A Solenoidal Finite Element Approach for Prediction of Radar Cross-Sections

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This report considers the solution of problems that involve the scattering of plane electromagnetic waves by perfectly conducting obstacles. Such problems are governed by the Maxwell equations.

An interesting facet of the solution of Faraday's law and Ampere's law, which on their own form a complete equation set for the determination of the field intensity components, is that there are the additional conservation statements of Coulomb's law and Gauss's law, which appear to be in excess of requirements. Often, these additional constraints are neglected due to an inability to incorporate them into the solution scheme. With the successful development of a solenoidal finite element for the solution of viscous incompressible flows, such a device now offers a practical means for the solution of the full Maxwell equations. To demonstrate the validity of this assertion, a suitable solution scheme is presented, accompanied by sample results for various test problems.

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Key words: Maxwell equations, radar cross-section, solenoidal finite element method

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1 Introduction

In recent times, research into scattering of electromagnetic waves by complex objects has assumed great importance due to its relevance to radar applications, where the main objective is to identify targeted objects. A typical radar accomplishes this objective by processing the received signal, which generally consists of the desired scattered signal plus an undesired noise component, the former containing the information characteristic of the targeted object. The received scattered signal power is directly proportional to the radar cross-section (RCS) and hence the importance of the control of this quantity for different objects in different situations.

In designing stealth weapon systems such as military aircraft, control of their radar cross-section is of paramount importance. Aircraft in combat situations are threatened by enemy missiles. One countermeasure that is used to reduce this threat is to minimise the radar cross-section. On the other hand, there is a demand for the enhancement of the radar cross-section of civilian spacecraft. Operators of communication satellites often request a complicated differential radar cross-section in order to assist with the tracking of the satellite. To control the radar cross-section, an essential requirement is a capability for accurate prediction of electromagnetic scattering from complex objects.

One difficulty that is encountered in the development of suitable numerical solution schemes is the existence of constraints which are in excess of those needed for a unique solution. Rather than attempt to correct for these constraints once the solutions are calculated, or to include the constraints in the original equation sets, the novel approach which is suggested here involves the use of the finite element method and the construction of a specialised element in which the relevant solution variables are appropriately constrained by the nature of their interpolation functions. For many years, such an idea was claimed to be impossible \textsuperscript{10,12}. While the idea is not without its difficulties, its advantages far outweigh its disadvantages. The author has successfully developed such an element for primitive variable solutions to viscous incompressible flows \textsuperscript{4-8}.

2 Radar cross-sections

An electromagnetic wave consists of an electric field and a magnetic field. At all times, these are normal to each other, as shown in Figure 1. The wave travels at the speed of light $c$, with a frequency $f$ and a wavelength $\lambda$. These quantities are related by

$$c = f \lambda.$$  \hspace{1cm} (1)

A typical radar consists of an antenna which transmits electromagnetic waves and another antenna which detects the waves which are scattered by a target illuminated by the radar. Usually, the two antennae are collocated ('monostatic' radar), although some systems use separate locations ('bistatic' radar).

The extent to which a target returns electromagnetic energy is the target's radar cross-section $\sigma$. This depends on the properties (polarisation, wavelength) of the wave plus the form (size, shape, aspect) of the target. The radar cross-section is expressed as the area of a sphere which would return the same energy. The radar cross-section thus has units of area...
(m^2), or more commonly decibels (db m^2), where

\[ \sigma_{\text{db m}^2} = 10 \log_{10} (\sigma_{\text{m}^2}). \]  

(2)

Relative values for various targets are given in Figure 2.

Figure 1: An electromagnetic wave

<table>
<thead>
<tr>
<th>Radar cross-section, m^2</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^4</td>
<td>Ships</td>
</tr>
<tr>
<td>10^3</td>
<td>Large aircraft</td>
</tr>
<tr>
<td>10^2</td>
<td>Small aircraft</td>
</tr>
<tr>
<td>10</td>
<td>Stealth aircraft</td>
</tr>
<tr>
<td>1</td>
<td>Birds</td>
</tr>
<tr>
<td>10^{-1}</td>
<td>Insects</td>
</tr>
</tbody>
</table>

Figure 2: Relative values of radar cross-section for various targets
The radar cross-section is not a single value. It is different for each angle from the radar, known as the 'look' angle. When graphed in polar coordinates, the radar cross-section for an aircraft takes the form shown in Figure 3.

![Figure 3: A typical aircraft radar cross-section (reproduced from [11])](image)

As can be seen, the energy returned varies with direction by orders of magnitude. Directions from which the returned energy is high are termed 'spikes'. These are often normal to flat surfaces. Design for stealth therefore begins, not with eliminating these surfaces, but with 'aiming' the spikes. This means that surfaces are sloped at a particular angle, based on the direction of known threat and the direction of other spikes. Some problems that are studied in this report have been chosen with this in mind.

### 3 Maxwell equations

In any study of electromagnetic scattering phenomena, the governing equations are the Maxwell field equations. These are derived from various fundamental laws and, for linear isotropic media with no sources, can be expressed as

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t},$$

(3)

Ampere's law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t},$$

(4)
Coulomb's law
\[ \nabla \cdot \mathbf{E} = 0, \quad (5) \]

Gauss's law
\[ \nabla \cdot \mathbf{H} = 0, \quad (6) \]

where each of the variables has been non-dimensionalised and, in particular, \( \mathbf{E} \) is the electric field intensity and \( \mathbf{H} \) is the magnetic field intensity. For each field, the total field comprises the incident field plus the scattered field. Since the incident field already satisfies the governing equations, solutions need be obtained only for the scattered field part.

A unique solution is provided by the specification of initial conditions and boundary conditions. The boundary conditions considered here are those for a perfect conductor, where the tangential component of the total electric field is zero and the normal component of the total magnetic field is zero. The initial conditions are zero amplitude everywhere.

Maxwell's equations appear to provide more equations than unknowns but, in fact, the divergence constraints are not independent equations. Since the div of a curl is identically zero, each of the time-dependent equations implies one of the divergence constraints, provided the initial conditions satisfy the divergence constraints.

For the plane case, the time dependent equations decouple into two polarisations, expressible in flux-vector form as
\[
\frac{\partial \Phi}{\partial t} + \frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial y} = \mathbf{0},
\]

or in terms of components,
\[
\frac{\partial}{\partial t} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (8)
\]

For the transverse electric (TE) polarisation,
\[
\Phi = \begin{bmatrix} E_1 \\ E_2 \\ H_3 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 \\ H_3 \\ E_2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -H_3 \\ 0 \\ -E_1 \end{bmatrix}. \quad (9.1)
\]

For the transverse magnetic (TM) polarisation,
\[
\Phi = \begin{bmatrix} H_1 \\ H_2 \\ E_3 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 \\ -E_3 \\ -H_2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} E_3 \\ 0 \\ H_1 \end{bmatrix}. \quad (9.2)
\]

Every plane electromagnetic field can be resolved as the vector sum of these two
polarisations.

Considering for the moment all the equations as originally provided, for each polarisation, there is an additional constraint

\[
\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} = 0. \tag{10}
\]

As already noted, this is not an independent equation, since it is implied that

\[
\frac{\partial}{\partial t} \left( \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} \right) = 0, \tag{11}
\]

or

\[
\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} = c(x, y), \tag{12}
\]

where the constant \( c \) is determined by the initial conditions. If the initial conditions satisfy Equation (10), then so do all subsequent solutions.

While this is true in an analytical sense, the question remains whether it is true for the numerical solution. The work of Canupp in [2] and others suggest that it is not. If this is so then, from the numerical perspective, Equation (8) forms a complete solution set on its own, while Equation (10) is an additional constraint that the solution needs to satisfy, either implicitly or explicitly.

Traditionally, a finite volume method has been employed but, in recent times, the finite element method has begun to attract attention. Often, the approach has been to solve just Equation (8), as in Morgan et al [9]. One means of incorporating the extra constraint is to use a least squares approach, as in Wu et al [13]. The approach that is used here follows the Taylor-Galerkin procedure of [9] but imposes Equation (10) through the use of a solenoidal (zero divergence) finite element. This approach does not involve corrections to an existing solution for errors in field divergence. The solution is explicitly constrained to be solenoidal from the outset by the nature of the interpolation functions.

### 4 Solution procedure

Consider the Taylor expansion

\[
\Phi^{n+1} = \Phi^n + h \left[ \frac{\partial \Phi}{\partial t} \right]^n + \frac{h^2}{2} \left[ \frac{\partial^2 \Phi}{\partial t^2} \right]^n, \tag{13}
\]

where the superscript \( n \) denotes evaluation at time \( t = t^n \) and \( h \) is the constant time step. Using the governing equation (7), this can be rewritten as
\[
\Delta \Phi = \Phi^{n+1} - \Phi^n = -h \left[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right]^n + \frac{h^2}{2} \left[ \left( \frac{\partial}{\partial x} \frac{\partial U}{\partial \Phi} + \frac{\partial}{\partial y} \frac{\partial V}{\partial \Phi} \right) \left( \frac{\partial U}{\partial \Phi} + \frac{\partial V}{\partial \Phi} \right) \right]^n .
\] (14)

Let \( \Phi' = \Phi'(t) \) be the nodal parameters. The local behaviour of the solution is of the form

\[
\Phi = N \Phi',
\] (15)

where \( N \) represents the interpolation functions. Over the domain \( \mathcal{A} \), application of the Galerkin method to Equation (14) therefore gives

\[
K \Delta \Phi' = f
\] (16)

where

\[
K = \sum_e \int_{\mathcal{A}_e} N^T N dA,
\]

\[
f = -h \sum_e \int_{\mathcal{A}_e} N^T \left[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right]^n dA + \frac{h^2}{2} \sum_e \int_{\mathcal{A}_e} N^T \left[ \left( \frac{\partial}{\partial x} \frac{\partial U}{\partial \Phi} + \frac{\partial}{\partial y} \frac{\partial V}{\partial \Phi} \right) \left( \frac{\partial U}{\partial \Phi} + \frac{\partial V}{\partial \Phi} \right) \right]^n dA.
\] (17)

If an intermediate time step is introduced, at which

\[
\Phi^{n+\frac{1}{2}} = \Phi^n - \frac{h}{2} \left[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right]^n,
\] (18)

then using the similar formulae

\[
U^{n+\frac{1}{2}} = U^n - \frac{h}{2} \left[ \frac{\partial U}{\partial \Phi} \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right]^n
\]

\[
V^{n+\frac{1}{2}} = V^n - \frac{h}{2} \left[ \frac{\partial V}{\partial \Phi} \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right]^n
\] (19)

gives

\[
f = -h \sum_e \int_{\mathcal{A}_e} N^T \left[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right]^{n+\frac{1}{2}} dA
\] (20)

which, upon application of Green’s theorem, becomes
\[ f = h \sum_{e} \int_{\Lambda_e} \left( \frac{\partial N^T}{\partial x} U^{n+\frac{1}{2}} + \frac{\partial N^T}{\partial y} V^{n+\frac{1}{2}} \right) \, dA - h \sum_{e} \int_{s_e} N^T \left( \alpha U^{n+\frac{1}{2}} + \beta V^{n+\frac{1}{2}} \right) \, ds , \]  

(21)

where \( \alpha, \beta \) are the direction cosines of the normal to the boundary \( s \). The values at the intermediate time step are calculated for each integration point from their respective definitions in Equation (9). The procedure is now second-order rather than first-order. Thus the time domain solution is generated through a two-step finite element Taylor-Galerkin procedure \(^3\). An attraction of this procedure is that the coefficient matrix is constant with time.

## 5 Elements

A triangle is chosen as the shape of the element, for the ease of its decomposition of the domain and to facilitate isotropic polynomial expansions.

![Conventional linear element](image1)

\[ + \Phi_1', \Phi_2', \Phi_3' \]

Figure 4: Conventional linear element

![Conventional quadratic element](image2)

\[ + \Phi_1', \Phi_2', \Phi_3' \]

Figure 5: Conventional quadratic element

A conventional finite element is employed as a benchmark for the validation of the
approach. The 3-node element configuration is shown in Figure 4. The 6-node element configuration is shown in Figure 5. For these elements, Equation (15) takes the form

\[
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix} =
\begin{bmatrix}
N_{11} & 0 & 0 \\
0 & N_{22} & 0 \\
0 & 0 & N_{33}
\end{bmatrix}
\begin{bmatrix}
\Phi_1' \\
\Phi_2' \\
\Phi_3'
\end{bmatrix}.
\]

(22)

For the solenoidal element, expansions are required which provide a number of degrees of freedom in excess of that needed to be conforming, with which to impose the desired constraint. While a complete quartic was the initial choice, a more efficient element is constructed from a complete quadratic with a quartic bubble. This provides the element with the benefit of the higher order accuracy without the drawback of the associated higher storage requirements.

\[
\sum \Phi_i' + \Phi_1', \Phi_2', \Phi_3'
\]

\[
\frac{\partial \Phi_1'}{\partial x} + \frac{\partial \Phi_2'}{\partial y} = 0
\]

Figure 6: Solenoidal element

As it happens, it is not possible to construct a conforming element which is point-wise solenoidal, that is, one which satisfies Equation (10) at all points in the element. A compromise, then, is to impose the constraint on the average over the element, such that

\[
\int_{A_e} \left( \frac{\partial \Phi_1'}{\partial x} + \frac{\partial \Phi_2'}{\partial y} \right) dA = 0.
\]

(23)

This is achieved through the use of the Gauss points for a cubic polynomial, at each of the six of which Equation (10) is satisfied. In this manner, the constraint is imposed both in a collocation sense and in an integral sense. While this represents a relaxation of the constraint, this is the case with all discretised approaches. Equation (23) is a relaxation of Equation (10), just as Equation (16) is a relaxation of Equation (8). It is just that Equation (16) uses the Galerkin method while Equation (23), in effect, uses the subdomain method. The 6-node element configuration is shown in Figure 6. For this element, Equation (15) takes the form
\[
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix} =
\begin{bmatrix}
N_{11} & N_{12} & 0 \\
N_{21} & N_{22} & 0 \\
0 & 0 & N_{33}
\end{bmatrix}
\begin{bmatrix}
\Phi_1' \\
\Phi_2' \\
\Phi_3'
\end{bmatrix},
\]  

(24)

with the notable feature that the local behaviour is now cross-coupled.

6 Computer code

The finite element approach described here forms the basis of the SOLWAVE computer code. This computer code produces, for either polarisation, solutions for the field intensity components and the radar cross-section. Numerical integration is by means of Gauss quadrature. A sparse scheme is used for the storage of the coefficient matrix. Solution is by means, either of mass lumping and matrix iteration, or of a Lanczos scheme with symmetric successive over-relaxation pre-conditioning. Artificial diffusion is employed to compensate for the truncation of the far-field boundary. The radar cross-section is computed from the field intensity components or, more precisely, from their amplitude and their phase. For the details of this calculation, the reader is referred to the discussion in [9].

7 Sample results

The validity of the solenoidal element approach is tested by the application of the code to cylinders of various cross-section, starting with the benchmark problem of a circular cylinder. Cylindrical structures form the basis of many practical scatterers, such as the fuselages of aircraft and missiles. The circular cylinder also possesses an analytical solution for the radar cross-section, in terms of Bessel functions \(^1\). Other cylinders of interest, for the reasons that are mentioned in the discussion on radar cross-sections, are a square plus a diamond.

For all problems, the computational domain is bounded by a circle with a diameter of 8. In between this boundary and the particular object are generated quadrilateral elements, subdivided into triangles in such a way as to not bias the mesh. While the meshes used in this instance are structured, this is purely for ease of convergence testing. For a finite element approach, a mesh can be as unstructured as desired. In all cases, the incident wave comes from the left.

Circular cylinder

The diameter of the circular cylinder is 2. Figure 7 shows a mesh of 120 elements in the circumferential direction and 20 elements in the radial direction. Figure 8 shows a mesh of 240 elements in the circumferential direction and 40 elements in the radial direction. Unless stated otherwise, for all results, the wave length of the incident wave is 2/3 while the elapsed time is 2. Different values are employed for later results.

Figure 9 and Figure 10 relate to the conventional element. Figure 11 and Figure 12 relate to the solenoidal element. All represent the scattered field and use the finer mesh. In comparison with Figure 12, Figure 10 shows what appear to be quite reasonable field contours. However Figure 9 illustrates that, even with the finer mesh, the conventional
element produces not inconsiderable field divergence errors, in particular near the surface of the cylinder and in the wake. This reinforces the earlier statement about what happens with no explicit imposition of the divergence constraint. Figure 12 shows the high resolution obtained with the solenoidal element, firstly for the same third component and also for the other two.

![Image](image_url)

**Figure 7**: 120*20 mesh for circular cylinder

![Image](image_url)

**Figure 8**: 240*40 mesh for circular cylinder

The next two figures compare the radar cross-section from the numerical simulations with the analytical solution. These demonstrate that the radar cross-section is quite sensitive to any errors in the solution on the surface of the cylinder. On the other hand, this provides cause for much confidence in the solution when discrepancies are small. Figure 13 clearly shows the improvement in the solution for each element with mesh refinement, as is to be expected. Figure 14 shows that, for each mesh, the solenoidal element is more accurate than the conventional element. This means that, for similar accuracy, a coarser mesh can be employed with the solenoidal element. This again is not unexpected since, for a start, the solenoidal element is of higher order. In relation to this improvement, it would thus be useful to investigate how much is due to the order of the polynomial and how much is due to the imposition of the constraint.
Figure 9: Element-level field divergence with conventional element and 240*40 mesh

Figure 10: Component-3 field contours with conventional element and 240*40 mesh

Figure 11: Field vectors with solenoidal element and 240*40 mesh
Figure 12: Field contours with solenoidal element and 240*40 mesh

TE polarisation

TM polarisation
Figure 13: Radar cross-section ($\sigma$ versus $\theta$) as function of mesh
Figure 14: Radar cross-section ($\sigma$ versus $\theta$) as function of element

**TE polarisation**

**TM polarisation**
Square cylinder

The length of each face of the square cylinder is 2. Figure 15 shows a mesh of 240 elements in the circumferential direction and 40 elements in the radial direction.

![Figure 15: 240*40 mesh for square cylinder](image)

Diamond cylinder

The length of each face of the diamond cylinder is 2. Figure 16 shows a mesh of 240 elements in the circumferential direction and 40 elements in the radial direction.

![Figure 16: 240*40 mesh for diamond cylinder](image)

Discussion

The colour plates and the final two figures clearly confirm the earlier comments on electromagnetic scattering and radar cross-sections. The circular cylinder scatters the incident wave, both back to its source and, to a lesser extent, off in all directions. The square cylinder scatters the incident wave almost entirely along the axis of its source. The diamond cylinder...
scatters the incident wave most noticeably off to the side, although the sharp leading edge also produces a spike. These characteristics are also visible in the associated radar cross-sections. Hence the favouring of shapes for various parts of stealth aircraft which present a diamond-like plan view towards the expected direction of an enemy radar, that is, towards the front quarters and the direct rear.

8 Conclusions

This report presents a new application for solenoidal elements in computational mechanics, namely in computational electromagnetics. Here, the element provides a means for the satisfaction of a constraint which appears to be in excess of that required for a unique solution. In fact, this constraint is already implied by the other equations but, while this is true in an analytical sense, this is not necessarily so for the numerical solution. The solenoidal element ensures that it is.

So, what are the main features of the solenoidal element. It is conforming. Due to the imposition of the solenoidal constraint, it has cross-coupled local behaviour, which appears to be unique. Since a conforming point-wise solenoidal element is not possible, the constraint needs to be relaxed in some sense. The element presented here is solenoidal in a collocation sense and, since the collocation points are the Gauss points, it is also solenoidal in an integral sense.

The solenoidal element is incorporated into a two-step second-order Taylor-Galerkin procedure. To demonstrate its value, field intensity solutions are provided for various test problems while a detailed study is performed on the circular cylinder. Comparisons are made between a conventional element and the solenoidal element. Also, comparisons are made with the analytical solution for the radar cross-section.

The report demonstrates that a solenoidal element approach is not only possible but also desirable. Its employment constitutes an added bonus in that it forces a constraint to be satisfied which might otherwise have had to be ignored. The divergence does not even have to be zero. As recognised by previous researchers, controlling field divergence is vital for accurate simulation of electromagnetic problems. In fact, the solenoidal element approach is even better suited to electromagnetic applications than to fluid dynamic applications.

The ultimate goal of this research is to develop an approach for the numerical solution of problems which have divergence-like conservation constraints, zero or not, which are not able to be incorporated in an explicit manner into the solution scheme of more conventional approaches. Electromagnetics is just one such example. Even in electromagnetics, the application of the Maxwell equations seems to be inexhaustible. A prodigious number of numerical applications await researchers who are prepared to tackle them.

9 Acknowledgement

The author wishes to express his sincere appreciation to Professor L.N. Trefethen for the opportunity to undertake this research at the Oxford University Computing Laboratory.
Figure 17: Radar cross-section with solenoidal element for $\frac{\lambda}{1} = \frac{1}{3}$, $t = \frac{5}{3}$
Figure 18: Radar cross-section with solenoidal element for $\lambda=2/3$, $t=5/3$
References


Plate 1: Component-3 field contours with solenoidal element for $\lambda=1/3$, $t=5/3$
circular cylinder

square cylinder

diamond cylinder

<table>
<thead>
<tr>
<th>TE polarisation</th>
<th>TM polarisation</th>
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Plate 2: Component-3 field contours with solenoidal element for $\lambda=2/3$, $t=5/3$