox case, but shows already slightly shortened chains of Julia points. The phenomenon becomes obvious in the third picture with $s_u, s_d = 5.0$, where the beginning of the chain from $(-0.8, 0)$ is preserved, but breaks down after the first Julia point. Only remains of the original chains are left. The number of Julia points decays even more for the last case.

The corresponding behaviour for the basins of attraction is given in Fig. 19. In this picture, the orthodox case was omitted and a case with very small shift limits added to show the limit for the deformation of the basins of attraction. The four points close to the origin are artefacts of the dynamical $x$-shift scaling and disappear for slightly more restrictive shift factors.

### 4.3.2 Line Searches

In the fractal analysis of the modified method, line searches played an important part for establishing the breakdown of the two-dimensional mapping that propagates the Julia set. Analysing the shift scaling, we were able to see that by appropriate choice of the limits, it is possible to eliminate the fractal nature of the method. However, this was achieved at the expense of a slower convergence speed. We will now investigate the effect that line searches alone have on the fractal nature of the method and see if fractal nature can be removed by employing line searches only.

Due to the monotonicity of the $L_2$ residual on the real axis $y = 0$, it is obvious that line searches in the defined form cannot eliminate the set of Julia points on that axis. This would, however, be possible with a changed definition that does not accept the original Newton step if it showed any improvement, but would overstep in any case. Such a method will be discussed in 4.4.

In the analysis of section 4.2.2, it was established that for $|x| > 1$, no orthodox shift would cross the real axis or halt on it. Therefore, it suffices to examine the case $|x| \leq 1$ here.

We will consider the case of positive $x$ first. For such a starting value, the orthodox Newton shift will have to intersect the axis $x = 0$ as well as intersect or end on the real axis $y = 0$. Therefore, the intersecting $y$-coordinate $y_c$ has to satisfy the following constraint:

$$|y_c| < |y|. $$ (4.21)

From condition (4.15), we can obtain

$$y^2 < \sqrt{x} - x^2, $$ (4.22)
and from this for the maximum value of \( y \) in \( x \in [0, 1] \):

\[
y = \frac{\sqrt{3}}{\sqrt[3]{16}} \quad \text{at} \quad x = \frac{1}{\sqrt[3]{16}}.
\]

(4.23)

The next step is regarding the actual residual. From (4.5), it is straightforward to obtain that

\[
\|f\|_2 \geq 3 \quad \text{for} \quad (x < -\sqrt{2}) \land (y = 0).
\]

(4.24)

On the other hand, for \( |y| < 1 \) - which we established as necessary in (4.23) - it follows from (4.5) that

\[
\|f\|_2 \leq \sqrt{2} \quad \text{for} \quad (|y| < 1) \land (x = 0).
\]

(4.25)

Therefore, any line search that would cross or end on the negative real axis with \( x < -\sqrt{2} \) finds a smaller residual at least on the imaginary axis (\( x = 0 \)) and therefore would stop there instead. (In reality, there is a symmetric basin of even smaller residual around the roots in the second and third quadrant, so the line searches do not actually stop on the axis.) The interval \( x > -\sqrt{2} \) on the negative real axis is a local maximum in the \( L_2 \) norm and therefore cannot be reached by any line search.

For negative \( x \) with \( |x| < 1 \), the argument set forth in section 4.2.2 still holds - the second condition in (4.13) is violated due to the sign of the \( y \)-shift equaling that of \( y \) for small \( x \) (and \( y \) small enough to satisfy the first condition). Hence, no shift vector starting in that interval will intersect the negative real axis.

Therefore, even if line searches alone are employed, the fractal structure disappears as there is no two-dimensional propagation of the Julia set. The argument for the negative real axis holds for its images under rotation as well, as the rotational symmetry is not broken by the algorithm and prevails in the \( L_2 \) norm. Fig. 20 shows the basins of attraction for the use of line searches.

### 4.3.3 Norm Dependency

As the determination of the residual in a given norm is part of the line search strategy, the question arises whether the previous argument still holds for other possible choices of a norm.

For this, we consider the \( L_1 \) norm which can be stated as

\[
\|f\|_1 = \left| x^3 - 3xy^2 - 1 \right| + \left| y \left( 3x^2 - y^2 \right) \right|
\]

(4.26)

and the \( L_\infty \) norm, which is

\[
\|f\|_\infty = \max \left( \left| x^3 - 3xy^2 - 1 \right|, \left| y \left( 3x^2 - y^2 \right) \right| \right).
\]

(4.27)
Inspection of the $L_\infty$ norm shows that the line $y = 0$ is a local maximum for any $x < 0$, and therefore no line search will halt on the axis. The fractal character is removed by the use of this norm combined with line searches. The local maximum property can be seen from the following argument. For $y = 0$, the norm collapses to

$$\|f\|_\infty = \max \left( |x^3 - 1|, 0 \right).$$

The first term will dominate the residual and constitutes a local maximum at $y = 0$ with a value $\|f\|_\infty > 1$. The second term in the maximum operator of (4.27) will start at $\|f\|_\infty = 0$ and grow from there. So, in a vicinity of $y = 0$, there will be a local maximum determined by the first term in (4.28).

The $L_1$ norm, on the contrary, is minimal on the whole negative real axis. This follows from the symmetry of the norm with respect to $y \leftrightarrow -y$, and the fact that

$$-3xy^2 > 0 \quad \text{for} \quad x < 0,$$

$$\frac{\partial}{\partial y} \left( y (3x^2 - y^2) \right) \neq 0 \quad \text{for} \quad (x < 0) \land (y = 0).$$

(4.29)

The second line shows that this expression has no extremum at $y = 0$, but by inspection obviously changes sign - therefore the modulus operator generates a minimum there. By a similar argument to that for the $L_2$ norm, it is however possible to show that again the line searches will be intercepted by basins with lower residual and therefore the fractal character breaks down again.

We have established that for both the $L_1$ and the $L_\infty$ norm, line searches will remove the fractal behaviour of the orthodox Newton method. However, as the differences in the analysis suggest, it is very advisable to carefully examine the norm that is used with line searches. It cannot be assumed that each
norm automatically possesses the properties needed for the breakdown of fractal characteristics.

This is particularly true for functions which are not members of the family of polynomials examined here, and in fact the choice of norm has great influence on the convergence behaviour with other, more complicated functions.

### 4.4 A Modified Newton Method with Empty Julia Set

Investigating the nature of the singular origin, it is easy to confirm that in the $L_2$ norm, it is neither maximal nor minimal, but a saddle point. Thus, there are points in its vicinity that have a lower residual and therefore no line search is forced to stop on the origin, if overstepping is allowed in any case.

The reason while the modified method with line searches still halted on the origin was due to its definition which tried to rely as little on line searches as possible. A definition of this kind is useful in a context where there are no appropriate tools for dealing with local minima, on which line searches tend to stop. The simple two-way extension of the orthodox Newton method described in this work definitely belongs to that class and fails badly by relying heavily on line searches with general functions that might have oscillatory perturbations. In such a case, the orthodox method might miss most local minima and step in a useful direction in general, whereas every line search is almost certain to end on a Jacobian singularity and yield some random direction from there on.

However, with a non-oscillatory function as the complex cubic, there is no reason why the method should not rely on line searches more heavily and therefore circumvent the singular origin. If appropriate coding takes the origin singularity into account, this point doesn't belong to the Julia set any longer. The theoretically infinite Newton shift is in any case dealt with by the shift scaling.

As shown in the previous section, the only remainder of the original fractal structure is the chain of Julia points on the real axis and its images under rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. As for any line search along the real axis $y = 0$, the origin is not a local minimum, no point will be mapped into it by a modified Newton iteration with overstepping. Therefore, the chain of Julia points ceases to exist. By symmetry, this holds for the rotated images as well. As the shift scaling is dynamic, it also ensures that for $s_d, s_u > 1$, there is no shift that ends on the origin due to scaling.

To restrict a line search into the basin of attraction where it started, a monotonicity condition is attached to the overstepping. The search for a minimum only continues as long as the residual keeps decreasing and is stopped as soon as it is increasing.

**Definition 4.5** We define a modified line search in the following way.

- By stepping $\alpha \in (0, 1]$ in $\frac{1}{50}$ increments, determine the $x_t$ with

$$x_t = x^{(n)} + \alpha x_s$$  \hspace{1cm} (4.30a)
• By stepping $\alpha \in (1, 10]$ in $\frac{1}{10}$ increments, aborting if $\|f(x_t)\|$ increases, determine

$$x_t = x^{(n)} + \alpha x_s$$  \hspace{1cm} (4.30b)

• Determine

$$\alpha_{\text{min}} = \{\alpha ||f(x_t)|| = \min\}$$  \hspace{1cm} (4.30c)

• Update the shift vector

$$x_s \mapsto \alpha_{\text{min}} x_s$$  \hspace{1cm} (4.30d)

The $L_2$ norm is used for the residual evaluations.

With the origin singularity remedied by the algorithm, we therefore have found an algorithm with an empty Julia set. It is equal to the modified algorithm proposed in 4.1, with the only change that overstepping is done unconditionally for every Newton shift as long as monotonicity is preserved. The resulting minimum of the line search within the orthodox shift and the overstepping is used for updating the shift vector.

![Figure 21: Basins of attraction for the method with empty Julia set](image.png)

A plot of the basins of attraction is given in Fig. 21. It is interesting that even with an empty Julia set, the basins of attraction are no simple $\frac{2\pi}{3}$ wedges. It is possible to construct modifications that have wedge-like basins of attraction - however, these variations are numerically problematic.
5 Concluding Remarks

Considering the test case of the complex cubic, we have examined the properties of Newton’s method employed to find the roots numerically. It was possible to explain the fractal structure that emerges when the basins of attraction for the roots are plotted. Fractal properties, i.e. global and local symmetries and scale factors have been established, leading to a quantitative understanding of the structure and an approximation to the fractal dimension.

Stabilising modifications to Newton’s method that were developed for other problems have been examined and their influence on the fractal nature has been discussed. It was possible to show that suitable modifications did gradually remove the self-similar structure and a method with an empty Julia set has been introduced.

Despite being restricted to a relatively simple model problem, considerable insights about the fractal behaviour of Newton’s method were gained. Many of these insights generalise to higher-order polynomials of the form (2.1). Of course, a closed form of the general inverse mapping with all its benefits cannot be expected for $\nu > 4$.

As Newton’s method is a widely used numerical method for the solution of nonlinear systems, it seems useful to relate the results of this work to numerical concepts. Therefore, the last section will contain remarks on the convergence of the orthodox Newton method in a fractal context. Also, convergence comparisons with the modified method that has been introduced to remedy the fractal character will be given. It will emerge that the stabilisations enhance global convergence without impairing local convergence speed, if suitably chosen. In any case, the predictability of a non-fractal method is superior to one with fractal behaviour.

5.1 Generalisation to Higher-order Polynomials

As an analytical solution for the general inverse mapping determined by (3.2) is extremely tedious for $\nu = 4$ and does not exist for the general $\nu > 4$, most of the technical discussion in chapter 3 cannot be extended beyond $\nu = 3$. It seems likely, however, that the fractal emerges due to the same mechanisms that were described in 3.3 - although it is not possible to state them as analytical expressions. This notion is confirmed by the fact that although the exact equations governing the fractal cannot be stated, most of the characteristic properties can be predicted. We will list these properties in the following paragraphs. As the concepts are similar to those discussed in detail for $\nu = 3$, we can state the results quite briefly. A colour plot of the case $\nu = 5$ in Fig. 15 illustrates many of the results, particularly those on symmetry, in an aesthetically appealing fashion.
Attractive Circle  It can be easily obtained from (3.2) that the first-order Julia points which determine the attractive circle are

\[ \chi_{1,k} = \frac{1}{\sqrt{\nu - 1}} \cdot e^{\frac{2\pi i k}{\nu}}, \quad k = 0, 1, \ldots, \nu - 1. \]  

(5.1)

Therefore, the radius of the attractive circle for the \( \nu^\text{th} \) order Newton fractal is

\[ r = \frac{1}{\sqrt{\nu - 1}}, \]

(5.2)

approaching 1 for large \( \nu \).

Global Symmetry  By inspection, it can be established that (3.2) is invariant under the coordinate transformation

\[ z \mapsto z e^{\frac{2\pi i m}{\nu}}, \quad z_0 \mapsto z_0 e^{\frac{2\pi i m}{\nu}}, \quad m \in \mathbb{N}. \] 

(5.3)

This is equivalent to subsequent rotations by \( \frac{2\pi}{\nu} \) with the origin as a centre. Furthermore, we can see that (3.2) is invariant to

\[ z \mapsto z^*, \quad z_0 \mapsto (z_0)^*, \]

(5.4)

with \( z^* \) denoting the conjugate of \( z \). Therefore, the Newton fractal of order \( \nu \) has a \( \nu \)-fold global rotational symmetry and a reflective symmetry regarding the real axis.

Global Scale Factor  For large \( z, z_0 (|z|, |z_0| \gg 1) \), the governing equation (3.2) can be expanded as

\[ z_0 = \frac{\nu - 1}{\nu} z + \mathcal{O}(z^{1-\nu}). \]

(5.5)

As both \( z \) and \( z_0 \) are contained in the fractal, the fractal is invariant under the scaling

\[ z \mapsto \lambda_{\nu, \infty} z, \quad \lambda_{\nu, \infty} = \frac{\nu}{\nu - 1}. \]

(5.6)

It has to be noted that this exact invariance is only achieved in the limit \( |z| \to \infty \). As an approximation, it may however be used much earlier. We therefore state that the Newton fractal of order \( \nu \) has a global scale factor of \( \lambda_{\nu, \infty} \).
Local Symmetry Following the discussion in 3.4.1 and the result on global symmetry, we define the global symmetry axes $\mathcal{L}_s$ to be the lines through the origin and

$$|z_s| = \frac{1}{\sqrt{\nu - 1}}, \quad \arg z_s = \frac{\pi}{\nu} + \frac{2\pi}{\nu} s, \quad s = 0, 1, \ldots, (\nu - 1). \tag{5.7}$$

We parametrise the axes by

$$z_0 = \zeta e^{i\theta_s}, \quad \theta_s = \frac{\pi}{\nu} + \frac{2\pi}{\nu} s, \quad 0 \leq \zeta < \infty. \tag{5.8}$$

Approximating the solution to (3.2) for large $z_0$ by

$$z = \frac{1}{\nu - \sqrt{\nu z_0}} + \mathcal{O}\left(\frac{2z_0 - 1}{\nu - 1}\right), \tag{5.9}$$

we obtain for the $s^{th}$ global symmetry axis and $\zeta \gg 1$

$$z_{sm}(\zeta) \approx \frac{1}{\sqrt{\nu \zeta}} \cdot e^{i \frac{2\pi}{\nu} s} \cdot e^{i \frac{2\pi}{\nu} m}, \quad m = 0, 1, \ldots, (\nu - 2). \tag{5.10}$$

Again, the $z_{sm}$ asymptotically approach the origin on straight lines for large $\zeta$. The succession of Julia points on these lines will be discussed in relation with local scale factors. The angle $\omega_{sm}$, from which the $z_{sm}$ approach the origin, can immediately be written as

$$\omega_{sm} = \frac{\pi}{\nu(\nu - 1)} \left[1 + 2(m\nu + s)\right]. \tag{5.11}$$

By inspection, $m\nu + s$ ranges from 0 to $\nu(\nu - 1) - 1$, and therefore the angles are

$$\omega_k = \frac{\pi}{\nu(\nu - 1)} + \frac{2\pi}{\nu(\nu - 1)} k, \quad k = 0, 1, \ldots, \nu(\nu - 1) - 1. \tag{5.12}$$

It can be seen that the angles $\omega_k = 0$ and $\omega_k = \pi$ never occur. The interpretation of this formula is that $\nu(\nu - 1)$ locally identical branches of the fractal approach the origin with an angle of

$$\theta_{\nu} = \frac{2\pi}{\nu(\nu - 1)} \tag{5.13}$$

between neighbouring branches. Again, we see that the local symmetry is an integer multiple of the global symmetry. This property is characteristic for all Newton fractals. We note that, as each Julia point is a locally conformal image of the origin, the local symmetry holds throughout the fractal. We can summarize that the Newton fractal of order $\nu$ is locally invariant to rotations by $\theta_{\nu}$ and exhibits $\nu(\nu - 1)$-fold local rotational symmetry at each point.
Local Scale Factors  We note that (5.9) yields small $z$ for large $z_0$ and therefore is suitable for approximating the local behaviour at the origin. Considering the geometric progression

$$z_0, \lambda_{\nu,\infty} z_0, (\lambda_{\nu,\infty})^2 z_0, \ldots \tag{5.14}$$

that describes Julia points for a suitable starting point $|z_0| \gg 1$, the progression mapped close to the origin by (5.9) is

$$\frac{1}{\sqrt[\nu]{\lambda_{\nu,\infty}^\nu z_0}}, \frac{1}{\sqrt[\nu]{(\lambda_{\nu,\infty})^2 \nu z_0}}, \frac{1}{\sqrt[\nu]{(\lambda_{\nu,\infty})^3 \nu^2 z_0}}, \ldots \tag{5.15}$$

This again is a geometric progression; the scale factor can be determined as

$$\lambda_\nu = \frac{1}{\sqrt[\nu]{\nu - 1}} = \frac{1}{\sqrt[\nu]{(\lambda_{\nu,\infty})^{-1}}}. \tag{5.16}$$

We see that $\lambda_\nu$ is rapidly approaching 1 as $\nu$ grows -

$$\lambda_\nu = \left(1 - \frac{1}{\nu}\right)^{\frac{1}{\nu^2}} \approx 1 - \frac{1}{\nu(\nu - 1)}. \tag{5.17}$$

The geometrical interpretation of this result is that the Julia points on a chain of order $n$ (obtained from the negative real axis after $n$ inverse Newton mappings) approach the Julia point of order $(n-1)$ in a geometrical progression with factor $\lambda_\nu$. This holds particularly for the straight lines approaching the origin that were described in the preceding paragraph. As $\nu$ is growing, the Julia points on that branch are more equally spaced and, as there is an infinite number of them on each branch, they appear more densely crowded.

Analogous to section 3.4.2, we can determine the cyclic fixed points of cycle $\nu$ by

$$z(\zeta) = \zeta \cdot e^{2\pi i k/\nu}, \quad k = 1, 2, \ldots, \nu - 1. \tag{5.18}$$

Substituting this into (3.82), we obtain after some arithmetic for the scale factor

$$\Lambda_{\nu,2,k} = \frac{1}{2(\nu - 1) \sin \frac{2\pi k}{\nu}}, \quad k = 1, 2, \ldots, \nu - 1. \tag{5.19}$$

The cyclic fixed points themselves can be determined via (3.2), solving

$$(\nu - 1)\zeta^\nu - \nu\zeta^\nu \cdot e^{-2\pi ik/\nu} + 1 = 0 \tag{5.20}$$

for $\zeta$. This yields $\nu(\nu - 1)$ fixed points

$$\zeta_{kl} = \frac{1}{\sqrt[\nu]{\nu^2 - 2\nu(\nu - 1) \cos \frac{2\pi k}{\nu} (\nu - 1)^2}} \cdot e^{\theta_{kl}}$$

$$\theta_{kl} = \left(\frac{\nu - 1}{\nu} \arctan \frac{\nu \sin \frac{2\pi k}{\nu}}{2\nu \sin^2 \frac{2\pi k}{\nu} - 1} + \frac{2\pi l}{\nu}\right)i \tag{5.21}$$

with $k = 1, 2, \ldots, \nu - 1$, $l = 0, 1, 2, \ldots, \nu - 1$. 


We note that for $\nu > 3$, there exist several distinct local scale factors $\Lambda_{\nu,2,k}$. Due to the symmetry of (5.19), there exist $\lfloor \frac{1}{2}(\nu - 1) \rfloor$ scale factors which are different from each other. They are linked with the fact that there are several fractal chains running on each global branch of the fractal.

The fixed points of smaller cycle than $\nu$ cannot be determined in a general fashion as specific properties of the third-order Newton polynomial were used for their derivation which do not generalise easily.

However, we can give an argument for the number of different cross-chain scale factors that should occur. It can be noted that the blobs of the $\nu^{th}$ order Newton-fractal consist of $(\nu - 1)$ branches which are arranged symmetrical around the centre line of the blob. If $\nu$ is even, the centre line coincides with one of the branches. Furthermore, the blobs on each branch (which are relevant for the next step of the inverse mapping) are composed of $(\nu - 1)$ sub-branches. This leads to a total number of $(\nu - 1)^2$ sub-branches. We now take the symmetry with respect to the centre line into account, in particular the fact that for even $\nu$, one (the centre) branch is symmetrical in itself. This reduces the number of different sub-branches to

$$b_{\nu} = \frac{1}{2} \lfloor (\nu - 1)^2 + 1 \rfloor,$$

with each of which a cross-chain scale factor should be associated.

Again, due to the local conformity of the inverse Newton mapping, we can generalise these results for the origin for all points of the fractal. The Julia points of order $n$ approach the point of order $(n - 1)$ on a branch with local scale factor $\lambda_n$ as a geometrical progression. There are $\lfloor \frac{1}{2}(\nu - 1) \rfloor$ different cross-chain scale factors $\Lambda_{\nu,2,k}$ for cycle $\nu$ and $b_{\nu}$ different cross-chain scale factors in general.

**Fractal Dimension**

The fractal dimension for $\nu = 3$ was approximated via the local scale factors both within a chain of Julia points of order $n$ and between chains of different order. For $\nu > 3$, the cross-chain scale factors cannot be determined in a straightforward manner and therefore the fractal dimension cannot be approximated in the same way. There is, however, no reason why numerical values for these scale factors should not be used. With these (e.g. obtained by numerical integration along arcs similar to that in Fig. 6), the dimension of the $\nu^{th}$ order Newton fractal can be approximated in the same way as suggested in 3.4.3. Denoting the $b_{\nu}$ cross-chain scale factors with $\Lambda_{\nu,k}$, we conjecture the fractal dimension $d$ to be determined by the equation

$$\sum_{k=1}^{b_{\nu}} (\Lambda_{\nu,k})^d + \lambda_{\nu}^d = 1.$$ 

An interesting question is how the fractal dimension and the associated scale factors vary with $\nu$ getting large and whether some limits can be established. We are working on this question and hope to give some results in the near future.
Further study seems to be appropriate to examine general polynomials of degree \( \nu \) and functions that can be generated from polynomials of the form (2.1) via defined mappings. The way in which the fractal properties mentioned above are changed by such mappings is being currently investigated and we hope it will give rise to further results which can be of practical relevance.

5.2 Numerical Relevance

After establishing the properties of Newton fractals, we now turn to the question how these properties relate to the application of Newton’s method as a solver for non-linear systems. The starting points for the numerical experiments are given in the following table. With \( d \), we depict the distance from the nearest root in the \( L_2 \) norm.

<table>
<thead>
<tr>
<th>( z_A )</th>
<th>( x )</th>
<th>( y )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.449327648485755</td>
<td>-0.403332754355275</td>
<td>0.6909</td>
<td></td>
</tr>
<tr>
<td>( z_1 )</td>
<td>-0.593573840026322</td>
<td>-0.146431149219563</td>
<td>0.7257</td>
</tr>
<tr>
<td>( z_2 )</td>
<td>0.538607827019894</td>
<td>-0.417205032532133</td>
<td>0.6220</td>
</tr>
<tr>
<td>( z_3 )</td>
<td>0.538607827019894</td>
<td>-0.35</td>
<td>0.5791</td>
</tr>
<tr>
<td>( z_a )</td>
<td>0.538607827019894</td>
<td>-0.417205031032133</td>
<td>0.6220</td>
</tr>
<tr>
<td>( z_b )</td>
<td>0.538607829019894</td>
<td>-0.417205031032133</td>
<td>0.6220</td>
</tr>
</tbody>
</table>

Table 1: Starting Points for Numerical Experiments

The points were determined from the inverse Newton mapping (3.27) for the complex cubic using double precision. The convergence criterion was a residual of \( 10^{-13} \) in the \( L_2 \) norm.

Singular Points One of the commonly known reasons for a breakdown of Newton’s method is encountering a point with singular Jacobian. In a numerical context, an iterate in the vicinity of such a point will cause the shifts to become huge and the method to diverge unpredictably. If the solution algorithm has no way of coping with such huge shifts, it might actually break down completely instead of just wandering off very far. Fig. 22 gives a convergence history where an iterate comes very close to the origin.

It can be seen that in step 14, the iterate approached the origin and a very large shift occurred. It then takes over 50 iterations to recover and finally approach the root \((1,0)\). An important point about Newton’s method with fractal boundaries for the basins of attraction is that there are infinitely many of these potentially singular points, namely the Julia points which all are images of the origin. Therefore, an iteration that starts too close to a Julia point will after
some seemingly well-behaved iterations approach the origin and suddenly behave unpredictably.

The vicinity of Julia points We now consider the three points $z_2$, $z_a$ and $z_b$. As can be seen from Table 1, they only differ after the 8th digit of one of their coordinates. Their convergence path is depicted in Fig. 22.

As expected, they seem to converge in exactly the same way for the first iterations. In iteration 18, the path of their iterates leaves the attractive circle on what can be thought of as the inverse of a prolongation step, hence the increase in residual. However, towards the end of the convergence history, their residuals begin to differ. It emerges that they converge to different roots (hence the inverse prolongation was by the same length on different branches of the fractal), namely

$$z_2 \to (1, 0), \quad z_a \to \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad z_b \to \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \quad (5.24)$$
This interesting observation is not a numerical artefact of finite precision, but the immediate consequence of Lemma 3.10, which stated that each Julia point is bordering the basins of attraction of all roots of the polynomial. This might be employed in a suitable way by a numerical algorithm that tries to map out the set of roots of a given polynomial. The formulation of an efficient algorithm is supported by the results of section 5.1 regarding local symmetry. These translate directly into a clever choice for a stepping angle in the vicinity of a Julia point. On the other hand, it is obvious from the same Lemma 3.10 that once in the fractal region, the root to which the iterates converge cannot be easily established. As the closest root to all three points is $(1, 0)$, the final convergence point also cannot be inferred from a ‘reasonably close’ distance to the root - as long as there is no sufficient knowledge about the function.

The attractive circle In the derivation of the local scale factors, we have established fixed points within the attractive circle which act as attractors under the inverse mapping. This means that starting from a vicinity of these points, the iterates of Newton’s method stay inside the attractive circle (and in a vicinity of the fixed points) for a considerable time without the influence of any singularities. The residual stagnates in that case. With the fixed points being transcendental, any iterate will start with a small perturbation in a finite precision context. Therefore, the limiting case of infinite iterations will never be attained, and the iterates will reach a root eventually. It is obvious from the discussion in section 3.3.3 and the shape of the function that any change of branch inside the attractive circle will leave the residual more or less unchanged. The only step that can affect the residual considerably is the inverse prolongation step.

This reveals a general pattern of convergence for starting points within the region of Julia points, which can be summarized as follows.

- A stationary residual with magnitude 1, when the iterates stay inside the attractive circle and constantly change branches.

- A sharp increase in residual when the iterates leave the attractive circle with a change of branch. This has to be followed by

- Linear convergence with an approximate rate $(\lambda_{3,\infty})^{-1}$ as the iterates move laterally along the branch. This is the only type of convergence possible outside the attractive circle for large iterates.

- Quadratic convergence once the iterates get sufficiently close to the root for classical Newton-Kantorovitch stability to hold.

The described behaviour can be observed in Fig. 24. The iterates starting from $z_2$ stay inside the attractive circle until iteration 18, when they leave for the fractal branch with angle $\pi$ and after some lateral backward movement along
Figure 24: Convergence history for starting points inside the attractive circle that branch converge to the root (1, 0). The sequence of iterates starting from $z_1$ leaves the attractive circle four times before finally arriving on the right branch and converging to (1, 0). The linear convergence during the lateral movements emerges quite clearly. Finally, $z_3$ is an example of a point within the attractive circle that converges immediately. This is due to the fact that it was chosen to be outside the region of Julia points (as can be confirmed in the fractal plot or via (3.31b) and the approximation in 3.3.3). Therefore, convergence is unimpaired and the convergence history leads straight to the root without any of the characteristics connected with the fractal behaviour.

**Convergence Radius** As quantitative approximations to the fractal structure are known (e.g. (3.31b)) and its formation is understood, suitably sharp geometrical bounds on the area where Julia points occur, can be established. One of these bounds was the sector of the attractive circle in 3.3.3. Using this or similar approximations, it is quite easy to establish a radius of convergence around the roots and therefore decide in advance, to which root a starting point in the complex plane will converge or whether it lies in the 'unpredictable' fractal region.

**Behaviour close to the origin** In addition to the singularity at the origin, it becomes obvious from (3.2) that for small $z$, $z_0$ will become very large in a vicinity of the origin. From (3.2), we can approximate $z_0$ for small $z$ by

$$|z_0| \approx \frac{1}{\nu|z|^{\nu-1}}.$$  

(5.25)

For example, setting $\nu = 10$ and $|z| = 0.01$, we get $|z_0| \approx 10^{17}$ - a value from which it will take a long time coming back to the root on the unit circle with only a linear convergence rate permissible. As $\nu$ gets larger, so does the ill-conditioned circle around the origin with its radius being 1 in the far limit.
Comparison with the Modified Method  

After explaining how the fractal character affects the orthodox Newton method, we might now ask whether the non-fractal modified method has any numerical benefits to it. It is obvious that one benefit of the modified method is its greater predictability as the boundaries of the convergence basins of the roots are well behaved. The prohibition of huge shifts eradicates the near-singular circle at the origin for higher-order polynomials. Also, due to the removal of the singularities, the stability of the method is enhanced. The question is, however, whether this was achieved at the cost of computational speed. To examine this, an array of $240 \times 240$ starting points was cast over the region $\{[-2,2] \times [-2,2]\}$ and each point converged to a root. The number of iterations for that was averaged to give a measure on how fast the method converges and computation times were taken. Table 2 gives the results. The limits given refer to the upward and downward limits of chapter 4 and are given in the form $s_d - s_u$. The method denoted by empty-$\mathcal{J}$ is the method presented in 4.4. The hardware used was an Intel-80486/100 based system.

<table>
<thead>
<tr>
<th>Method</th>
<th>limits</th>
<th>line searches</th>
<th>average iteration #</th>
<th>time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthodox</td>
<td>-</td>
<td>-</td>
<td>7.997</td>
<td>23.4</td>
</tr>
<tr>
<td>empty-$\mathcal{J}$</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>5.958</td>
<td>235.0</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>yes</td>
<td>5.877</td>
<td>1172.0</td>
</tr>
<tr>
<td>modified</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>6.439</td>
<td>237.0</td>
</tr>
<tr>
<td></td>
<td>2.0 - 10.0</td>
<td>yes</td>
<td>6.605</td>
<td>243.0</td>
</tr>
<tr>
<td></td>
<td>1.1 - 1.1</td>
<td>yes</td>
<td>7.213</td>
<td>268.0</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>yes</td>
<td>6.528</td>
<td>238.0</td>
</tr>
<tr>
<td>modified</td>
<td>10.0 - 10.0</td>
<td>no</td>
<td>7.529</td>
<td>27.4</td>
</tr>
<tr>
<td></td>
<td>1.1 - 1.1</td>
<td>no</td>
<td>7.667</td>
<td>27.7</td>
</tr>
<tr>
<td></td>
<td>1.01 - 1.01</td>
<td>no</td>
<td>8.194</td>
<td>30.0</td>
</tr>
</tbody>
</table>

Table 2: Comparison of Convergence for various Newton methods, $z^3 = 1$

We immediately see that the line searches in the modified methods decrease the number of iterations by up to 25%. However, the computation time increases dramatically. This is due to the fact that line searches involve a considerable number of function evaluations. In the two-dimensional case, the solution of the linear system involving the Jacobian is available at very low cost, and therefore the additional function evaluations add a high proportion of computational burden. In higher-dimensional systems, however, the solution of the system is a non-trivial matter and it may take as much as 95% of the total computation time. The additional function evaluations for line searches present less cost than what is gained by fewer iterations. It should also be mentioned that the present straightforward coding of the line searches is certainly a very expensive strategy and can be greatly improved by smarter algorithms.
The modified method which favours the Newton step against overstepping if it improves the residual seems to offer no advantage over the empty-$\mathcal{J}$ method which is superior both in iteration count and execution time.

By depleting the fractal chain as described in 4.3.1, shift scalings decrease the fractal behaviour of the method, removing it gradually as they tend to 1. As the scaling is computationally inexpensive, there are no severe effects on execution time. However, with very restrictive limits, both execution time and the number of iterations increase. The advantage of mere shift scaling is that it allows a smooth control over 'how much fractal there is left' which can be traded against increase in runtime and convergence speed. Due to the ease with which the orthodox Newton method still can handle the cubic case, even the shift scaling has no positive effect in runtime. This already changes with the quartic problem $z^4 = 1$ where the ill-conditioned region around the origin is growing larger. Table 3 gives the results. A breakdown denotes the failure to converge within 500 iterations. Breakdowns are excluded from the calculation of the average iteration count (favouring the method with breakdowns).

<table>
<thead>
<tr>
<th>Method</th>
<th>limits</th>
<th>line search</th>
<th>av. iteration #</th>
<th>time [sec]</th>
<th>breakdowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthodox</td>
<td>-</td>
<td>-</td>
<td>10.948</td>
<td>36.7</td>
<td>84</td>
</tr>
<tr>
<td>empty-$\mathcal{J}$</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>7.257</td>
<td>347.0</td>
<td>0</td>
</tr>
<tr>
<td>modified</td>
<td>5.0 - 5.0</td>
<td>no</td>
<td>8.910</td>
<td>36.4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1.1 - 1.1</td>
<td>no</td>
<td>8.789</td>
<td>36.3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1.01 - 1.01</td>
<td>no</td>
<td>9.458</td>
<td>39.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Comparison of Convergence for various Newton methods, $z^4 = 1$

The above remarks are supported by these results. The modified method with suitable shift scaling offers not only higher stability, but also a slightly better execution time. Again, the increase in execution time and number of iterations is visible for shift limits that are too restrictive. To check for influences of the hardware, two cases were also run on a Sun SparcClassic using optimised Gnu C in double precision. The results obtained are shown in table 4.

Due to their denial of excessive shifts where the internal representation of numbers is more likely to have an influence, the stabilised methods also show a higher degree of portability.

We come to the conclusion that the stabilising modifications improve the number of iterations to convergence quite dramatically if line searches are employed. Mere shift scaling does not improve the iteration count dramatically, but stays competitive in runtime. Therefore, we advocate the use of line searches for systems where the solution of the Jacobian system is dominant and shift scaling for low-dimensional problems where the Jacobian solution is cheap. If line
<table>
<thead>
<tr>
<th>Method</th>
<th>limits</th>
<th>line search</th>
<th>av. iteration</th>
<th>time [sec]</th>
<th>break-downs</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthodox</td>
<td>-</td>
<td>-</td>
<td>10.966</td>
<td>26.2</td>
<td>79</td>
</tr>
<tr>
<td>empty-$J$</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>7.257</td>
<td>272.0</td>
<td>0</td>
</tr>
<tr>
<td>modified</td>
<td>5.0 - 5.0</td>
<td>no</td>
<td>8.910</td>
<td>25.7</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 4: Convergence on a Sun SparcClassic, $z^4 = 1$

searches are employed, we recommend aggressive overstepping rather than retaining a Newton step that improved the residual. Smart line search algorithms which minimise the number of function evaluations are definitely necessary for competitive execution times.

### 5.3 Beyond Polynomials

We note that the results established in this work can be transferred to other polynomials that emerge from the polynomials (2.1) through conformal mappings. Also, linear combinations of the polynomials (2.1) will have fractal characteristics, albeit of a more complicated nature. In general, it is reasonable to assume that general complex polynomials of degree $\nu \geq 3$ will exhibit fractal behaviour in conjunction with Newton’s method. It is also quite likely that despite the existence of a Julia set and fractal characteristics, a quantitative description will be even harder than for the family of polynomials (2.1). However, we may assume that some of the general principles for fractal formation will still hold and therefore the modified methods will yield improvement in convergence.

As an example of a transcendental function that as such is not a finite polynomial, we use

$$\sin z - \frac{z}{2} = 0$$

which can be rewritten as a two-dimensional system, splitting in real and imaginary part $z = x + iy$,

$$\mathbf{f} = \left( \begin{array}{c} \sin x \cosh y - \frac{z}{2} \\ \cos x \sinh y - \frac{z}{2} \end{array} \right).$$

It can be established that for $|z| < 2$, this system has the trivial solutions $(0, 0)$ and two symmetrical solutions being defined by $\{(\sin x = \frac{z}{2}) \wedge (y = 0)\}$. For larger $|z|$, there is an infinite number of solutions to (5.27). For the Jacobian, we obtain

$$\mathbf{J} = \begin{bmatrix} \cos x \cosh y - \frac{1}{x} & \sin x \sinh y \\ -\sin x \sinh y & \cos x \cosh y - \frac{1}{y} \end{bmatrix}.$$
Of course, (5.26) can be expanded to obtain

\[ \frac{z}{2} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \ldots = 0, \]

with an infinite sum of polynomials on the left side. Therefore, we hope to use the modified methods with some success. The results of the numerical experiment are given in Table 5. The parameters of the experiment were the same as described in the previous sections for polynomials.

<table>
<thead>
<tr>
<th>Method</th>
<th>limits</th>
<th>line search</th>
<th>av. iteration</th>
<th>time [sec]</th>
<th>break-downs</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthodox</td>
<td>-</td>
<td>-</td>
<td>6.087</td>
<td>77.7</td>
<td>30</td>
</tr>
<tr>
<td>empty-$\mathcal{F}$</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>4.689</td>
<td>588.0</td>
<td>0</td>
</tr>
<tr>
<td>modified</td>
<td>10.0 - 10.0</td>
<td>yes</td>
<td>5.336</td>
<td>644.0</td>
<td>0</td>
</tr>
<tr>
<td>modified</td>
<td>10.0 - 10.0</td>
<td>no</td>
<td>5.918</td>
<td>78.4</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>5.0 - 5.0</td>
<td>no</td>
<td>5.845</td>
<td>77.2</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1.1 - 1.1</td>
<td>no</td>
<td>5.911</td>
<td>74.8</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5: Comparison of Convergence for various Newton methods, $\sin z = \frac{z}{2}$

Obviously, the modified methods yield the same results as with polynomials. In particular, we can achieve a 'fractal depletion' by tightening the shift limits just as with polynomials. Fig. 25 gives an account of that process. For clarity, the black area has been chosen as the union of the basins of attraction of the two roots on the real axis which differ from zero. The shift limits $s_d, s_u$ have both been assigned the value given in each picture. The fractal character of the problem with the orthodox Newton method is depicted in Fig. 16, showing the 'non-depleted' case.

Of course, the results of this experiment can only present a notion that the methods will be useful in a more general context. More work in that area remains to be done, and might induce useful and interesting results.

The principles of fractal formation and their numerical remedy generalise beyond the case of simple polynomials of the form $z^n = 1$. We have shown that some analysis is possible for simple polynomials, but more complicated systems are difficult to analyse. We do not suggest that all problems associated with difficult convergence of Newton’s method can be attributed to a fractal structure of the boundaries of basins of attraction. Nevertheless, there are characteristics such as linear convergence after a large jump in residual or a prolonged stagnation period of the residual which ought to be taken as very suggestive of the presence of fractal boundaries. It is also the case that remedies such as those described in the modified Newton method are effective in removing difficult convergence.
Figure 25: Basins of attraction of non-zero roots for \( \sin z = \frac{z}{2} \) with various shift limits.
problems, although at a cost of increased computation time if not chosen with consideration of the problem structure. However, it is also possible to choose stabilisations that increase stability and decrease computation time. We feel that, bearing fractal characteristics of model problems and their analysis in mind, improved adaptations of Newton’s method for engineering and physics applications can be devised.
References


