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A Posteriori Error Analysis for Linear Convection–Diffusion Problems Under Weak Mesh Regularity Assumptions

Paul Houston Endre Süli

In this paper we consider the generalisation of standard *a posteriori* error estimates, derived for unsteady problems, to arbitrary space–time meshes. In particular, we derive an *a posteriori* error bound for the discontinuity capturing Lagrange–Galerkin method applied to an unsteady (linear) convection–diffusion problem, assuming only that the underlying mesh is *non-degenerate*. The proof of this error estimate will be based on strong stability estimates of an associated dual problem, together with the Galerkin orthogonality of the finite element method.

Key words and phrases: A posteriori error analysis, quasi-interpolation operators, Lagrange–Galerkin finite element methods

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Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England OX1 3QD

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1 Introduction

The derivation of *a posteriori* error estimates for unsteady problems are generally based on assuming that the spatial mesh function $h \in C^1(\bar{\Omega})$ and that the gradient of h is uniformly bounded by a constant μ, where μ is assumed to be *sufficiently small*, cf. [10, 11, 15, 16]. Clearly, these conditions do not preclude the use of non-uniform meshes, however, the change in the size of the elements in the mesh must be very smooth, i.e. so that the change in h is sufficiently small to guarantee that $|\boldsymbol{\nabla} h(\mathbf{x})| \leq \mu \quad \forall \mathbf{x} \in \bar{\Omega}$. For practical computations in two or three space–dimensions, such a restriction on h may be unrealistic.

The objective of this paper is to derive an *a posteriori* error estimate for the discontinuity capturing Lagrange–Galerkin method under weaker mesh regularity assumptions. To achieve this, the key part of the proof of the error estimate relies on the definition in space of the function ϕ_h introduced using the Galerkin orthogonality of the finite element method. We note that, for a given choice of ϕ_h there are essentially two main requirements that must be satisfied: firstly, that ϕ_h is stable in the $L^2(\Omega)$ norm, i.e. there exists a positive constant C, independent of h, such that

$$\|\phi_h\| \leq C\|\phi\|. \tag{1.1}$$

Secondly, that the approximation error between the solution of the corresponding dual problem, ϕ, and ϕ_h (measured in some appropriate norm) can be bounded *locally* on an element κ (or on a patch of elements surrounding κ), so that the spatial mesh may be non-uniform.

In [10, 11, 15, 16], ϕ_h is defined to be the space–time L^2-projection of ϕ. Clearly, this choice automatically satisfies (1.1) with $C = 1$. However, the spatial projection error cannot be bounded locally due to the global nature of projecting onto continuous piecewise polynomial functions. Hence, it is first necessary to bound the weighted projection error (weighted with powers of h) by the weighted interpolation error. Then, the latter can be locally estimated on each element κ in the mesh. Unfortunately, this process assumes that the weighting factor inside the norm satisfies certain regularity assumptions, which in turn induce the restrictions on mesh function h.

In this paper we propose to define ϕ_h to be the L^2-projection of ϕ in time, but in space we shall define ϕ_h to be a *quasi-interpolant* of ϕ. This quasi-interpolation operator will be constructed in such a way that (1.1) will hold, and we shall show that optimal approximation results hold on local patches surrounding a particular element domain. Moreover, based on this choice of ϕ_h a global *a posteriori* error estimate will be established assuming only that the triangulation of the spatial domain Ω is *non-degenerate*.

The outline of this paper is as follows: in Section 2 we summarise some of the notational conventions that we shall use. In Section 3 we develop the necessary quasi-interpolation theory needed for the proceeding *a posteriori* error analysis.

In Section 4 we state the model problem to be considered and formulate the discontinuity capturing Lagrange–Galerkin method for this problem. In Section 5 we derive an *a posteriori* error estimate for our model problem, assuming only that the underlying mesh is non-degenerate. Finally, in Section 6 we summarise the work presented in this paper.

2 Notation and basic definitions

Let $\mathbf{Z}$ denote the set of integers, $\mathbf{N}$ the set of positive integers, $\mathbf{N}_0$ the set of non-negative integers, $\mathbf{R}$ the set of real numbers and $\mathbf{R}^+$ the set of positive real numbers.

Let ω be a bounded open subset of $\mathbf{R}^d$ ($d \in \mathbf{N}$) with boundary $\partial\omega$. We shall say that ω is *star–shaped* with respect to every point in a set of positive measure $B \subset \omega$, if for every $\mathbf{x} \in B$, every $\mathbf{y} \in \omega$ and every $\theta \in [0,1]$ we have that $\mathbf{x} + \theta(\mathbf{y} - \mathbf{x}) \in \omega$.

For $1 \leq p \leq \infty$, let $L^p(\omega)$ denote the usual Lebesgue space of real-valued functions with norm $\|\cdot\|_{L^p(\omega)}$. For $p = 2$ and for $u, v \in L^2(\omega)$ we denote by $(\cdot,\cdot)_\omega$ the $L^2(\omega)$ inner product defined as

$$(u, v)_\omega := \int_\omega u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}.$$

For $\omega = \Omega$, where Ω will be specified later, we denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$, and $(\cdot,\cdot)_\Omega$ by $(\cdot,\cdot)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, with each $\alpha_i \in \mathbf{N}_0$, $i = 1, \ldots, d$; we write

$$|\alpha| := \sum_{i=1}^{d} \alpha_i \quad \text{and} \quad \alpha! := \prod_{i=1}^{d} \alpha_i!.$$

Further, given a vector $\mathbf{x} = (x_1, \ldots, x_d)$, we write

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

Let $D^\alpha := D_1^{\alpha_1} \ldots D_d^{\alpha_d}$ and $D_j = \partial/\partial x_j$ for $1 \leq j \leq d$. For $m \in \mathbf{N}_0$, we denote by $C^m(\omega)$ the set of all continuous real-valued functions defined on ω such that $D^\alpha u$ is continuous on ω for all $|\alpha| \leq m$. $C^m(\bar{\omega})$ will denote the set of all u in $C^m(\omega)$ such that $D^\alpha u$ can be extended from ω to a continuous function on $\bar{\omega}$ for all $|\alpha| \leq m$. In particular, when $m = 0$ we simply write $C(\bar{\omega})$ instead of $C^0(\bar{\omega})$. The subspace $C_0^m(\bar{\omega})$ will denote the set of functions in $C^m(\bar{\omega})$ which have compact support in ω.

For $m \in \mathbf{N}_0$, let $W^{m,p}(\omega)$ denote the classical Sobolev space endowed with the norm $\|\cdot\|_{W^{m,p}(\omega)}$ and the semi-norm $|\cdot|_{W^{m,p}(\omega)}$ (cf. Adams [1]). Further,

$W_0^{m,p}(\omega)$ will denote the closure of $C_0^\infty(\omega)$ in the norm of $W^{m,p}(\omega)$. For $p = 2$ we write $H^m(\omega)$ and $H_0^m(\omega)$ for $W^{m,2}(\omega)$ and $W_0^{m,2}(\omega)$, respectively. In addition, the dual space of $H_0^m(\omega)$ will be denoted by $H^{-m}(\omega)$.

Let X be any of the spaces just defined. Then X^2 will denote the topological product $X \times X$.

3 Quasi-interpolation theory

Quasi-interpolation operators are generally constructed when the function under consideration is not sufficiently smooth to guarantee the existence of the interpolant; for example, if $v \in H^1(\Omega)$, where Ω is a two–dimensional domain, then pointwise values of v cannot be defined and consequently no interpolation is possible, see Bernardi [4], Brenner & Scott [5], Clément [7] and Scott & Zhang [19, 20], for example. The construction of such generalised interpolation operators is usually based on some form of *local* regularisation of the function. For instance, Clément [7] proposed a quasi-interpolation operator based on locally L^2-projecting a function onto the patch of elements surrounding each node in the mesh; then pointwise values are obtained by evaluating the projection at each node.

In this section we shall construct a quasi-interpolation operator $\tilde{\mathcal{I}}$ based on a modification of the generalised interpolation operators developed by Scott & Zhang [19] and Brenner & Scott [5], Section 4.8. The nodal values will be defined by locally averaging the function over an element κ; however, as in [7] these nodal values will be modified in order to fit homogeneous boundary conditions. To establish optimal approximation results for $\tilde{\mathcal{I}}$, we shall also need to construct a further quasi-interpolation operator $\tilde{\mathcal{I}}'$. This latter operator will be defined in a similar manner as the operator $\tilde{\mathcal{I}}$; except that for nodes lying on the boundary $\partial\Omega$ of the domain, the nodal values will be obtained by averaging along an edge τ such that $\tau \subset \partial\Omega$.

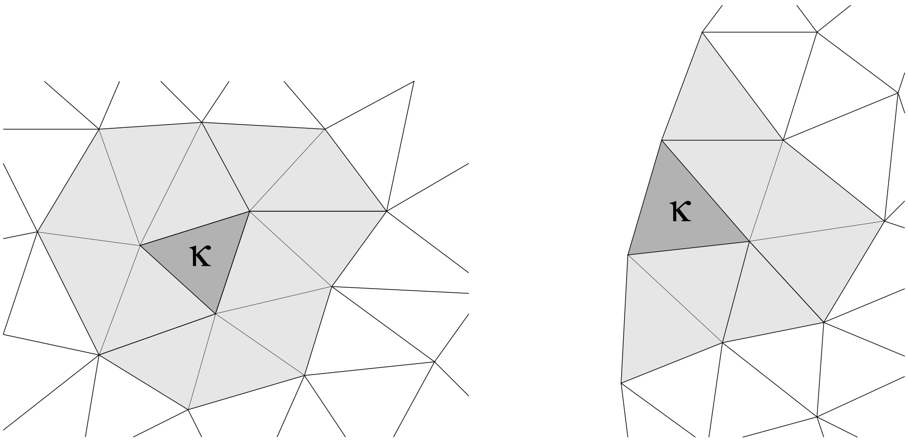
3.1 Preliminaries

Let Ω be a connected, open, bounded polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$. Let $\mathcal{T} = \{\kappa\}$ be an admissible subdivision of Ω into closed triangles κ, with corresponding mesh function h satisfying

$$c_1 h_\kappa^2 \leq \mathrm{meas}(\kappa) \quad \forall \kappa \in \mathcal{T}, \tag{3.1a}$$

$$c_2 h_\kappa \leq h(\mathbf{x}) \leq h_\kappa \quad \forall \mathbf{x} \in \kappa \ \ \forall \kappa \in \mathcal{T}, \tag{3.1b}$$

where $h_\kappa = \mathrm{diam}(\kappa)$ and c_1 and c_2 are positive constants independent of h.



(a) (b)

Figure 1: Examples of the set S_κ for: (a) κ an internal element; (b) κ a boundary element.

Further, we define $E = \{\tau\}$ to consist of the set of line segments in $\mathbf{R}^2$ which appear as an edge of some $\kappa \in \mathcal{T}$.

To each $\kappa \in \mathcal{T}$, we associate the set S_κ, which we define by

$$S_\kappa = \text{interior}\left(\bigcup\{\tilde{\kappa} \in \mathcal{T} : \tilde{\kappa} \cap \kappa \neq \emptyset\}\right),$$

see Figure 1, for example. Moreover, for each S_κ, $\kappa \in \mathcal{T}$, we assume that S_κ is star–shaped with respect to a ball $B_{S_\kappa} \subset S_\kappa$, and that the following conditions hold:

$$c_3 h_{\tilde{\kappa}} \leq h_\kappa \leq c_4 h_{\tilde{\kappa}} \qquad \forall \tilde{\kappa} \in S_\kappa \quad \forall S_\kappa, \tag{3.2a}$$

$$c_5 h_\kappa \leq \text{radius}(B_{S_\kappa}) \quad \forall S_\kappa, \tag{3.2b}$$

$$\text{diam}(S_\kappa) \leq c_6 h_\kappa \qquad \forall S_\kappa, \tag{3.2c}$$

where c_3, c_4, c_5 and c_6 are positive constants independent of h. In addition, we define

$$c_7 = \sup_{\tilde{\kappa} \in \mathcal{T}} \{\text{card}\,\{\kappa \in \mathcal{T} : \tilde{\kappa} \in S_\kappa\}\} < \infty. \tag{3.3}$$

We note that each of the constants c_3, c_4, c_5, c_6 and c_7 will depend on c_1, since the mesh conditions (3.2) and (3.3) follow from the fact that $\mathcal{T}$ is a non-degenerate triangulation of Ω, i.e. that (3.1a) holds.

Remark 3.1 *Here, we note that the assumption that S_κ is star–shaped with respect to a ball B_{S_κ} is necessary to apply the form of the Bramble–Hilbert lemma stated in Lemma 3.9. However, non-constructive versions of this result, i.e. where a generic constant is given, may be proved without such a condition on the domain S_κ, cf. Ciarlet [6], Theorem 3.1.1, for example.*

For $r \in \mathbf{N}$, we shall consider the following finite element spaces

$$\begin{aligned} \mathcal{V}^h &= \{v \in C(\Omega) : v|_\kappa \in \mathbf{P}_r(\kappa)\ \forall \kappa \in \mathcal{T}\}, \\ \mathcal{V}_0^h &= \{v \in C_0(\Omega) : v|_\kappa \in \mathbf{P}_r(\kappa)\ \forall \kappa \in \mathcal{T}\}, \end{aligned}$$

where $\mathbf{P}_r(\kappa)$ denotes the set of polynomials of degree at most r over κ.

Let $\mathcal{N}_h = \{\mathbf{a}_i\}_{i=1}^L$ denote the set of interpolation nodes of $\mathcal{T}$ and $\mathcal{N}_h^{\text{int}} \subset \mathcal{N}_h$ the set consisting of all interior nodes. Further, we let $\{\phi_i\}_{i=1}^L$ denote the set of nodal basis functions of $\mathcal{V}^h$.

We note that by (3.1a), we have a family of Lagrange finite elements $(\kappa, \mathcal{P}, \Sigma)$, where $\mathcal{P} = \mathbf{P}_r(\kappa)$ and Σ consists of point evaluations at appropriate points ('nodes'). We denote by $(\hat{\kappa}, \hat{\mathcal{P}}, \hat{\Sigma})$ the reference/canonical element in $\hat{\mathbf{x}} = (\xi_1, \xi_2)$ plane with vertices (0,0), (1,0) and (0,1). Given any finite element $(\kappa, \mathcal{P}, \Sigma)$, where $\kappa \in \mathcal{T}$, there exists a unique invertible affine mapping

$$F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{x}_0, \tag{3.4}$$

i.e. where B is an invertible 2×2 matrix and $\mathbf{x}_0$ is a vector in $\mathbf{R}^2$, such that $(\kappa, \mathcal{P}, \Sigma)$ and $(\hat{\kappa}, \hat{\mathcal{P}}, \hat{\Sigma})$ are *affine–equivalent* finite elements, cf. Ciarlet [6]. The effects of this change of variable are described in the following lemma.

Lemma 3.1 *Let κ and $\hat{\kappa}$ be two affine–equivalent (closed) simplices in $\mathbf{R}^2$. If $v \in W^{m,p}(\kappa)$ for some $m \in \mathbf{N}_0$ and some $p \in [1, \infty]$, then $\hat{v} = v \circ F$ belongs to $W^{m,p}(\hat{\kappa})$, and in addition, there exists a constant $C^{\text{map}} = C^{\text{map}}(m,p)$ such that*

$$\forall v \in W^{m,p}(\kappa), \quad |\hat{v}|_{W^{m,p}(\hat{\kappa})} \leq C^{\text{map}} |B|^m |det(B)|^{-1/p} |v|_{W^{m,p}(\kappa)}, \tag{3.5}$$

where B is the matrix occurring in the mapping F defined in (3.4) and $|B|$ denotes the matrix two-norm of B.

Analogously, we have

$$\forall \hat{v} \in W^{m,p}(\hat{\kappa}), \quad |v|_{W^{m,p}(\kappa)} \leq C^{\text{map}} |B^{-1}|^m |det(B)|^{1/p} |\hat{v}|_{W^{m,p}(\hat{\kappa})}. \tag{3.6}$$

Proof The proof of this lemma may be found in Ciarlet [6], Theorem 3.1.2. We note that a constructive proof of this lemma, i.e. where the size of the constant C^{map} is known, is presented in [14], Lemma 2.1; there, it was shown that if $|B|$ and $|B^{-1}|$ are replaced by $|B|_\infty$ and $|B^{-1}|_\infty$, respectively, where $|\cdot|_\infty$ denotes the matrix ∞-norm, then

$$C^{\text{map}}(m,p) = 2 \sup_{|\beta|=m} \left\{ \prod_{k=1}^{2} \max_{|\alpha|=\beta_k} \frac{\beta_k!}{\alpha!} \right\} \text{card} \left\{ \alpha \in \mathbf{N}_0^2 : |\alpha| = m \right\}^{1+1/p}.$$

□

In the following, it will be desirable to evaluate the norm of B^{-1} in terms of local geometric quantities. To this end, we give the following bound

$$|B^{-1}| \leq h_\kappa / \text{meas}(\kappa), \tag{3.7}$$

cf. [15]. In addition, we observe that

$$|\det(B)| = \frac{\text{meas}(\kappa)}{\text{meas}(\hat{\kappa})} = 2\text{meas}(\kappa). \tag{3.8}$$

Finally, we end this section with the following trace inequality.

Lemma 3.2 *Let $\kappa \in \mathcal{T}$, where $\mathcal{T}$ satisfies conditions (3.1). If $v \in W^{1,1}(\kappa)$, then there exists a positive constant C^t such that*

$$\int_\tau |v| \mathrm{d}s \leq C^t \left(\int_\kappa |\boldsymbol{\nabla} v| \mathrm{d}\mathbf{x} + h_\kappa^{-1} \int_\kappa |v| \mathrm{d}\mathbf{x} \right), \quad \tau \subset \partial\kappa \ \forall \kappa \in \mathcal{T},$$

where $C^t = 4\sqrt{2}/c_1$.

Proof See [15], Lemma A.1. □

3.2 Construction of the quasi-interpolation operator $\tilde{\mathcal{I}}$

For each node $\mathbf{a}_i \in \mathcal{N}_h$ we choose an element domain κ such that $\mathbf{a}_i \in \kappa$, and we let $\sigma_i = \kappa$. We note that there may be many such element domains, but we pick just one. Let us denote by n_1 the dimension of $\mathbf{P}_r(\sigma_i)$. Further, let $\mathbf{a}_{i,1} = \mathbf{a}_i$ and $\{\mathbf{a}_{i,j}\}_{j=1}^{n_1}$ be the set of nodal points in σ_i. For the nodal basis $\{\phi_{i,j}\}_{j=1}^{n_1}$ for σ_i, we have the corresponding $L^2(\sigma_i)$-dual basis $\{\psi_{i,j}\}_{j=1}^{n_1}$ defined by

$$\int_{\sigma_i} \psi_{i,j}(\mathbf{x}) \phi_{i,k}(\mathbf{x}) \mathrm{d}\mathbf{x} = \delta_{jk}, \quad \text{for } j,k = 1,2,\ldots,n_1, \tag{3.9}$$

where δ_{jk} is the Kronecker delta. To simplify notation, we let

$$\psi_i = \psi_{i,1} \quad \forall \mathbf{a}_i \in \mathcal{N}_h.$$

Hence, it follows that for any nodal basis function ϕ_j of $\mathcal{V}^h$, we have

$$\int_{\sigma_i} \psi_i(\mathbf{x}) \phi_j(\mathbf{x}) \mathrm{d}\mathbf{x} = \delta_{ij}, \quad \text{for } i,j = 1,2,\ldots,L. \tag{3.10}$$

We now define the quasi-interpolation operator $\tilde{\mathcal{I}} : L^1(\Omega) \to \mathcal{V}_0^h$ by

$$\tilde{\mathcal{I}} v(\mathbf{x}) = \sum_{i=1}^{L} \tilde{\mathcal{I}} v(\mathbf{a}_i) \phi_i(\mathbf{x}), \tag{3.11}$$

where

$$\tilde{\mathcal{I}} v(\mathbf{a}_i) = \chi_\Omega(\mathbf{a}_i) \int_{\sigma_i} \psi_i(\mathbf{y}) v(\mathbf{y}) \mathrm{d}\mathbf{y}, \tag{3.12}$$

and χ_Ω is the characteristic function for Ω, i.e. if $\mathbf{a}_i \in \partial\Omega$ then $\tilde{\mathcal{I}} v(\mathbf{a}_i) \equiv 0$.

Remark 3.2 *As an example, we let* σ_i *be the canonical element* $\hat{\kappa}$ *and* $\{\hat{\phi}_{i,j}\}_{j=1}^3$ *be the set of linear basis functions on* $\hat{\kappa}$ *given by*

$$\hat{\phi}_{i,1} = 1 - \xi_1 - \xi_2, \quad \hat{\phi}_{i,2} = \xi_1, \quad \hat{\phi}_{i,3} = \xi_2;$$

then, the corresponding $L^2(\hat{\kappa})$*-dual basis is defined as follows:*

$$\hat{\psi}_{i,1} = -24\xi_1 - 24\xi_2 + 18, \quad \hat{\psi}_{i,2} = 24\xi_1 - 6, \quad \hat{\psi}_{i,3} = 24\xi_2 - 6.$$

We shall now proceed to prove the following results for the quasi-interpolation operator $\tilde{\mathcal{I}}$.

Lemma 3.3 *The quasi-interpolation operator* $\tilde{\mathcal{I}}$ *defined in (3.11) is a projection from* $L^1(\Omega)$ *to* $\mathcal{V}_0^h$*.*

Proof Suppose that $v \in \mathcal{V}_0^h$, then using the definition of $\tilde{\mathcal{I}}$ and (3.10) gives, for each $\mathbf{x} \in \bar{\Omega}$,

$$\begin{aligned}\tilde{\mathcal{I}}v(\mathbf{x}) &= \sum_{i=1}^{L} \chi_\Omega(\mathbf{a}_i)\phi_i(\mathbf{x}) \left[\int_{\sigma_i} \psi_i(\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y}\right] \\ &= \sum_{i=1}^{L} \chi_\Omega(\mathbf{a}_i)\phi_i(\mathbf{x}) \left[\sum_{j=1}^{L} v(\mathbf{a}_j) \int_{\sigma_i} \psi_i(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}\mathbf{y}\right] \\ &= \sum_{i=1}^{L} \chi_\Omega(\mathbf{a}_i)v(\mathbf{a}_i)\phi_i(\mathbf{x}) \\ &\equiv v(\mathbf{x}),\end{aligned}$$

by assumption. □

Lemma 3.4 *Let* $\mathbf{a}_i \in \mathcal{N}_h$*, then*

$$\|\psi_i\|_{L^\infty(\sigma_i)} \le \tilde{C}_1 h_{\sigma_i}^{-2},$$

where $\tilde{C}_1 = \|\hat{\psi}_i\|_{L^\infty(\hat{\kappa})}/(2c_1)$*.*

Proof By definition, we have

$$\int_{\sigma_i} \psi_i(\mathbf{x})\phi_j(\mathbf{x})\mathrm{d}\mathbf{x} = \delta_{ij}, \quad \text{for } i,j = 1,\dots,L,$$

cf. (3.10). Further, using the affine mapping (3.4), it follows that

$$\int_{\hat{\kappa}} \psi_i(B\hat{\mathbf{x}} + \mathbf{x}_0)\phi_j(B\hat{\mathbf{x}} + \mathbf{x}_0)\det(B)\mathrm{d}\hat{\mathbf{x}} = \delta_{ij}, \quad \text{for } i,j = 1,\dots,L.$$

Since $\hat{\phi}_j(\hat{\mathbf{x}}) = \phi_j(B\hat{\mathbf{x}} + \mathbf{x}_0)$, it follows that the corresponding $L^2(\hat{\kappa})$-dual basis is given by

$$\hat{\psi}_i(\hat{\mathbf{x}}) = \det(B)\psi_i(B\hat{\mathbf{x}} + \mathbf{x}_0), \quad \text{for } i = 1,\dots,L.$$

Using (3.8) and (3.1a), we have that

$$\begin{aligned}\|\psi_i\|_{L^\infty(\sigma_i)} &\le \frac{1}{\det(B)}\|\hat{\psi}_i\|_{L^\infty(\hat{\kappa})}\\ &\le (2c_1)^{-1}h_{\sigma_i}^{-2}\|\hat{\psi}_i\|_{L^\infty(\hat{\kappa})},\end{aligned}$$

as required. □

Lemma 3.5 *Let $v \in L^p(\Omega)$, $p \in [1,\infty]$, and let $\kappa \in \mathcal{T}$, where $\mathcal{T}$ satisfies conditions (3.1). Then,*

$$|\tilde{\mathcal{I}}v|_{W^{m,q}(\kappa)} \le \tilde{C}_2 h_\kappa^{-m+2/q-2/p}\|v\|_{L^p(S_\kappa)},$$

where $\tilde{\mathcal{I}}$ is defined in (3.11),

$$\tilde{C}_2 = C^{\mathrm{map}}(1/2)^{2-1/p}c_1^{-(m+1)}c_4^{2/p}n_1 \max_{1\le i\le n_1}\|\hat{\psi}_i\|_{L^\infty(\hat{\kappa})}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})},$$

$q \in [1,\infty]$ and $m \in \mathbf{N}_0$.

Proof Without loss of generality, we assume that the set $\{\mathbf{a}_i : 1 \le i \le n_1\}$ comprise the nodal points for the element κ. Using the triangle inequality, Lemma 3.1, (3.7), (3.8) and (3.1a), we have

$$\begin{aligned}|\tilde{\mathcal{I}}v|_{W^{m,q}(\kappa)} &\le \sum_{i=1}^{n_1}|\tilde{\mathcal{I}}v(\mathbf{a}_i)||\phi_i|_{W^{m,q}(\kappa)}\\ &\le C^{\mathrm{map}}c_1^{-m}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q}\sum_{i=1}^{n_1}|\tilde{\mathcal{I}}v(\mathbf{a}_i)|,\end{aligned}$$

for $q \in [1,\infty]$ and $m \in \mathbf{N}_0$. Further, using (3.12), Hölder's inequality and Lemma 3.4 gives

$$\begin{aligned}|\tilde{\mathcal{I}}v|_{W^{m,q}(\kappa)} &\le C^{\mathrm{map}}c_1^{-m}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q}\sum_{i=1}^{n_1}\left|\chi_\Omega(\mathbf{a}_i)\int_{\sigma_i}\psi_i(\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y}\right|\\ &\le C^{\mathrm{map}}c_1^{-m}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q}\sum_{i=1}^{n_1}\|\psi_i\|_{L^\infty(\sigma_i)}\|v\|_{L^1(\sigma_i)}\\ &\le C^{\mathrm{map}}\tilde{C}_1'c_1^{-m}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q}\sum_{i=1}^{n_1}h_{\sigma_i}^{-2}\|v\|_{L^1(\sigma_i)},\end{aligned} \tag{3.13}$$

where $\tilde{C}_1' = \max_{1\le i\le n_1}\|\hat{\psi}_i\|_{L^\infty(\hat{\kappa})}/(2c_1)$.

By Hölder's inequality, we have

$$\|v\|_{L^1(\sigma_i)} \le (1/2)^{1-1/p}h_{\sigma_i}^{2-2/p}\|v\|_{L^p(\sigma_i)}, \tag{3.14}$$

for any $v \in L^p(\Omega)$, $p \in [1,\infty]$. Using (3.14) and (3.2a), (3.13) becomes

$$\begin{aligned}|\tilde{\mathcal{I}}v|_{W^{m,q}(\kappa)} &\le C^{\mathrm{map}}\tilde{C}_1'(1/2)^{1-1/p}c_1^{-m}\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q}\sum_{i=1}^{n_1}h_{\sigma_i}^{-2/p}\|v\|_{L^p(\sigma_i)}\\ &\le C^{\mathrm{map}}\tilde{C}_1'(1/2)^{1-1/p}c_1^{-m}c_4^{2/p}n_1\max_{1\le i\le n_1}|\hat{\phi}_i|_{W^{m,q}(\hat{\kappa})}\,h_\kappa^{-m+2/q-2/p}\|v\|_{L^p(S_\kappa)},\end{aligned}$$

as required. □

Corollary 3.6 *Let $v \in L^p(\Omega)$, $p \in [1, \infty]$, and suppose that $\mathcal{T}$ satisfies conditions (3.1). Then,*

$$\|\tilde{\mathcal{I}}v\|_{L^p(\Omega)} \leq C_1^{\tilde{i}} \|v\|_{L^p(\Omega)},$$

where $C_1^{\tilde{i}} = \tilde{C}_2 c_7^{1/p}$.

Proof Let us suppose that $p \in [1, \infty)$, since the proof is greatly simplified in the case $p = \infty$. Then, using Lemma 3.5 with $m = 0$ and $q = p$, we have

$$\|\tilde{\mathcal{I}}v\|_{L^p(\kappa)} \leq \tilde{C}_2 \|v\|_{L^p(S_\kappa)} \quad \forall \kappa \in \mathcal{T}. \tag{3.15}$$

Therefore, raising (3.15) to the power p and summing over $\kappa \in \mathcal{T}$ gives

$$\begin{aligned} \|\tilde{\mathcal{I}}v\|_{L^p(\Omega)}^p &\leq (\tilde{C}_2)^p \sum_{\kappa \in \mathcal{T}} \|v\|_{L^p(S_\kappa)}^p \\ &= (\tilde{C}_2)^p \sum_{\kappa \in \mathcal{T}} \sum_{\tilde{\kappa} \in S_\kappa} \|v\|_{L^p(\tilde{\kappa})}^p \\ &\leq (\tilde{C}_2)^p c_7 \sum_{\kappa \in \mathcal{T}} \|v\|_{L^p(\kappa)}^p, \end{aligned}$$

where c_7 is defined by (3.3), as required. $\square$

3.3 Construction of the quasi-interpolation operator $\tilde{\mathcal{I}}'$

In this section we shall construct a further quasi-interpolation operator $\tilde{\mathcal{I}}'$, in order that optimal approximation results may be established for the operator $\tilde{\mathcal{I}}$ defined in the previous section. This operator $\tilde{\mathcal{I}}'$ will not be defined for as broad a class of functions as $\tilde{\mathcal{I}}$, since it will be necessary to assume the existence of a well defined trace of the function under consideration. A consequence of this assumption is that $\tilde{\mathcal{I}}'$ is not stable in the $L^2(\Omega)$ norm, and hence is not a suitable choice of operator for the definition of ϕ_h in the *a posteriori* error analysis presented in Section 5. It is worth noting that the operator $\tilde{\mathcal{I}}'$ is a special case of the quasi-interpolant introduced by Scott & Zhang [19]; moreover, the approximation results proved in [19] are equally applicable to $\tilde{\mathcal{I}}'$.

We construct the operator $\tilde{\mathcal{I}}'$ as follows: for internal nodes, i.e. for $\mathbf{a}_i \in \mathcal{N}_h^{\text{int}}$, we define $\tilde{\mathcal{I}}'$ in precisely the same manner as $\tilde{\mathcal{I}}$, cf. (3.11) and (3.12). However, boundary nodes will be treated in a more natural way; let us first recall that E denotes the set of line segments in $\mathbf{R}^2$ which appear as an edge of some $\kappa \in \mathcal{T}$. If $\mathbf{a}_i \in \partial\Omega$, then we choose an edge $\tau \in E$ such that

$$\mathbf{a}_i \in \tau \subset \partial\Omega,$$

and we let $\sigma_i' = \tau$. Let us denote by n_2 the dimension of $\mathbf{P}_r(\sigma_i')$. Further, let $\mathbf{a}_{i,1} = \mathbf{a}_i$ and $\{\mathbf{a}_{i,j}\}_{j=1}^{n_2}$ be the set of nodal points in σ_i'. The set $\{\phi_{i,j}\}_{j=1}^{n_2}$ will denote the nodal basis for σ_i'; we note that these nodal basis functions will be the restrictions to σ_i' of the corresponding basis functions defined on an element κ, where $\sigma_i' \subset \partial\kappa$.

For the nodal basis $\{\phi_{i,j}\}_{j=1}^{n_2}$ for σ_i', we have the corresponding $L^2(\sigma_i')$-dual basis $\{\psi_{i,j}'\}_{j=1}^{n_2}$ defined by

$$\int_{\sigma_i'} \psi_{i,j}'(\mathbf{x})\phi_{i,k}(\mathbf{x})\mathrm{d}s = \delta_{jk}, \quad \text{for } j,k = 1,\ldots,n_2.$$

Again, to simplify notation, let

$$\psi_i' = \psi_{i,1}' \quad \forall \mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\mathrm{int}}.$$

Hence, it follows that

$$\int_{\sigma_i'} \psi_i'(\mathbf{x})\phi_j(\mathbf{x})\mathrm{d}s = \delta_{ij}, \quad \text{for } j = 1,\ldots,L, \text{ for } i \in \{k : \mathbf{a}_k \in \mathcal{N}_h \backslash \mathcal{N}_h^{\mathrm{int}}\}. \tag{3.16}$$

We now define the quasi-interpolation operator $\tilde{\mathcal{I}}' : W^{1,1}(\Omega) \to \mathcal{V}^h$ by

$$\tilde{\mathcal{I}}'v(\mathbf{x}) = \sum_{i=1}^{L} \tilde{\mathcal{I}}'v(\mathbf{a}_i)\phi_i(\mathbf{x}), \tag{3.17}$$

where

$$\tilde{\mathcal{I}}'v(\mathbf{a}_i) = \begin{cases} \int_{\sigma_i} \psi_i(\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y}, \text{ if } \mathbf{a}_i \in \mathcal{N}_h^{\mathrm{int}}, \\ \\ \int_{\sigma_i'} \psi_i'(\mathbf{y})v(\mathbf{y})\mathrm{d}s, \text{ if } \mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\mathrm{int}}. \end{cases} \tag{3.18}$$

Remark 3.3 *Following on from Remark 3.2, if we let σ_i' be the canonical edge $\hat{\tau} = (0,1)$ and $\{\hat{\phi}_{i,j}\}_{j=1}^{2}$ denote the set of linear basis functions on $\hat{\tau}$ given by*

$$\hat{\phi}_{i,1}|_{\hat{\tau}} = 1 - \xi_1, \quad \hat{\phi}_{i,2}|_{\hat{\tau}} = \xi_1;$$

then, the corresponding $L^2(\hat{\tau})$-dual basis is defined by

$$\hat{\psi}_{i,1}' = -6\xi_1 + 4, \quad \hat{\psi}_{i,2}' = 6\xi_1 - 2.$$

We shall now proceed to prove the following results for the quasi-interpolation operator $\tilde{\mathcal{I}}'$.

Lemma 3.7 *The quasi-interpolation operator $\tilde{\mathcal{I}}'$ defined in (3.17) is a projection from $W^{1,1}(\Omega)$ to $\mathcal{V}_h$, with the property that $W_0^{1,p}(\Omega)$, $p \geq 1$, is mapped to $\mathcal{V}_0^h$.*

Proof Let $v \in \mathcal{V}^h$; then using (3.10) and (3.16) gives, for each $\mathbf{x} \in \bar{\Omega}$,

$$\begin{aligned}
\tilde{\mathcal{I}}'v(\mathbf{x}) &= \sum_{i=1}^{L} \tilde{\mathcal{I}}'v(\mathbf{a}_i)\phi_i(\mathbf{x}) \\
&= \sum_{\mathbf{a}_i \in \mathcal{N}_h^{\text{int}}} \phi_i(\mathbf{x}) \int_{\sigma_i} \psi_i(\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y} + \sum_{\mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\text{int}}} \phi_i(\mathbf{x}) \int_{\sigma_i'} \psi_i'(\mathbf{y})v(\mathbf{y})\mathrm{d}s \\
&= \sum_{\mathbf{a}_i \in \mathcal{N}_h^{\text{int}}} \phi_i(\mathbf{x}) \left[\sum_{j=1}^{L} v(\mathbf{a}_j) \int_{\sigma_i} \psi_i(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}\mathbf{y} \right] \\
&\qquad + \sum_{\mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\text{int}}} \phi_i(\mathbf{x}) \left[\sum_{j=1}^{L} v(\mathbf{a}_j) \int_{\sigma_i'} \psi_i'(\mathbf{y})\phi_j(\mathbf{y})\mathrm{d}s \right] \\
&= \sum_{\mathbf{a}_i \in \mathcal{N}_h^{\text{int}}} v(\mathbf{a}_i)\phi_i(\mathbf{x}) + \sum_{\mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\text{int}}} v(\mathbf{a}_i)\phi_i(\mathbf{x}) \\
&\equiv v(\mathbf{x}).
\end{aligned}$$

Hence, $\tilde{\mathcal{I}}'$ is a projection from $W^{1,1}(\Omega)$ to $\mathcal{V}^h$.

If $v \in W_0^{1,p}(\Omega)$, $p \geq 1$, then $v|_{\partial\Omega} = 0$ in $L^1(\partial\Omega)$, i.e. $\|v\|_{L^1(\partial\Omega)} = 0$. By choosing $\sigma_i' \subset \partial\Omega$ when $\mathbf{a}_i \in \partial\Omega$ in the definition of $\tilde{\mathcal{I}}'$, we deduce that

$$v \in W_0^{1,p}(\Omega) \Rightarrow \tilde{\mathcal{I}}'v(\mathbf{a}_i) = 0 \quad \forall \mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\text{int}}. \tag{3.19}$$

□

Lemma 3.8 *Let* $\mathbf{a}_i \in \mathcal{N}_h \backslash \mathcal{N}_h^{\text{int}}$, *then*

$$\|\psi_i'\|_{L^\infty(\sigma_i')} \leq \tilde{C}_3 h_\kappa^{-1},$$

where $\tilde{C}_3 = \|\hat{\psi}_i'\|_{L^\infty(\hat{\tau})}/(2c_1)$, $\kappa \in \mathcal{T}$ *is an element such that* $\sigma_i' \subset \partial\kappa$ *and* $\hat{\tau}$ *is the canonical edge (0,1).*

Proof This proof is omitted, since it is essentially the same as the proof presented above for Lemma 3.4. □

3.4 Approximation theory

In this section we shall derive optimal order approximation results for the quasi-interpolation operator $\tilde{\mathcal{I}}$. However, before we proceed we first need the following results.

Lemma 3.9 *Let* $\omega \subset \mathbf{R}^d$ *be an open bounded set, which is star–shaped with respect to every point in a set of positive measure* $B \subset \omega$. *Let* $p \in (1, \infty)$ *and*

$0 \le m \le k$. If $v \in W^{k,p}(\omega)$, then there is a constant $C^{\mathrm{BH}} = C^{\mathrm{BH}}(k,m,p,d)$ such that

$$|v - Q_k v|_{W^{m,p}(\omega)} \le C^{\mathrm{BH}}(diam(\omega))^{k-m}|v|_{W^{k,p}(\omega)},$$

where

$$Q_k v(\mathbf{y}) = \frac{1}{meas(B)}\int_B \sum_{|\alpha|<k} D^\alpha v(\mathbf{x})\frac{(\mathbf{y}-\mathbf{x})^\alpha}{\alpha!}\mathrm{d}\mathbf{x},$$

$$C^{\mathrm{BH}} = (card\{\alpha : |\alpha| = m\})\frac{(diam(\omega))^{d/p}}{(meas(B))^{1/p}}\frac{(k-m)p}{(p-1)d^{1/p}}\,\omega_d^{1/p}\left(\sum_{|\beta|=k-m}(\beta!)^{-q}\right)^{1/q}$$

for $0 \le m < k$ and $C^{\mathrm{BH}} = 1$ for $m = k$, ω_d denotes the measure of the unit sphere in $\mathbf{R}^d$ and q is the exponent conjugate to p, i.e. $1/p + 1/q = 1$.

Proof This is a slight modification of the result proved by Durán [9]; here, we have adopted a different definition for the $W^{m,p}(\omega)$ semi-norm. □

Remark 3.4 *We note that in the following, we shall apply Lemma 3.9 over the domain $S_\kappa \subset \mathbf{R}^2$, in which case $B = B_{S_\kappa}$. Moreover, using (3.2b) and (3.2c) we may define the constant C^{BH} for $0 \le m < k$, as follows*

$$C^{\mathrm{BH}} = (card\{\alpha : |\alpha| = m\})\frac{c_6^{2/p}}{(\pi c_5^2)^{1/p}}\frac{(k-m)p}{(p-1)2^{1/p}}\,\omega_2^{1/p}\left(\sum_{|\beta|=k-m}(\beta!)^{-q}\right)^{1/q}.$$

Lemma 3.10 *Let $v \in W^{1,p}(\Omega)$, $p \in [1,\infty]$ and let $\kappa \in \mathcal{T}$, where $\mathcal{T}$ satisfies conditions (3.1). Then,*

$$|\tilde{\mathcal{I}}'v - \tilde{\mathcal{I}}v|_{W^{m,p}(\kappa)} \le \tilde{C}_4\left[h_\kappa^{1-m}|v-w|_{W^{1,p}(S_\kappa)} + c_4 h_\kappa^{-m}\|v-w\|_{L^p(S_\kappa)}\right] \quad (3.20)$$

$\forall w \in W_0^{1,p}(\Omega)$ and $m \in \mathbf{N}_0$; where

$$\tilde{C}_4 = C^{\mathrm{map}}C^t(1/2)^{1-1/p}c_1^{-(m+1)}c_4c_3^{-2(1-1/p)}\beta_{\max} \times \max_{1\le i\le n_2}\|\hat{\psi}_i'\|_{L^\infty(\hat{\tau})}\max_{1\le i\le\beta_{\max}}|\hat{\phi}_i|_{W^{m,p}(\hat{\kappa})},$$

$$\beta_{\max} = \sup_{\kappa\in\mathcal{T}}\{card\{\mathbf{a}_i \in \kappa : \mathbf{a}_i \in \partial\Omega\}\},$$

and $C^t = 4\sqrt{2}/c_1$.

Proof This is based on a proof presented by Clément [7]. As in the proof of Lemma 3.5, we assume that the set $\{\mathbf{a}_i : 1 \le i \le n_1\}$ comprises the nodal points for the element κ. Further, we assume that $\mathbf{a}_i \in \partial\Omega$ for $i = 1, 2, \ldots, \beta$ and for $\beta < i \le n_1$, $\mathbf{a}_i \notin \partial\Omega$. Here, we assume that $\beta > 0$, since otherwise $\tilde{\mathcal{I}}v|_\kappa \equiv \tilde{\mathcal{I}}'v|_\kappa$.

Using the definitions of $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}'$ (see (3.11) and (3.17), respectively), the triangle inequality and Lemma 3.1, we have

$$\begin{aligned} |\tilde{\mathcal{I}}'v - \tilde{\mathcal{I}}v|_{W^{m,p}(\kappa)} &= \left|\sum_{i=1}^{\beta} \tilde{\mathcal{I}}'v(\mathbf{a}_i)\phi_i\right|_{W^{m,p}(\kappa)} \\ &\le \sum_{i=1}^{\beta} |\tilde{\mathcal{I}}'v(\mathbf{a}_i)||\phi_i|_{W^{m,p}(\kappa)} \\ &\le C^{\mathrm{map}} c_1^{-m} \max_{1\le i\le\beta} |\hat{\phi}_i|_{W^{m,p}(\hat{\kappa})}\, h_\kappa^{-m+2/p} \sum_{i=1}^{\beta} |\tilde{\mathcal{I}}'v(\mathbf{a}_i)|. \end{aligned} \tag{3.21}$$

Using (3.18), Hölder's inequality and Lemma 3.8, we have

$$\begin{aligned} |\tilde{\mathcal{I}}'v(\mathbf{a}_i)| &= \left|\int_{\sigma_i'} \psi_i'(\mathbf{y})v(\mathbf{y})\mathrm{d}s\right| \\ &\le \|\psi_i'\|_{L^\infty(\sigma_i')}\|v\|_{L^1(\sigma_i')} \\ &\le \tilde{C}_3 h_{\tilde{\kappa}}^{-1}\|v\|_{L^1(\sigma_i')}, \end{aligned}$$

where $\tilde{\kappa}$ is an element such that $\sigma_i' \subset \partial\tilde{\kappa}$; in the following, we label this element κ_i.

Let $w \in W_0^{1,p}(\Omega)$, then

$$\|w\|_{L^1(\sigma_i')} = 0 \Rightarrow \|v\|_{L^1(\sigma_i')} = \|v - w\|_{L^1(\sigma_i')}.$$

Hence, using Lemma 3.2 and Hölder's inequality, we have

$$\begin{aligned} |\tilde{\mathcal{I}}'v(\mathbf{a}_i)| &\le \tilde{C}_3 h_{\kappa_i}^{-1}\|v - w\|_{L^1(\sigma_i')} \\ &\le \tilde{C}_3 C^t h_{\kappa_i}^{-1}\left[\int_{\kappa_i} |\boldsymbol{\nabla}(v-w)|\mathrm{d}\mathbf{x} + h_{\kappa_i}^{-1}\int_{\kappa_i} |v-w|\mathrm{d}\mathbf{x}\right] \\ &\le 2\tilde{C}_3 C^t (1/2)^{1-1/p} h_{\kappa_i}^{1-2/p}\left[|v-w|_{W^{1,p}(\kappa_i)} + h_{\kappa_i}^{-1}\|v-w\|_{L^p(\kappa_i)}\right] \end{aligned} \tag{3.22}$$

$\forall w \in W_0^{1,p}(\Omega)$. Hence, substituting (3.22) into (3.21) and using (3.2a) gives

$$\begin{aligned} |\tilde{\mathcal{I}}'v - \tilde{\mathcal{I}}v|_{W^{m,p}(\kappa)} &\le 2C^{\mathrm{map}}\tilde{C}_3' C^t (1/2)^{1-1/p} c_1^{-m} \max_{1\le i\le\beta} |\hat{\phi}_i|_{W^{m,p}(\hat{\kappa})}\, h_\kappa^{-m+2/p} \\ &\qquad \times \sum_{i=1}^{\beta} h_{\kappa_i}^{1-2/p}\left[|v-w|_{W^{1,p}(\kappa_i)} + h_{\kappa_i}^{-1}\|v-w\|_{L^p(\kappa_i)}\right] \\ &\le 2C^{\mathrm{map}}\tilde{C}_3' C^t (1/2)^{1-1/p} c_1^{-m} c_4 c_3^{-2(1-1/p)} \beta \max_{1\le i\le\beta} |\hat{\phi}_i|_{W^{m,p}(\hat{\kappa})}\, h_\kappa^{1-m} \\ &\qquad \times \left[|v-w|_{W^{1,p}(S_\kappa)} + c_4 h_\kappa^{-1}\|v-w\|_{L^p(S_\kappa)}\right], \end{aligned}$$

where $\tilde{C}_3' = \max_{1\le i\le n_2} \|\hat{\psi}_i'\|_{L^\infty(\hat{\tau})}/(2c_1)$, as required. □

We now give the following approximation theorem for the quasi-interpolation operator $\tilde{\mathcal{I}}$.

Theorem 3.11 *Let $v \in W_0^{1,p}(\Omega) \cap W^{k,p}(\Omega)$, for $p \in (1,\infty)$ and $1 \leq k \leq r+1$, and let $\mathcal{T}$ satisfy conditions (3.1). Then,*

$$\left(\sum_{\kappa \in \mathcal{T}} h_\kappa^{p(m-k)} |v - \tilde{\mathcal{I}} v|^p_{W^{m,p}(\kappa)} \right)^{1/p} \leq C^{\tilde{i}} |v|_{W^{k,p}(\Omega)}, \quad 0 \leq m \leq k,$$

where

$$C^{\tilde{i}} = c_6^k c_7^{1/p} \left(C_1^{\mathrm{BH}} c_6^{-m} + \tilde{C}_2 C_2^{\mathrm{BH}} + \tilde{C}_4 C_3^{\mathrm{BH}} c_6^{-1} + \tilde{C}_4 C_2^{\mathrm{BH}} c_4 \right),$$

$C_1^{\mathrm{BH}} = C^{\mathrm{BH}}(k,m,p,2)$, $C_2^{\mathrm{BH}} = C^{\mathrm{BH}}(k,0,p,2)$ *and* $C_3^{\mathrm{BH}} = C^{\mathrm{BH}}(k,1,p,2)$.

Proof Let $\kappa \in \mathcal{T}$, where $\mathcal{T}$ satisfies conditions (3.1), and let $Q_k v \in \mathbf{P}_{k-1}(S_\kappa)$ be as defined in Lemma 3.9, where $1 \leq k \leq r+1$. Then, using Lemma 3.7 and the triangle inequality, we have for $0 \leq m \leq k$,

$$\begin{aligned} |v - \tilde{\mathcal{I}} v|_{W^{m,p}(\kappa)} &\leq |v - Q_k v|_{W^{m,p}(\kappa)} + |\tilde{\mathcal{I}}(Q_k v - v)|_{W^{m,p}(\kappa)} + |\tilde{\mathcal{I}}' Q_k v - \tilde{\mathcal{I}} Q_k v|_{W^{m,p}(\kappa)} \\ &\equiv \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{3.23}$$

Let us first consider term I: using Lemma 3.9 and (3.2c) gives

$$\begin{aligned} \mathrm{I} &\leq |v - Q_k v|_{W^{m,p}(S_\kappa)} \\ &\leq C_1^{\mathrm{BH}} c_6^{k-m} h_\kappa^{k-m} |v|_{W^{k,p}(S_\kappa)}, \quad 0 \leq m \leq k. \end{aligned} \tag{3.24}$$

Next, consider term II: using Lemma 3.5, Lemma 3.9 and (3.2c) gives

$$\begin{aligned} \mathrm{II} &\leq \tilde{C}_2 h_\kappa^{-m} \|Q_k v - v\|_{L^p(S_\kappa)} \\ &\leq \tilde{C}_2 C_2^{\mathrm{BH}} c_6^k h_\kappa^{k-m} |v|_{W^{k,p}(S_\kappa)}, \quad 0 \leq m \leq k. \end{aligned} \tag{3.25}$$

Finally, consider term III: using Lemma 3.10, we have

$$\mathrm{III} \leq \tilde{C}_4 h_\kappa^{1-m} |Q_k v - w|_{W^{1,p}(S_\kappa)} + \tilde{C}_4 c_4 h_\kappa^{-m} \|Q_k v - w\|_{L^p(S_\kappa)} \tag{3.26}$$

$\forall w \in W_0^{1,p}(\Omega)$, $0 \leq m \leq k$. Choosing $w = v$ in (3.26) and applying Lemma 3.9 (along with (3.2c)) to each of the terms on the right–hand side yields

$$\mathrm{III} \leq \left(\tilde{C}_4 C_3^{\mathrm{BH}} c_6^{k-1} + \tilde{C}_4 C_2^{\mathrm{BH}} c_4 c_6^k \right) h_\kappa^{k-m} |v|_{W^{k,p}(S_\kappa)}, \quad 0 \leq m \leq k. \tag{3.27}$$

If we now substitute (3.24), (3.25) and (3.27) into (3.23), then proceeding as in the proof of Corollary 3.6 gives the desired result. □

Corollary 3.12 *Let $v \in W_0^{1,p}(\Omega) \cap W^{k,p}(\Omega)$, for $p \in (1,\infty)$ and $1 \leq k \leq r+1$, and let $\mathcal{T}$ satisfy conditions (3.1). Then,*

$$|\tilde{\mathcal{I}} v|_{W^{k,p}(\Omega)} \leq C_2^{\tilde{i}} |v|_{W^{k,p}(\Omega)},$$

where $C_2^{\tilde{i}} = 1 + C^{\tilde{i}}$.

Proof This result follows by letting $m = k$ in Theorem 3.11 and applying the triangle inequality. $\square$

Remark 3.5 *We note that the above results can be extended to include both domains $\Omega \subset \mathbf{R}^d$, $d \geq 2$, and finite element spaces of the form $\{v \in \mathcal{V}^h : v|_\Gamma = 0\}$, where $\Gamma \subset \partial\Omega$, provided that the triangulation (subdivision) of Ω matches Γ appropriately.*

4 Model problem and discretisation

Given a final time $T > 0$, we shall consider the following unsteady convection–diffusion problem: given that $f \in L^2(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, find u such that:

$$
\begin{aligned}
u_t + \mathbf{a} \cdot \boldsymbol{\nabla} u - \epsilon \Delta u &= f, && \mathbf{x} \in \Omega, \quad t \in (0, T], && (4.1a)\\
u(\cdot, t) &= 0, && \mathbf{x} \in \partial\Omega, \ t \in [0, T], && (4.1b)\\
u(\mathbf{x}, 0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, && (4.1c)
\end{aligned}
$$

where Ω is a bounded convex polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$. Further, we assume that the diffusion coefficient $\epsilon > 0$, the velocity vector $\mathbf{a}$ lies in the function space $C([0, T]; C_0^1(\bar{\Omega})^2)$ and that $\mathbf{a}$ is incompressible, i.e.

$$\boldsymbol{\nabla} \cdot \mathbf{a} = 0 \quad \forall \mathbf{x} \in \Omega, \ t \in (0, T).$$

In this section we shall formulate the discontinuity capturing Lagrange–Galerkin method for problem (4.1). However, we first need to introduce the following notation.

Let $0 = t^0 < t^1 < \ldots < t^M < t^{M+1} = T$ be a subdivision (not necessarily uniform) of $[0, T]$, with corresponding time intervals $I^n = (t^{n-1}, t^n]$ and time steps $k_n = t^n - t^{n-1}$. For each n, $0 \leq n \leq M + 1$, let $\mathcal{T}^n = \{\kappa\}$ be an admissible subdivision of Ω into closed triangles κ, with corresponding mesh function h_n satisfying the conditions

$$
\begin{aligned}
c_1 h_\kappa^2 &\leq \operatorname{meas}(\kappa) && \forall \kappa \in \mathcal{T}^n, && (4.2a)\\
c_2 h_\kappa \leq h_n(\mathbf{x}) &\leq h_\kappa && \forall \mathbf{x} \in \kappa \ \ \forall \kappa \in \mathcal{T}^n, && (4.2b)
\end{aligned}
$$

where $h_\kappa = \operatorname{diam}(\kappa)$ and c_1 and c_2 are positive constants independent of h_n. Further, h is assumed to be the global mesh function defined by $h(\mathbf{x}, t) = h_n(\mathbf{x})$, for $(\mathbf{x}, t) \in \Omega \times I^n$ and we define the corresponding time step function $k = k(t)$ by $k(t) = k_n, \ t \in I^n$.

For some $n \in \mathbf{N}_0$, we associate with $\mathcal{T}^n$ the set $E^n = \{\tau\}$ consisting of those line segments in $\mathbf{R}^2$ which appear as an edge of some $\kappa \in \mathcal{T}^n$. We also denote by E_i^n, those τ in E^n which are interior to $\bar{\Omega}$ (i.e. not part of $\partial\Omega$).

Let $S^n = \Omega \times I^n$; for $r \in \mathbf{N}$ and $s \in \mathbf{N}_0$ we define the following finite element spaces

$$S^{h_n} = \{v \in C_0(\Omega) : v|_\kappa \in \mathbf{P}_r(\kappa)\ \forall\kappa \in \mathcal{T}^n\},$$
$$V^{h_n} = \{v : v(\mathbf{x},t)|_{S^n} = \sum_{j=0}^{s} t^j v_j,\ v_j \in S^{h_n}\},$$
$$V^h = \{v : v(\mathbf{x},t)|_{S^n} \in V^{h_n},\ n = 1,\dots,M+1\}.$$

As in [15, 16] we shall let $s = 0$; however, here it will not be necessary to restrict the size of r. We note that if $v \in V^h$, then v is continuous in space at any time, but may be discontinuous in time at the discrete time levels t^n. To account for this, we introduce the notation

$$v_\pm^n := \lim_{s\to 0^+} v(t^n \pm s) \quad \text{and} \quad [v^n] := v_+^n - v_-^n.$$

In addition, for each edge $\tau \in E_i^n$, let $\mathbf{n}_\tau$ denote the unit normal to τ in the outward direction to κ, and define for $v \in S^{h_n}$ (for some $n \in \mathbf{N}_0$),

$$\left[\frac{\partial v}{\partial \mathbf{n}_\tau}\right] = \lim_{s\to 0^+} (\boldsymbol{\nabla} v(\mathbf{x}+s\mathbf{n}_\tau) - \boldsymbol{\nabla} v(\mathbf{x}-s\mathbf{n}_\tau)) \cdot \mathbf{n}_\tau, \quad \mathbf{x} \in \tau,$$

that is, $[\partial v/\partial \mathbf{n}_\tau]$ is the jump across τ in the normal component of $\boldsymbol{\nabla} v$. Finally, we introduce the discrete second derivatives

$$D_h^2 v|_\kappa = \sum_{\tau\in\partial\kappa\cap E_i^n} \left\|\left[\frac{\partial v}{\partial \mathbf{n}_\tau}\right]\right\|_{L^\infty(\tau)} \frac{1}{h_\kappa}, \quad \kappa \in \mathcal{T}^n, \tag{4.3a}$$

$$\overset{\circ}{D}{}_h^2 v|_\kappa = \sum_{\tau\in\partial\kappa\cap E^n} \left\|\left[\frac{\partial v}{\partial \mathbf{n}_\tau}\right]\right\|_{L^\infty(\tau)} \frac{1}{h_\kappa}, \quad \kappa \in \mathcal{T}^n, \tag{4.3b}$$

and in (4.3b) we define $\boldsymbol{\nabla} v(\mathbf{x}+s\mathbf{n}_\tau) = 0$ if $\tau \in \partial\Omega$.

To construct the Lagrange–Galerkin method, we first define the particle trajectories (or characteristics) associated with problem (4.1): the path of a particle located at position $\mathbf{x} \in \bar{\Omega}$ at time $s \in [0,T]$ is defined as the solution of the initial value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(\mathbf{x},s;t) &= \mathbf{a}(\mathbf{X}(\mathbf{x},s;t),t), \\ \mathbf{X}(\mathbf{x},s;s) &= \mathbf{x}. \end{aligned}$$

The Lagrange–Galerkin method makes use of the *material derivative* $D_t u$, which is defined, for u smooth enough, as follows:

$$D_t u(\mathbf{x},t) := \frac{\mathrm{d}}{\mathrm{d}t} u(\mathbf{X}(\mathbf{x},s;t),t)\,|_{s=t}$$

$$= \frac{\partial}{\partial t}u(\mathbf{x},t) + \mathbf{a}(\mathbf{x},t)\cdot\boldsymbol{\nabla}u(\mathbf{x},t) \quad \forall \mathbf{x}\in\Omega,\ t\in(0,T).$$

Hence, using the material derivative, equation (4.1a) may be rewritten in the following (weak) form: find $u(t)\in V$, such that

$$\begin{aligned}(D_t u(\cdot,t),v) + (\epsilon\boldsymbol{\nabla}u(\cdot,t),\boldsymbol{\nabla}v) &= (f(\cdot,t),v) \quad \forall v\in V,\\ (u(\cdot,0),v) &= (u_0(\cdot),v) \quad \forall v\in V,\end{aligned}$$

where $V = H_0^1(\Omega)$ and, for the sake of simplicity, we shall assume that $f \in C([0,T];L^2(\Omega))$. The Lagrange–Galerkin time–discretisation involves approximating the material derivative by a divided difference operator. The simplest appropriate discretisation is the backward Euler method, giving for $n = 0,\ldots,M$:

$$\begin{aligned}\left(\frac{u(\cdot,t^{n+1}) - u(\mathbf{X}(\cdot,t^{n+1};t^n),t^n)}{k_{n+1}},v\right)&\\ +(\epsilon\boldsymbol{\nabla}u(\cdot,t^{n+1}),\boldsymbol{\nabla}v) &\approx (f(\cdot,t^{n+1}),v) \quad \forall v\in V, \quad (4.6a)\\ (u(\cdot,0),v) &= (u_0(\cdot),v) \quad \forall v\in V. \quad (4.6b)\end{aligned}$$

If we now define u_h^n to be the Galerkin finite element approximation to $u(\cdot,t^n)$ at time t^n; then applying the finite element method to (4.6) yields the Lagrange–Galerkin discretisation of (4.1) as follows: find $u_h^{n+1}\in S^{h_{n+1}}$ for $0\le n\le M$ such that

$$\begin{aligned}\left(\frac{u_h^{n+1} - u_h^n(\mathbf{X}(\cdot,t^{n+1};t^n))}{k_{n+1}},v\right)&\\ +(\epsilon\boldsymbol{\nabla}u_h^{n+1},\boldsymbol{\nabla}v) &= (f^{n+1},v) \quad \forall v\in S^{h_{n+1}}, \quad (4.7a)\\ (u_h^0,v) &= (u_0,v) \quad \forall v\in S^{h_0}, \quad (4.7b)\end{aligned}$$

where $f^{n+1}(\cdot) := f(\cdot,t^{n+1})$. This is the same approach as that used by Bercovier *et al.* [2, 3, 13], Douglas & Russell [8], Lesaint [17], Pironneau [18] and Süli [21, 22].

Further, integrating (4.7) with respect to t over I^{n+1} and adding an artificial diffusion term, we obtain the discontinuity capturing Lagrange–Galerkin method: find u_h such that, for $n = 0,1,\ldots,M$, $u_h|_{S^{n+1}}\in V^{h_{n+1}}$ and satisfies

$$\begin{aligned}(D_t^h u_h,v)_{n+1} + ((\epsilon+\hat\epsilon)\boldsymbol{\nabla}u_h,\boldsymbol{\nabla}v)_{n+1} &= (\bar f,v)_{n+1} \quad \forall v\in V^{h_{n+1}}, \quad (4.8a)\\ (u_{h-}^0,v) &= (u_0,v) \quad \forall v\in S^{h_0}, \quad (4.8b)\end{aligned}$$

where

$$D_t^h u_h|_{S^{n+1}} := (u_{h-}(\mathbf{X}(\mathbf{x},t^{n+1};t^{n+1}),t^{n+1}) - u_{h-}(\mathbf{X}(\mathbf{x},t^{n+1};t^n),t^n))/k_{n+1},$$

$\bar{f}|_{S^{n+1}} := f(\cdot, t^{n+1})$, and for $v, w \in L^2(I^{n+1}; L^2(\Omega))$, we have used the notation

$$(v, w)_{n+1} := \int_{t^n}^{t^{n+1}} (v, w)\mathrm{d}t.$$

Furthermore, $\hat{\epsilon} = F_h(u_h, h)$ and, in the case $r = 1$, $F_h|_\kappa$ for $\kappa \in \mathcal{T}^{n+1}$ is defined as follows:

$$F_h(u_h, h)|_\kappa = \max(C_1^{\hat{\epsilon}} h^3 \overset{\circ}{D}{}_h^2 \breve{u}_h^n - \epsilon, C_2^{\hat{\epsilon}} h^3 \overset{\circ}{D}{}_h^2 u_h^n - \epsilon, 0),$$

where $\breve{u}_h^n = u_h(\mathbf{X}(\mathbf{x}, t^{n+1}; t^n), t^n)$ and $C_1^{\hat{\epsilon}}$, $C_2^{\hat{\epsilon}}$ are positive constants, cf. [16]; for $r > 1$ we have not studied the choice of a suitable artificial diffusion model $F_h(u_h, h)$.

5 A posteriori error analysis

In this section we shall derive an *a posteriori* estimate for the error $e = u - u_h$, in the $L^2(0, T; L^2(\Omega))$ norm, where u and u_h are the solutions of (4.1) and (4.8), respectively. However, before we proceed, we first introduce the following notation: for $v, w \in L^2(0, T; L^2(\Omega))$ we define

$$\begin{aligned} (v, w)_Q &:= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (v, w)\mathrm{d}t, \\ \|v\|_Q &:= \left((v, v)_Q\right)^{1/2}, \end{aligned}$$

where $Q := \Omega \times (0, T)$.

5.1 Error representation

The (backward) dual problem takes the form: find ϕ such that

$$\begin{aligned} -\phi_t - \boldsymbol{\nabla} \cdot (\mathbf{a}\phi) - \epsilon\Delta\phi &= e \equiv u - u_h, \quad \mathbf{x} \in \Omega, \quad t \in [0, T), && (5.1a) \\ \phi(\cdot, t) &= 0, \quad \mathbf{x} \in \partial\Omega, \ t \in [0, T], && (5.1b) \\ \phi(\mathbf{x}, T) &= 0, \quad \mathbf{x} \in \Omega. && (5.1c) \end{aligned}$$

We shall now proceed to prove the following error representation: multiplying (5.1a) by e and integrating by parts in both space and time, we get

$$\|e\|_Q^2 = \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (e, -\phi_t - \boldsymbol{\nabla} \cdot (\mathbf{a}\phi) - \epsilon\Delta\phi)\mathrm{d}t$$

$$
\begin{aligned}
&= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (e_t + \mathbf{a}\cdot\boldsymbol{\nabla} e, \phi)\mathrm{d}t + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (\epsilon\boldsymbol{\nabla} e, \boldsymbol{\nabla}\phi)\mathrm{d}t \\
&\quad - \sum_{n=0}^{M} ([u_h^n], \phi(t^n)) + (u_0 - u_{h-}^0, \phi(0)) \\
&= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (f - \mathbf{a}\cdot\boldsymbol{\nabla} u_h, \phi)\mathrm{d}t - \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (\epsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}\phi)\mathrm{d}t \\
&\quad - \sum_{n=0}^{M} ([u_h^n], \phi(t^n)) + (u_0 - u_{h-}^0, \phi(0)),
\end{aligned}
$$

where we have used (4.1a). If we now let $\phi_h \in V^h$, then using (4.8) we have

$$
\begin{aligned}
\|e\|_Q^2 &= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \sum_{\kappa\in\mathcal{T}^{n+1}} ([u_h^n]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h - \epsilon\Delta u_h - f, \phi_h - \phi)_\kappa \,\mathrm{d}t \\
&\quad + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa\in\mathcal{T}^{n+1}} (\epsilon\Delta u_h, \phi_h - \phi)_\kappa + (\epsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}(\phi_h - \phi)) \right) \mathrm{d}t \\
&\quad + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (D_t^h u_h - ([u_h^n]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h), \phi_h)\mathrm{d}t \\
&\quad + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} ([u_h^n]/k_{n+1}, \phi - \phi(t^n))\mathrm{d}t + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (f - \bar{f}, \phi_h)\mathrm{d}t \\
&\quad + (u_0 - u_{h-}^0, \phi(0)) + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (\hat{\epsilon}\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}\phi_h)\mathrm{d}t \\
&\equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} \quad \forall \phi_h \in V^h. \qquad (5.2)
\end{aligned}
$$

Remark 5.1 *We note that for $r \in \mathbf{N}$, we must include the term $\epsilon\Delta u_h$ in terms I and II of the error representation formula (5.2), since if $r > 1$ then $\Delta u_h|_\kappa$ may not be zero.*

5.2 Interpolation/projection estimates for the dual problem

We shall now choose $\phi_h \in V^h$ in (5.2) to be the quasi-interpolant of ϕ in space and the L^2-projection of ϕ in time, i.e. we first define the operators

$$
\begin{aligned}
\tilde{\mathcal{I}}_n &: L^1(\Omega) \to S^{h_n}, \\
\pi_n &: L^2(I^n) \to \mathbf{P}_0(I^n),
\end{aligned}
$$

in space and in time, by (3.11) and

$$
\int_{t^{n-1}}^{t^n} (\pi_n\phi - \phi)v\mathrm{d}t = 0 \quad \forall v \in \mathbf{P}_0(I^n),
$$

respectively. Then, we define (locally) $\phi_h|_{S^n} \in V^{h_n}$ by letting

$$\phi_h|_{S^n} = \tilde{\mathcal{I}}_n \pi_n \phi = \pi_n \tilde{\mathcal{I}}_n \phi \in V^{h_n},$$

where $\phi = \phi|_{S^n}$. Further, we introduce $\tilde{\mathcal{I}}$ and π by

$$(\tilde{\mathcal{I}}\phi)|_{S^n} = \tilde{\mathcal{I}}_n(\phi|_{S^n}), \tag{5.3a}$$
$$(\pi\phi)|_{S^n} = \pi_n(\phi|_{S^n}); \tag{5.3b}$$

then we let $\phi_h \in V^h$ be

$$\phi_h = \tilde{\mathcal{I}}\pi\phi = \pi\tilde{\mathcal{I}}\phi \in V^h. \tag{5.4}$$

Based on this definition of ϕ_h, we give the following result.

Lemma 5.1 *Suppose that $\mathcal{T}^n$ satisfies conditions (4.2) for $n = 0, 1, \ldots, M+1$. If ϕ_h is defined by (5.4), then*

$$\|\phi_h\|_Q \leq C_1^{\tilde{i}} \|\phi\|_Q, \tag{5.5a}$$
$$\|\boldsymbol{\nabla}\phi_h\|_Q \leq C_2^{\tilde{i}} \|\boldsymbol{\nabla}\phi\|_Q. \tag{5.5b}$$

Proof The estimate (5.5a) follows from Corollary 3.6 and the properties of the projection operator π. Similarly, estimate (5.5b) follows from Corollary 3.12. □

We now give the following error estimates for the operators $\tilde{\mathcal{I}}$ and π in order to estimate $\phi - \phi_h = \phi - \tilde{\mathcal{I}}\pi\phi$. However, we first require the following result.

Lemma 5.2 *Suppose that Ω is a bounded convex polygonal domain in $\mathbf{R}^2$; then*

$$|v|_{H^2(\Omega)} \leq \|\Delta v\| \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{5.6}$$

Proof See Grisvard [12], Theorem 2.2.1 on p. 43; also, see [15], Lemma 4.2. □

Lemma 5.3 *Suppose that $n \in \mathbf{N}_0$ and $\mathcal{T}^n$ satisfies conditions (4.2). If $v \in H_0^1(\Omega) \cap H^2(\Omega)$, then there exists positive constants $C_3^{\tilde{i}}$ and $C_4^{\tilde{i}}$ such that*

$$\left(\sum_{\kappa \in \mathcal{T}^n} h_\kappa^{-4} \|v - \tilde{\mathcal{I}}_n v\|_{L^2(\kappa)}^2 \right)^{1/2} \leq C_3^{\tilde{i}} |v|_{H^2(\Omega)},$$
$$\left(\sum_{\kappa \in \mathcal{T}^n} h_\kappa^{-2} |v - \tilde{\mathcal{I}}_n v|_{H^1(\kappa)}^2 \right)^{1/2} \leq C_4^{\tilde{i}} |v|_{H^2(\Omega)},$$

respectively.

Proof These bounds follow from Theorem 3.11. □

Lemma 5.4 *Suppose that* $R \in L^2(0,T;L^2(\Omega))$ *and* $v \in V^h$ *then*

$$|(R, \tilde{\mathcal{I}}\phi - \phi)_Q| \le C_1^{\tilde{p}} \|h^2 R\|_Q \|\Delta\phi\|_Q, \tag{5.7a}$$

$$\left| \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa \in \mathcal{T}^{n+1}} (\epsilon \Delta v, \tilde{\mathcal{I}}_{n+1}\phi - \phi)_\kappa + (\epsilon \boldsymbol{\nabla} v, \boldsymbol{\nabla}(\tilde{\mathcal{I}}_{n+1}\phi - \phi)) \right) \mathrm{d}t \right| \\ \le C_2^{\tilde{p}} \|h^2 D_h^2 v\|_Q \|\epsilon \Delta\phi\|_Q, \tag{5.7b}$$

where $C_1^{\tilde{p}} = C_3^{\tilde{i}} c_2^{-2}$, $C_2^{\tilde{p}} = C^t(C_3^{\tilde{i}} c_2^{-1} + C_4^{\tilde{i}})/(2c_2^2)$ *and* $C^t = 4\sqrt{2}/c_1$.

Proof First, we consider (5.7a): using the Cauchy–Schwarz inequality, we get

$$|(R, \tilde{\mathcal{I}}\phi - \phi)_Q| \le \|h^2 R\|_Q \|h^{-2}(\tilde{\mathcal{I}}\phi - \phi)\|_Q.$$

By definition, we have that

$$\|h^{-2}(\tilde{\mathcal{I}}\phi - \phi)\|_Q = \left(\sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \|h_{n+1}^{-2}(\tilde{\mathcal{I}}_{n+1}\phi - \phi)\|^2 \mathrm{d}t \right)^{1/2}. \tag{5.8}$$

Using (4.2b), Lemma 5.3 and Lemma 5.2, we have

$$\begin{aligned} \|h_{n+1}^{-2}(\tilde{\mathcal{I}}_{n+1}\phi - \phi)\|^2 &\le \sum_{\kappa \in \mathcal{T}^{n+1}} c_2^{-4} h_\kappa^{-4} \|\tilde{\mathcal{I}}_{n+1}\phi - \phi\|_{L^2(\kappa)}^2 \\ &\le (C_3^{\tilde{i}})^2 c_2^{-4} |\phi|_{H^2(\Omega)}^2 \\ &\le (C_3^{\tilde{i}})^2 c_2^{-4} \|\Delta\phi\|^2. \end{aligned} \tag{5.9}$$

Substituting (5.9) into (5.8) gives the desired result.

Next, we consider (5.7b): first, let $\mathcal{A}$ denote the left–hand side of (5.7b) inside the modulus signs and let $\rho = \tilde{\mathcal{I}}\phi - \phi$. Then, by integrating by parts in space, we have

$$\begin{aligned} \mathcal{A} &= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa \in \mathcal{T}^{n+1}} (\epsilon \Delta v, \rho)_\kappa + \sum_{\kappa \in \mathcal{T}^{n+1}} \left(\int_{\partial\kappa} \epsilon \frac{\partial v}{\partial \mathbf{n}_\kappa} \rho \mathrm{d}s - (\epsilon \Delta v, \rho)_\kappa \right) \right) \mathrm{d}t \\ &= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa \in \mathcal{T}^{n+1}} \left(\sum_{\tau \in \partial\kappa \cap E_i^{n+1}} \frac{\epsilon}{2} \int_\tau \left[\frac{\partial v}{\partial \mathbf{n}_\tau} \right] \rho \mathrm{d}s \right) \right) \mathrm{d}t. \end{aligned}$$

Using Hölder's inequality and the trace inequality (cf. Lemma 3.2), we have

$$\begin{aligned} |\mathcal{A}| &\le \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa \in \mathcal{T}^{n+1}} \left(\sum_{\tau \in \partial\kappa \cap E_i^{n+1}} \frac{\epsilon}{2} \left\| \left[\frac{\partial v}{\partial \mathbf{n}_\tau} \right] \right\|_{L^\infty(\tau)} \int_\tau |\rho| \mathrm{d}s \right) \right) \mathrm{d}t \\ &\le \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa \in \mathcal{T}^{n+1}} C^t \frac{\epsilon}{2} D_h^2 v \int_\kappa (h_\kappa |\boldsymbol{\nabla}\rho| + |\rho|) \, \mathrm{d}\mathbf{x} \right) \mathrm{d}t, \end{aligned}$$

where we have used the above notation. Further, using (4.2b) and the Cauchy–Schwarz inequality we have

$$|\mathcal{A}| \le \frac{1}{2} C^t c_2^{-1} \|h^2 D_h^2 v\|_Q \|\epsilon(h^{-1}|\nabla\rho| + h^{-2}|\rho|)\|_Q. \tag{5.10}$$

Let us now consider the second term on the right–hand side of (5.10): using the triangle inequality, we have

$$\begin{aligned}\|\epsilon(h^{-1}|\nabla\rho| + h^{-2}|\rho|)\|_Q &\le \|\epsilon h^{-1}\nabla\rho\|_Q + \|\epsilon h^{-2}\rho\|_Q \\ &\equiv \mathrm{I} + \mathrm{II}.\end{aligned}$$

In the first part of this lemma, we have already proved that

$$\mathrm{II} \le C_1^{\tilde{p}} \|\epsilon\Delta\phi\|_Q. \tag{5.11}$$

Let us now consider term I: by definition, we have

$$\|\epsilon h^{-1}\nabla\rho\|_Q = \epsilon \left(\sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \|h_{n+1}^{-1}\nabla\rho\|^2 \mathrm{d}t \right)^{1/2}. \tag{5.12}$$

Using (4.2b), Lemma 5.3 and Lemma 5.2, we have

$$\begin{aligned}\|h_{n+1}^{-1}\nabla\rho\|^2 &\le \sum_{\kappa\in\mathcal{T}^{n+1}} c_2^{-2} h_\kappa^{-2} \|\nabla(\tilde{\mathcal{I}}_{n+1}\phi - \phi)\|_{L^2(\kappa)}^2 \\ &\le (C_4^{\tilde{i}})^2 c_2^{-2} |\phi|_{H^2(\Omega)}^2 \\ &\le (C_4^{\tilde{i}})^2 c_2^{-2} \|\Delta\phi\|^2.\end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.12) and using (5.11) gives

$$\|\epsilon(h^{-1}|\nabla\rho| + h^{-2}|\rho|)\|_Q \le \left(C_1^{\tilde{p}} + C_4^{\tilde{i}} c_2^{-1} \right) \|\epsilon\Delta\phi\|_Q,$$

which proves the lemma. □

Lemma 5.5 *Suppose that $R \in L^2(0,T;L^2(\Omega))$ and $v \in V^h$ then*

$$\begin{aligned}|(R, \tilde{\mathcal{I}}(\pi\phi - \phi))_Q| &\le C_3^{\tilde{p}} \|kR\|_Q \|\phi_t\|_Q, \\ \left| \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} (R, \phi^n - \phi)\mathrm{d}t \right| &\le C_5^{i} \|kR\|_Q \|\phi_t\|_Q, \\ (\epsilon\nabla v, \nabla\tilde{\mathcal{I}}(\pi\phi - \phi))_Q &= 0, \\ \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa\in\mathcal{T}^{n+1}} (\epsilon\Delta v, \tilde{\mathcal{I}}_{n+1}(\pi_{n+1}\phi - \phi))_\kappa \right) \mathrm{d}t &= 0,\end{aligned}$$

where $C_3^{\tilde{p}} = C_1^{\tilde{i}} C_4^{i}$, $C_4^{i} = 1/\sqrt{2}$, $C_5^{i} = 1$ and $\phi^n = \phi(\mathbf{x}, t^n)$.

Proof This is a slight generalisation of Lemma 4.6 proved in [15]. □

5.3 Strong stability of the dual problem

This section summarises the strong stability estimates for the solution ϕ of the dual problem (5.1) in terms of the forcing function e (see [14, 15] for details).

Lemma 5.6 *Let ϕ be the solution of (5.1); then there is a constant $C_1^s = C_1^s(T, \Omega, \epsilon)$ such that*

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\epsilon^{1/2}\boldsymbol{\nabla}\phi\|^2_Q \leq C_1^s\|e\|^2_Q, \tag{5.15}$$

where $C_1^s = 2\min\{c_*^2/\epsilon, \mathrm{e}^T\}$ and $c_* = c_*(\Omega)$.

Lemma 5.7 *Let ϕ be the solution of (5.1); then there is a constant $C_2^s = C_2^s(T, \Omega, \mathbf{a}, \epsilon)$ such that*

$$\|\epsilon^{1/2}\boldsymbol{\nabla}\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\epsilon\Delta\phi\|^2_Q \leq C_2^s\|e\|^2_Q, \tag{5.16}$$

where $C_2^s = 4\min\left\{exp\left(2\|\mathbf{a}\|^2_{L^2(0,T;L^\infty(\Omega))}/\epsilon\right), \left(1 + C_1^s\|\mathbf{a}\|^2_{L^\infty(0,T;L^\infty(\Omega))}/\epsilon\right)\right\}$ and C_1^s is as defined in Lemma 5.6.

Lemma 5.8 *Let ϕ be the solution of (5.1); then there is a constant $C_3^s = C_3^s(T, \Omega, \mathbf{a}, \epsilon)$ such that*

$$\|\phi_t\|^2_Q + \|\epsilon^{1/2}\boldsymbol{\nabla}\phi(0)\|^2 \leq C_3^s\|e\|^2_Q, \tag{5.17}$$

where $C_3^s = \left(2 + 2\min\left\{C_1^s\|\mathbf{a}\|^2_{L^\infty(0,T;L^\infty(\Omega))}/\epsilon, C_2^s\|\mathbf{a}\|^2_{L^2(0,T;L^\infty(\Omega))}/\epsilon\right\}\right)$, and C_1^s and C_2^s are as defined in Lemma 5.6 and Lemma 5.7, respectively.

In practice, the strong stability constants (factors) C_1^s, C_2^s and C_3^s are computed numerically. Hence, for efficiency it is desirable to bound only the terms on the left–hand sides of (5.15)-(5.17) required in the error analysis by $\|e\|_Q$. For this purpose, we shall redefine the strong stability factors to be the smallest constants which satisfy the following bounds,

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_{1,1}^s\|e\|^2_Q, \tag{5.18a}$$
$$\|\epsilon^{1/2}\boldsymbol{\nabla}\phi\|^2_Q \leq C_{1,2}^s\|e\|^2_Q, \tag{5.18b}$$
$$\|\epsilon\Delta\phi\|^2_Q \leq C_2^s\|e\|^2_Q, \tag{5.18c}$$
$$\|\phi_t\|^2_Q \leq C_3^s\|e\|^2_Q. \tag{5.18d}$$

5.4 Completion of the proof of the a posteriori error estimate

We shall now proceed to estimate the terms I-VII on the right–hand side of (5.2). For the first term I, we have

$$\begin{aligned}
\mathrm{I} &= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \sum_{\kappa\in\mathcal{T}^{n+1}} ([u_h^n]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h - \epsilon\Delta u_h - f, \phi_h - \phi)_\kappa \,\mathrm{d}t \\
&\equiv (R_1, \phi_h - \phi)_Q \\
&= (R_1, \tilde{\mathcal{I}}\phi - \phi)_Q + (R_1, \tilde{\mathcal{I}}(\pi\phi - \phi))_Q \\
&\equiv \mathrm{I}_1 + \mathrm{I}_2,
\end{aligned}$$

where $\tilde{\mathcal{I}}$ and π are as defined by (5.3), and

$$R_1|_\kappa = [u_h^n]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h - \epsilon\Delta u_h - f, \quad \text{for } \kappa\in\mathcal{T}^{n+1}.$$

By Lemma 5.4 and (5.18c), we have

$$|\mathrm{I}_1| \le C_1^{\tilde{p}} \|h^2 R_1\|_Q \|\Delta\phi\|_Q \le \frac{C_1^{\tilde{p}}\sqrt{C_2^s}}{\epsilon}\|h^2 R_1\|_Q\|e\|_Q.$$

Similarly, using Lemma 5.5 and (5.18d), we have

$$|\mathrm{I}_2| \le C_3^{\tilde{p}}\|kR_1\|_Q\|\phi_t\|_Q \le C_3^{\tilde{p}}\sqrt{C_3^s}\,\|kR_1\|_Q\|e\|_Q.$$

Hence,

$$|\mathrm{I}| \le \frac{C_1^{\tilde{p}}\sqrt{C_2^s}}{\epsilon}\|h^2 R_1\|_Q\|e\|_Q + C_3^{\tilde{p}}\sqrt{C_3^s}\,\|kR_1\|_Q\|e\|_Q.$$

Analogously, we have

$$\begin{aligned}
\mathrm{II} &= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa\in\mathcal{T}^{n+1}} (\epsilon\Delta u_h, \phi_h - \phi)_\kappa + (\epsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}(\phi_h - \phi)) \right) \mathrm{d}t \\
&= \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \left(\sum_{\kappa\in\mathcal{T}^{n+1}} (\epsilon\Delta u_h, \tilde{\mathcal{I}}_{n+1}\phi - \phi)_\kappa + (\epsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}(\tilde{\mathcal{I}}_{n+1}\phi - \phi)) \right) \mathrm{d}t \\
&\quad + \sum_{n=0}^{M} \int_{t^n}^{t^{n+1}} \Bigg(\sum_{\kappa\in\mathcal{T}^{n+1}} (\epsilon\Delta u_h, \tilde{\mathcal{I}}_{n+1}(\pi_{n+1}\phi - \phi))_\kappa \\
&\qquad\qquad + (\epsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}\tilde{\mathcal{I}}_{n+1}(\pi_{n+1}\phi - \phi)) \Bigg) \mathrm{d}t \\
&\equiv \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

By Lemma 5.4 and (5.18c), we have

$$|\mathrm{II}_1| \le C_2^{\tilde{p}}\|h^2 D_h^2 u_h\|_Q\|\epsilon\Delta\phi\|_Q \le C_2^{\tilde{p}}\sqrt{C_2^s}\,\|h^2 D_h^2 u_h\|_Q\|e\|_Q.$$

Also, by Lemma 5.5 we have

$$\text{II}_2 = 0.$$

Thus, we have that

$$|\text{II}| \leq C_2^{\tilde{p}} \sqrt{C_2^s}\, \|h^2 D_h^2 u_h\|_Q \|e\|_Q.$$

Next, we consider term III: applying the Cauchy–Schwarz inequality, Lemma 5.1 and (5.18a), we have

$$\begin{aligned} |\text{III}| &\leq \|kR_3\|_Q \|\phi_h\|_Q \leq C_1^{\tilde{i}} \|kR_3\|_Q \|\phi\|_Q \\ &\leq C_1^{\tilde{i}} \sqrt{T}\, \|kR_3\|_Q \|\phi\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C_1^{\tilde{i}} \sqrt{C_{1,1}^s T}\, \|kR_3\|_Q \|e\|_Q, \end{aligned}$$

where

$$R_3|_{S^{n+1}} = (D_t^h u_h - ([u_h^n]/k_{n+1} + \mathbf{a} \cdot \boldsymbol{\nabla} u_h))/k_{n+1}.$$

Next, we consider term IV: using Lemma 5.5 and (5.18d), we have

$$|\text{IV}| \leq C_5^i \|kR_4\|_Q \|\phi_t\|_Q \leq C_5^i \sqrt{C_3^s}\, \|kR_4\|_Q \|e\|_Q,$$

where

$$R_4|_{S^{n+1}} = [u_h^n]/k_{n+1} = (u_h^{n+1} - u_h^n)/k_{n+1}.$$

Next, we consider term V: using the Cauchy–Schwarz inequality, Lemma 5.1 and (5.18a), we get

$$\begin{aligned} |\text{V}| &\leq \|f - \bar{f}\|_Q \|\phi_h\|_Q \leq C_1^{\tilde{i}} \|f - \bar{f}\|_Q \|\phi\|_Q \\ &\leq C_1^{\tilde{i}} \sqrt{T}\, \|f - \bar{f}\|_Q \|\phi\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C_1^{\tilde{i}} \sqrt{C_{1,1}^s T}\, \|f - \bar{f}\|_Q \|e\|_Q. \end{aligned}$$

Next, we consider term VI: using the Cauchy–Schwarz inequality and (5.18a), we get

$$\begin{aligned} |\text{VI}| &\leq \|u_0 - u_{h-}^0\| \|\phi(0)\| \leq \|u_0 - u_{h-}^0\| \|\phi\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq \sqrt{C_{1,1}^s}\, \|u_0 - u_{h-}^0\| \|e\|_Q. \end{aligned}$$

Finally, we consider term VII: using the Cauchy–Schwarz inequality, Lemma 5.1 and (5.18b), we have

$$\begin{aligned} |\text{VII}| &\leq \|\hat{\epsilon} \boldsymbol{\nabla} u_h\|_Q \|\boldsymbol{\nabla} \phi_h\|_Q \leq C_2^{\tilde{i}} \|\hat{\epsilon} \boldsymbol{\nabla} u_h\|_Q \|\boldsymbol{\nabla} \phi\|_Q \\ &\leq C_2^{\tilde{i}} \sqrt{\frac{C_{1,2}^s}{\epsilon}}\, \|\hat{\epsilon} \boldsymbol{\nabla} u_h\|_Q \|e\|_Q. \end{aligned}$$

We have now proved the main result of this paper:

Theorem 5.9 *Let u and u_h be solutions of (4.1) and (4.8), respectively. If $\mathcal{T}^n$, $0 \le n \le M+1$, satisfies conditions (4.2); then*

$$\|e\|_Q = \|u - u_h\|_Q \le \mathring{\mathcal{E}}(u_h, h, k, f), \tag{5.19}$$

where

$$\begin{aligned} \mathring{\mathcal{E}}(u_h, h, k, f) &= \mathcal{E}(u_h, h, k, f) + \mathcal{E}_0(u_0, u^0_{h-}, h), \\ \mathcal{E}(u_h, h, k, f) &= C_1\|h^2 R_1\|_Q + C_2\|kR_1\|_Q + C_3\|h^2 R_2\|_Q \\ &\quad + C_4\|kR_3\|_Q + C_5\|kR_4\|_Q + C_6\|kR_5\|_Q + C_8\|R_6\|_Q, \\ \mathcal{E}_0(u_0, u^0_{h-}, h) &= C_7\|u_0 - u^0_{h-}\|, \end{aligned}$$

and

$$\begin{aligned} R_1|_\kappa &= [u^n_h]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h - \epsilon\Delta u_h - f, \quad \textit{for } \kappa \in \mathcal{T}^{n+1}, \\ R_2 &= D^2_h u_h, \\ R_3|_{S^{n+1}} &= (D^h_t u_h - ([u^n_h]/k_{n+1} + \mathbf{a}\cdot\boldsymbol{\nabla} u_h))/k_{n+1}, \\ R_4|_{S^{n+1}} &= [u^n_h]/k_{n+1}, \\ R_5 &= (f - \bar{f})/k, \\ R_6 &= \hat{\epsilon}\boldsymbol{\nabla} u_h, \\ C_1 &= \frac{C^{\tilde{p}}_1\sqrt{C^s_2}}{\epsilon} = \frac{C^{\tilde{i}}_3\sqrt{C^s_2}}{c^2_2\epsilon}, \\ C_2 &= C^{\tilde{p}}_3\sqrt{C^s_3} = C^{\tilde{i}}_1 C^i_4\sqrt{C^s_3}, \\ C_3 &= C^{\tilde{p}}_2\sqrt{C^s_2} = C^t\sqrt{C^s_2}\,(C^{\tilde{i}}_3 c_2^{-1} + C^{\tilde{i}}_4)/(2c^2_2), \\ C_4 &= C^{\tilde{i}}_1\sqrt{C^s_{1,1}T}, \\ C_5 &= C^i_5\sqrt{C^s_3}, \\ C_6 &= C_4 = C^{\tilde{i}}_1\sqrt{C^s_{1,1}T}, \\ C_7 &= \sqrt{C^s_{1,1}}, \\ C_8 &= C^{\tilde{i}}_2\sqrt{\frac{C^s_{1,2}}{\epsilon}}. \end{aligned}$$

Remark 5.2 *We note that by exploiting the properties of the quasi-interpolation operator constructed in Section 3, the above a posteriori error estimate was proved assuming only that the underlying mesh was non-degenerate. Moreover, the term arising from the artificial diffusion model added to the Lagrange–Galerkin method can be treated in a much simpler manner than in [16].*

Further, we note that the exponents of h and k in the *a posteriori* error estimate (5.19) are essentially determined by the strong stability properties of the dual problem (5.1), rather than the approximation properties of the finite element space V^h; assuming, of course, that the space V^h is adequately equipped so that the order of the approximation error is at least equal to the order of the highest derivatives of the dual solution that are boundable by the error e. Here, the temporal discretisation of (4.1) involved a first–order time–stepping procedure along the characteristics of the reduced (hyperbolic) problem, hence, the error estimate (5.19) is optimal with respect to k. However, the *a posteriori* error estimate (5.19) will only be optimal with respect to h, if the degree of the approximating polynomial in the spatial variable (denoted by r) is equal to one, i.e. when $u_h(t)$ is a piecewise linear function. For $r > 1$ we expect that the residual of the underlying partial differential equation will decrease 'faster' as r increases in order that the improved approximation properties of the finite element space V^h are correctly reflected by (5.19); this will be numerically investigated in future work.

6 Closing remarks

In this paper we have constructed a quasi-interpolation operator $\tilde{\mathcal{I}}$ based on a modification of the generalised interpolation operators introduced by Scott & Zhang [19] and Brenner & Scott [5]. Moreover, we have shown that this operator is both stable with respect to the $L^p(\Omega)$ norm and optimally accurate in the $W^{m,p}(\Omega)$ semi-norm.

Based on this quasi-interpolation operator we have derived an *a posteriori* error estimate in the $L^2(0,T;L^2(\Omega))$ norm for the discontinuity capturing Lagrange–Galerkin method applied to a linear (unsteady) convection–diffusion problem. Moreover, this error estimate was established assuming *only* that the underlying mesh was *non-degenerate*. The theory developed in this paper can equally be applied to problems which lack the regularity needed for an interpolant to exist. In particular, we have derived *a posteriori* error estimates for the discontinuity capturing Lagrange–Galerkin discretisation of multi-dimensional scalar (first–order) hyperbolic equations, see [23].

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