Density and Trace for Graph Spaces of First-Order Linear Operators

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Abstract

We define and analyse graph spaces of first-order linear differential operators. In particular we consider the density of the set of smooth functions and the construction of a trace operator.
Introduction

Let us begin with an example. Suppose that $\Omega$ is the open square $(0,1)^2 \subset \mathbb{R}^2$ on which we define the differential operator

$$\mathcal{L} : L^1(\Omega) \to \mathcal{D}'(\Omega), v \mapsto \partial_1 v + \partial_2 v,$$

$\mathcal{D}'(\Omega)$ denoting the space of distributions. Then the characteristics of $\mathcal{L}$ pass at an angle of $45^\circ$ or $\pi/4$ through the domain, as indicated on the left figure below. We wish to consider the boundary value problem for data prescribed on the set

$$\partial_- \Omega := \partial \Omega \cap \{(x_1, x_2) : x_1 = 0\} \cap \{(x_1, x_2) : x_2 = 0\}.$$

The method of characteristics allows one to solve this problem for smooth and also nonsmooth data on $\partial_- \Omega$. Clearly, discontinuities in the boundary data lead to discontinuous solutions which do not lie in $W^{1,q}(\Omega)$. Therefore we focus our attention on the graph space $W_0^q(\Omega)$ of $\mathcal{L}$, which is defined as the vector space of all functions $v \in L^q(\Omega)$ for which the value $\mathcal{L}(v)$ lies in $L^q(\Omega)$ as well. In the context of this example it means that we require the existence of a weak derivative in the characteristic direction only, while admitting discontinuities in other directions.

In the main part of this report we address the issue of density of $C^\infty$-functions for general first-order linear differential operators on Lipschitz domains. We are also interested in defining a trace for members of $W_0^q(\Omega)$. However we notice that certain restrictions emerge through the weaker requirements on differentiability, compared with $W^{1,q}(\Omega)$; considering, for instance, the characteristic boundary at the upper left corner of the domain depicted in the right plot; on the restriction to the slice $\overline{ab}$ in normal direction, a solution of the boundary value problem is identical to the boundary data on $\overline{ab} \subset \partial_- \Omega$, neglecting the compression factor $\cos(\pi/4)$. Therefore when the boundary data lie merely in $L^q(\partial \Omega)$ it appears unreasonable to expect existence of a trace of the solution in the sense of $W^{1,q}$-functions.

Our aim in this report is to formalise these observations and to introduce the graph space and its associated trace operator in a rigorous manner. We shall conclude the discussion with a few comments about related publications.
Definition and Density

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial \Omega$ and unit outward normal $\nu = (\nu_1, \ldots, \nu_n)$. Choose a conjugate pair $q, q'$, i.e., $q, q' \in \mathbb{R}$ such that

$$1 < q < \infty, \quad q' = \frac{q}{q - 1}.$$

Given a tensor $\mathbf{B} \in [W^{1,\infty}(\Omega)]^{m \times n}$ and a matrix $\mathbf{C} \in [W^{1,\infty}(\Omega)]^{m \times m}$, we are interested in the graph space of the differential operator

$$\mathcal{L} : [L^q(\Omega)]^m \to [\mathcal{D}^q(\Omega)]^m, \mathbf{v} \mapsto \partial_k (B_{ij} \nu_j) + C_{ij} \nu_j.$$

Here and throughout the text we employ the Einstein summation convention. We also follow the notational convention to typeset entries of tensors, including vectors and matrices, in italic letters; tensors will be typeset in bold letters. Further, we set, for every manifold $M$,

$$L^q(M) := [L^q(\Omega)]^m, \quad W^{r,q}(M) = [W^{r,q}(\Omega)]^m.$$

Then the graph space of $\mathcal{L}$ is the set

$$W^r_q(\Omega) := \{ \mathbf{v} \in L^q(\Omega) : \mathcal{L}(\mathbf{v}) \in L^q(\Omega) \},$$

which is normed by the mapping

$$\| \mathbf{v} \|_{\mathcal{L}_q} = \| \mathbf{v} \|_{L^q(\Omega)} + \| \mathcal{L}(\mathbf{v}) \|_{L^q(\Omega)}.$$

Our first investigations concern the density of smooth functions in $W^r_q(\Omega)$. Consider the approximate identity $\delta \mapsto \psi_\delta$, $\delta > 0$, with non-negative functions $\psi_\delta(x) = \delta^{-n} \psi_1(\delta^{-1} x)$, satisfying the condition that the support of $\psi_1$ lies within the unit ball centred at the origin. We write, for $\mathbf{v} \in W^r_q(\Omega)$,

$$\mathbf{v}^\delta = (v_1^\delta, \ldots, v_n^\delta) = (v_1 * \psi_\delta, \ldots, v_n * \psi_\delta).$$

**Theorem 1**: Let $\Omega' \subset \subset \Omega$ and suppose that, for $\mathbf{v} \in W^r_q(\Omega)$, $1 < q < \infty$, the support $\text{supp}(\mathbf{v})$ is a subset of $\Omega'$. Then, for every $\varepsilon > 0$ there exists a function $\mathbf{v}_\varepsilon \in [C^\infty_0(\Omega')]^m$ with

$$\| \mathbf{v} - \mathbf{v}_\varepsilon \|_{\mathcal{L}_q} < \varepsilon.$$

**Proof. Step 1**: Transformation of $\partial(\mathbf{Bv}) \ast \psi$

Let $\mathbf{u} \in L^q(\Omega')$ and assume $\delta < \text{dist}(\text{supp}(\mathbf{v}), \partial\Omega')$ and $\delta' < \text{dist}(\partial\Omega', \partial\Omega)$. We denote differentiation with respect to $\mathbf{x}$ by $\partial_k$, this implies

$$\partial_k \psi_\delta(x - \mathbf{x}) = -\partial_k \psi_\delta(x - \mathbf{x}).$$
Hence, if we extend \( u \) to \( \Omega \) by setting \( u(x) = 0 \) outside \( \Omega' \),
\[
\int_{\Omega} u_i \left( \partial_k (B_{ijk} v_j) * \psi_\delta \right) dx = \int_{\Omega} u_i(x) \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) v_j(\bar{x})) \psi_\delta(x - \bar{x}) d\bar{x} dx
\]
\[
= \int_{\Omega} \lim_{\delta \to 0} u_i^\delta(x) \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) v_j(\bar{x})) \psi_\delta(x - \bar{x}) d\bar{x} dx
\]
\[
= \lim_{\delta \to 0} \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) v_j(\bar{x})) \int_{\Omega} u_i^\delta(x) \psi_\delta(x - \bar{x}) dx d\bar{x}
\]
\[
= \lim_{\delta \to 0} \int_{\Omega} u_i^\delta(x) \int_{\Omega} -B_{ijk}(\bar{x}) v_j(\bar{x}) \partial_k \psi_\delta(x - \bar{x}) dx d\bar{x}
\]
\[
= \int_{\Omega} u_i(x) \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) \psi_\delta(x - \bar{x})) v_j(\bar{x}) d\bar{x} dx.
\]

In the course of integration by parts we used that
\[
\left( \bar{x} \mapsto \int_{\Omega} u_i^\delta(x) \psi_\delta(x - \bar{x}) dx \right) \in \mathcal{D}(\Omega).
\]

As \( L^p(\Omega') \) is the dual space of \( L^q(\Omega') \) and \( \text{supp}(\psi_\delta * \partial_k (B_{ijk} v_j)) \subset \Omega' \), we obtain the identity of \( L^q(\Omega') \) functions
\[
\left( x \mapsto (\partial_k (B_{ijk} v_j) * \psi_\delta)(x) \right) = \left( x \mapsto \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) \psi_\delta(\bar{x} - \bar{x})) v_j(\bar{x}) d\bar{x} \right).
\]

**Step II: Transformation of \( \partial(Bv^\delta) \)**
As above, we have
\[
\int_{\Omega} u_i \partial_k (B_{ijk} (\psi_\delta * v_j)) dx = \int_{\Omega} u_i(x) \partial_k (B_{ijk}(x) \int_{\Omega} v_j(\bar{x}) \psi_\delta(x - \bar{x}) d\bar{x}) dx
\]
\[
= \int_{\Omega} u_i(x) \int_{\Omega} \partial_k (B_{ijk}(x) \psi_\delta(x - \bar{x})) v_j(\bar{x}) d\bar{x} dx.
\]

Therefore, in \( L^q(\Omega') \),
\[
\left( x \mapsto \partial_k (B_{ijk} (\psi_\delta * v_j))(x) \right) = \left( x \mapsto \int_{\Omega} \partial_k (B_{ijk}(x) \psi_\delta(x - \bar{x})) v_j(\bar{x}) d\bar{x} \right).
\]

**Step III: Boundedness of \( B(\partial\psi_\delta) v \)**
Suppose that \( \delta < \text{dist}(\text{supp}(v), \partial\Omega) \). We consider the operator
\[
T : L^1(\Omega) \to L^1(\Omega), \quad (T v)_i(x) = \int_{\Omega} (B_{ijk}(x) - B_{ijk}(x)) \partial_k \psi_\delta(x - \bar{x}) v_j(\bar{x}) d\bar{x}.
\]

Since \( B_{ijk} \) lies in \( \text{W}^{1,\infty}(\Omega) \), it is a Lipschitz continuous function for which
\[
\|B\| := \|B\|_{W^{1,\infty}(\Omega)}^{m \times m \times n}
\]
is a Lipschitz constant. Therefore we have continuity of $T$ in the $L^1$-vector norm:

$$
\|Tv\|_{L^1(\Omega)} = \sum_{i=1}^m \int_{\Omega} \int_{\Omega} (B_{ijk}(\bar{x}) - B_{ij}(x)) \partial_k \psi_\delta(x - \bar{x}) v_j(x) \, dx \, dx
$$

$$
\leq \sum_{i=1}^m \int_{\Omega} \int_{\Omega} \left| \frac{B_{ijk}(\bar{x}) - B_{ij}(x)}{\delta} \right| \left| \partial_k \psi_\delta(x - \bar{x}) \right| |v_j(x)| \, dx \, dx
$$

$$
\leq \sum_{j=1}^m \|B\| \cdot \left( \int_{\Omega} \int_{\Omega} |\partial_k \psi_\delta(x - \bar{x})| |v_j(x)| \, dx \right)
$$

$$
= \|B\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|v\|_{L^1(\Omega)},
$$

where we used that $\delta \partial_k \psi_\delta(x) = \delta^{-n} (\partial_k \psi_1)(\delta^{-1}x)$ and the transformation of variables $x \mapsto \delta x$. In the case of $v \in L^\infty(\Omega)$, the same bound holds for the $L^\infty$-norm, since

$$
\|Tv\|_{L^\infty(\Omega)} = \max_{i} \sup_{\bar{x}} \int_{\Omega} (B_{ijk}(\bar{x}) - B_{ij}(x)) \partial_k \psi_\delta(x - \bar{x}) v_j(x) \, dx \, dx
$$

$$
\leq \max_{i} \sup_{\bar{x}} \int_{\Omega} \left| \frac{B_{ijk}(\bar{x}) - B_{ij}(x)}{\delta} \right| |\partial_k \psi_\delta(x - \bar{x})| |v_j(x)| \, dx \, dx
$$

$$
\leq \|B\| \cdot \|v\|_{L^\infty(\Omega)} \cdot \int_{\Omega} |\partial_k \psi_\delta(x - \bar{x})| \, dx
$$

$$
= \|B\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|v\|_{L^\infty(\Omega)}.
$$

Next we apply the Riesz-Thorin Interpolation Theorem to derive a bound for $v \in L^q(\Omega)$. For the reader’s convenience we have stated the theorem in the Appendix; we apply it with $\theta = 1 - 1/q = 1/q’$ and $p = q$, so that, for $v \in L^q(\Omega)$,

$$
\left\| \int_{\Omega} (B_{ijk}(\bar{x}) - B_{ij}(\cdot)) \partial_k \psi_\delta(\cdot - \bar{x}) v_j(\cdot) \, d\bar{x} \right\|_{L^q(\Omega)}
$$

$$
= \|Tv\|_{L^q(\Omega)} \leq 2 \cdot \|B\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|v\|_{L^q(\Omega)}.
$$

**Step IV: Boundedness of ($L\psi_\delta - L\psi_\delta^\delta$)**

Using that $\psi_\delta(x) \leq 0$ for all $x \in \mathbb{R}^n$, we deduce that

$$
\left\| \int_{\Omega} \partial_k (B_{ijk}(\bar{x}) - B_{ij}(\cdot)) \psi_\delta(\cdot - \bar{x}) v_j(\cdot) \, d\bar{x} \right\|_{L^q(\Omega)}
$$

$$
\leq \left( \sum_{i=1}^m \int_{\Omega} \left( \int_{\Omega} |\partial_k ((B_{ijk}(\bar{x}) - B_{ij}(x)) \psi_\delta(x - \bar{x})) v_j(x) \, dx \right)^q \, dx \right)^{1/q}
$$

$$
\leq 2 \cdot \|B\| \cdot \|\psi_\delta\| \cdot \|v\|_{L^q(\Omega)} \leq 2 \cdot \|B\| \cdot \|\psi_1\|_{L^1(\Omega)} \cdot \|v\|_{L^q(\Omega)}.
$$

Hence using the product rule

$$
\|\partial_k (B_{ijk} v_j) - \partial_k (B_{ij} v_j^\delta)\|_{L^q(\Omega)}
$$

$$
= \left\| \int_{\Omega} \partial_k ((B_{ijk}(\bar{x}) - B_{ij}(\cdot)) \psi_\delta(\cdot - \bar{x})) v_j(\bar{x}) \, d\bar{x} \right\|_{L^q(\Omega)}
$$

$$
\leq 2 \cdot \|B\| \cdot \|\psi_1\|_{W^{1,1}(\Omega)} \cdot \|v\|_{L^q(\Omega)}.
Finally as in (1)
\[
\| (C_{ij} v_j) * \psi_j - C_{ij} v_j^\delta \|_{L^p(\Omega)} = \left\| \int_\Omega (C_{ij}(\tilde{x}) - C_{ij}(\cdot)) \psi_j(\cdot - \tilde{x}) v_j(\tilde{x})\, d\tilde{x} \right\|_{L^p(\Omega)} \\
\leq 2 \cdot \|C\|_{L^\infty(\Omega)}^{m \times m} \cdot \|\psi_j\|_{L^1(\Omega)} \cdot \|\nu\|_{L^p(\Omega)}.
\]

**Step V: Weakly converging sequence**
The last step shows that the sequence
\[
s_1(\ell) = \mathcal{L}(\nu) * \psi_{1/\ell} - \mathcal{L}(\nu * \psi_{1/\ell})
\]
is bounded in $L^q(\Omega)$. Hence by the Banach-Alaoglu Theorem there exists a sequence $t_1 : \mathbb{N} \to \mathbb{N}$ such that $s_1 \circ t_1$ is weakly converging to an element $\varphi \in L^q(\Omega)$. But, for $w \in L^q(\Omega)$ and $\tilde{\ell} = 1/t_1(\ell)$,
\[
\int_\Omega \varphi_i w_i \, dx = \lim_{\delta \to 0} \int_\Omega \varphi_i^\delta w^\delta_i \, dx \\
= \lim_{\delta \to 0} \lim_{\ell \to \infty} \int_\Omega (\mathcal{L}(\nu) * \psi_{\tilde{\ell}} - \mathcal{L}(\nu * \psi_{\tilde{\ell}})) w_i^\delta \, dx \\
= \lim_{\delta \to 0} \left( \int_\Omega \mathcal{L}(\nu) w_i^\delta \, dx - \lim_{\ell \to \infty} \int_\Omega \mathcal{L}(\nu * \psi_{\tilde{\ell}}) w_i^\delta \, dx \right) \\
= \lim_{\delta \to 0} \lim_{\ell \to \infty} \int_\Omega (v_j - v_j * \psi_{\tilde{\ell}}) (C_{ij} u_i^\delta - B_{ijk} \partial_k u_i^\delta) \, dx \\
= \lim_{\delta \to 0} \lim_{\ell \to \infty} \lim_{\ell \to \infty} (v_j - v_j * \psi_{\tilde{\ell}}) (C_{ij} u_i^\delta - B_{ijk} \partial_k u_i^\delta) \, dx = 0,
\]
and so $\varphi = 0$.

**Step VI: Strongly converging sequence**
Given $\varepsilon > 0$, we can select a sequence $t_2 : \mathbb{N} \to \mathbb{N}$, such that for $t_3 = t_1 \circ t_2$ and $\tilde{\ell} = 1/t_3(\ell)$
\[
\|v - v * \psi_{\tilde{\ell}}\|_{L^p(\Omega)} < \frac{\varepsilon}{3 \cdot 2^\ell}, \quad \|\mathcal{L}(\nu) - \mathcal{L}(\nu) * \psi_{\tilde{\ell}}\|_{L^p(\Omega)} < \frac{\varepsilon}{3 \cdot 2^\ell}.
\]
Using Mazur’s Theorem, cf. [4, Theorem 3.13], there exists a finite convex combination
\[
v_\varepsilon = \sum_{\ell=1}^s \lambda_\ell v * \psi_{\tilde{\ell}}, \quad \sum_{\ell=1}^s \lambda_\ell = 1, \lambda_\ell \in [0,1], s \in \mathbb{N},
\]
such that
\[
\left\| \left( \sum_{\ell=1}^s \lambda_\ell \mathcal{L}(\nu) * \psi_{\tilde{\ell}} \right) - \mathcal{L}(v_\varepsilon) \right\|_{L^p(\Omega)} = \left\| \sum_{\ell=1}^s \lambda_\ell (\mathcal{L}(\nu) * \psi_{\tilde{\ell}} - \mathcal{L}(\nu * \psi_{\tilde{\ell}})) \right\|_{L^p(\Omega)} < \frac{\varepsilon}{3}.
\]
Hence
\[
\|v - v_\varepsilon\|_{L^p(\Omega)} \leq \sum_{\ell=1}^s \lambda_\ell \|v - v * \psi_{\tilde{\ell}}\|_{L^p(\Omega)} < \sum_{\ell=1}^\infty \frac{\varepsilon}{3 \cdot 2^\ell} < \frac{\varepsilon}{3}.
\]
Similarly, but by using the triangle inequality
\[ \| \mathcal{L}(v) - \mathcal{L}(v_\varepsilon) \|_{L^q(\Omega)} \leq \varepsilon \frac{\varepsilon}{3} + \varepsilon \frac{\varepsilon}{3}. \]

But then \( \| v - v_\varepsilon \|_{L^q(\Omega < \varepsilon} \) and \( v_\varepsilon \in [C^\infty_0(\Omega')]^m. \)

The transformations in Steps I, II and the first bound in Step III are based on ideas in [1].

We can extend the theorem to all functions in \( W^q_0(\Omega) \) using the techniques introduced by Meyers and Serrin in [2], see also [7, p. 54]. However, due to the existence of functions \( v \in W^q_0(\Omega) \) whose support does not lie compactly in \( \Omega \), it becomes necessary to admit \( C^\infty(\Omega) \) as class of approximating functions instead of \( C^\infty_0(\Omega) \), \( \Omega' \subset \subset \Omega \).

**Theorem 2:** The space \( [C^\infty(\Omega)]^m \cap W^q_0(\Omega) \) is dense in \( W^q_0(\Omega) \), \( 1 < q < \infty \).

**Proof.** Let \( \Omega_i \) be open subsets of \( \Omega \) such that \( \Omega_i \subset \subset \Omega_{i+1} \) and
\[ \bigcup_{i=1}^\infty \Omega_i = \Omega. \]

Let \( \mathcal{F} \) be a partition of unity of \( \Omega \) subordinate to the covering \( (\Omega_{i+1} \setminus \overline{\Omega_{i-1}})_{i \in \mathbb{N}} \), where \( \Omega_{-1} \)

is taken as the empty set. This means that \( \mathcal{F} \) is a family of functions \( f_j, j \in \mathbb{N} \), such that:

1. for each \( f_j \) there exists an \( i \) such that \( \text{supp}(f_j) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}} \);
2. each compact set \( K \subset \Omega \) intersects the support of finitely many \( f_j \) only;
3. for all \( x \in \Omega \) holds \( \sum f_j(x) = 1 \).

Let \( f_i \) be the sum of all \( f_j \in \mathcal{F} \) for which \( i \) is the smallest index such that \( \text{supp}(f_j) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}} \). Then the \( f_i \) sum to one, too. Choose \( \varepsilon > 0 \). For \( i \in \mathbb{N} \) and \( v \in W^q_0(\Omega) \) there exists, according to the last theorem, a function \( v_{\varepsilon,i} \in [C^\infty_0(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})]^m \) such that
\[ \| f_i \cdot v - v_{\varepsilon,i} \|_{L^q(\Omega)} \leq \varepsilon \frac{\varepsilon}{2^i}. \]

Since \( \overline{\Omega_j} \) is compact, the supports of only finitely many \( v_{\varepsilon,i} \) intersect \( \Omega_j \). Hence the sum
\[ v_\varepsilon = \sum_{i=1}^\infty v_{\varepsilon,i} \]

is defined and is a member of \( [C^\infty(\Omega)]^m \). Notice that because of the layout of the supports of the \( v_{\varepsilon,i} \), the sequence
\[ j \mapsto \left( v - \sum_{i=1}^j v_{\varepsilon,i} \right)_{+} \mid_{\Omega_j} = (v - v_\varepsilon)_+ |_{\Omega_j} \]

of \( L^q(\Omega) \) functions exhibits monotonic and pointwise convergence to \( (v - v_\varepsilon)_+ \) as \( j \to \infty \). Hence by the Monotonic Convergence Theorem \( (v - v_\varepsilon)_+ \) is \( L^q \)-integrable, as well. The
same argument employed on \((\mathbf{v} - \mathbf{v}_e)_-\), \((\partial_k B_{ijk}(v_j - (v_e)_j))_+\) and \((\partial_k B_{ijk}(v_j - (v_e)_j))_-\) asserts that \(\mathbf{v}_e \in \mathbf{W}^{q}_2(\Omega)\). We conclude

\[
\|\mathbf{v} - \mathbf{v}_e\|_{L^q} \leq \sum_{i=1}^{\infty} \|f_i \cdot \mathbf{v} - \mathbf{v}_e,i\|_{L^q} < \varepsilon.
\]

This proves the density of smooth functions in \(\mathbf{W}^{q}_2(\Omega)\).

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**Trace**

To define the trace of functions in \(\mathbf{W}^{q}_2(\Omega)\) we follow the construction for \(W^q(\text{div}, \Omega)\); compare, for example, with [5]. Necas [3] shows existence of a bounded extension operator

\[
\Gamma_e : W^{1-1/d',d'}(\partial \Omega) \to W^{1,d'}(\Omega), \quad g = (\Gamma_e g)|_{\partial \Omega}, \quad 1 < q < \infty.
\]

Since, for \(\mathbf{v} \in W^{1,q}(\Omega)\) and \(\mathbf{w} \in W^{1,d'}(\Omega)\),

\[
\int_{\partial \Omega} w_i B_{ijk} v_j \, dS = \int_{\Omega} (\partial_k w_i) B_{ijk} v_j \, dV + \int_{\Omega} w_i \partial_k (B_{ijk} v_j) \, dV,
\]

we obtain, for all \(\mathbf{w}\) in the image of \(\Gamma_e\) with \(\mathbf{w} = \Gamma_e \mathbf{g}\),

\[
\langle (\mathbf{g}, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v})_{\partial \Omega} \rangle \leq C \cdot \|\mathbf{g}\|_{W^{1-1/d',d'}(\partial \Omega)} \|\mathbf{B}(\mathbf{v})\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^q(\Omega)} \|\partial_k (B_{ijk} v_j)\|_{L^q(\Omega)}
\]

setting \(C = \|\Gamma_e\|\) and \(\mathbf{B}(\mathbf{v}) = (B_{ijk} v_j)_i\). This shows that, for all \(\mathbf{v} \in W^{1,q}(\Omega)\), the functional

\[
\langle \cdot, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v} \rangle : W^{1-1/d',d'}(\partial \Omega) \to \mathbb{R}, \quad \mathbf{g} \mapsto \langle \mathbf{g}, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v} \rangle_{\partial \Omega}
\]

is a continuous mapping. Therefore, \(\langle \cdot, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v} \rangle\) belongs to the dual of \(W^{1-1/d',d'}(\partial \Omega)\), which is the space

\[
(W^{1-1/d',d'}(\partial \Omega))^{' \prime} = [(W^{1-1/d',d'}(\partial \Omega))^{' \prime}]^m = W^{-1/q,q}(\partial \Omega).
\]

Also

\[
\|\langle \cdot, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v} \rangle\|_{W^{-1/q,q}(\partial \Omega)} \leq C \cdot \|B_{ijk} v_j\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^q(\Omega)} + \|\partial_k (B_{ijk} v_j)\|_{L^q(\Omega)}.
\]

Because

\[
|B_{ijk} v_j|_{L^q(\Omega)} \leq \|\mathbf{B}\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^q(\Omega)},
\]

\[
|\partial_k (B_{ijk} v_j)|_{L^q(\Omega)} \leq \|\mathbf{L}(\mathbf{v})\|_{L^q(\Omega)} + \|C\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^q(\Omega)},
\]

we obtain from (3) the continuity of the linear operator

\[
\Gamma_L : W^{1,q}(\Omega) \to W^{-1/q,q}(\partial \Omega), \mathbf{v} \mapsto \langle \cdot, \mathbf{B}(\mathbf{v}) \cdot \mathbf{v} \rangle_{\partial \Omega}.
\]
in \( \| \cdot \|_{L_q} \). As \( W^{1,q}(\Omega) \) is dense in \( W^2_q(\Omega) \), \( \Gamma_B \) extends to a continuous linear operator on the graph space.

**Definition:** The operator

\[
\Gamma_L : W^2_q(\Omega) \to W^{-1,q}(\partial\Omega), \quad v \mapsto \lim_{\varepsilon \to 0} \langle \cdot, B(\nu) \cdot v_\varepsilon \rangle_{\partial\Omega} = : \langle \cdot, B(\nu) \cdot v \rangle_{\partial\Omega}
\]

is called trace operator for \( W^2_q(\Omega) \); here \( \{v_\varepsilon\}_{\varepsilon > 0} \) is smooth and converging to \( v \) in \( W^2_q(\Omega) \).

Observe that the definition of the trace is independent of the extension operator \( \Gamma_\varepsilon \) as the choice of \( \Gamma_\varepsilon \) only influences the constant \( C \) in (3). As \( \partial_k(B_{ijk} v_j) \in L^q(\Omega) \), the identity (2) is meaningful for all elements of \( W^2_q(\Omega) \): Choose \( v_\varepsilon \) as above and let \( w \in W^{1,q}(\Omega) \), then

\[
\langle w, B(\nu) \cdot v \rangle_{\partial\Omega} = \lim_{\varepsilon \to 0} \langle w, B(\nu) \cdot v_\varepsilon \rangle_{\partial\Omega}
\]

\[
= \lim_{\varepsilon \to 0} \int_\Omega (\partial_k w_i) B_{ijk} v_\varepsilon, j \, dV + \lim_{\varepsilon \to 0} \int_\Omega w_i \partial_k(B_{ijk} v_\varepsilon, j) \, dV
\]

\[
= \int_\Omega (\partial_k w_i) B_{ijk} v_j \, dV + \int_\Omega w_i \partial_k(B_{ijk} v_j) \, dV.
\]

**Comments**

In [1] Friedrichs analyses the existence and uniqueness of linear symmetric hyperbolic systems with associated differential operator \( \mathcal{L} \). Initially he considers the weak and strong extension of these operators, but then shows that these extensions coincide. This allows him to apply a duality argument by which he proves the existence of a strong solution by uniqueness of weak solutions. To ensure coincidence of the weak and strong extension he introduces the integral operators

\[
K_\varepsilon u(x) = \int_\Omega k_\varepsilon(x, \xi) u(\xi) \, d\xi,
\]

which generalize the idea of mollifiers. Indeed \( k_\varepsilon \) is used in the calculation as a combination of an approximate identity with terms of \( \mathcal{L} \) and its dual \( \mathcal{L}' \). Friedrichs presents three properties of \( k_\varepsilon \) which are sufficient to prove convergence of certain smooth functions \( v_\varepsilon \) to \( v \). We only used one of the properties, namely boundedness. The other two properties concern a suitable choice of the support of \( k_\varepsilon \) and the mean of \( k_\varepsilon \). The analysis in [1] is given for lens-shaped domains since then symmetries in boundary terms can be exploited. In 1964, ten years after the publication of [1], Meyers and Serrin proved that for \( W^{r,q}(\Omega) \) the weak and strong extension are equivalent for general open sets \( \Omega \). The proof of Theorem 2 in this report uses very similar arguments to those in [2]. The only difference is that Meyers and Serrin were able to use mollified functions directly while for our problem the strongly convergent sequence obtained by Theorem 1 is required.
Appendix

We cite the Riesz-Thorin Interpolation Theorem in abbreviated form from [6].

**Theorem 3:** Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Further let $0 < \theta < 1$, and let $p$ and $q$ be defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$  

Let $\mu$ and $\nu$ be $\sigma$-finite measures. If $T$ is a linear mapping with

$$T : L^{p_0}(\mu) \to L^{q_0}(\nu) \quad \text{continuous with norm } M_0,$$

$$T : L^{p_1}(\mu) \to L^{q_1}(\nu) \quad \text{continuous with norm } M_1,$$

then

$$\|Tf\|_{L^q} \leq 2M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p} \quad \forall f \in L^{p_0}(\mu) \cap L^{p_1}(\mu).$$

Hence the operator is extendable to a continuous linear mapping

$$T : L^p(\mu) \to L^q(\nu)$$

with norm $2M_0^{1-\theta}M_1^{\theta}$.

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References


