One-parameter discontinuous Galerkin finite element discretisation of quasilinear parabolic problems

Andris Lasis Endre Süli

We consider the analysis of a one-parameter family of $hp$-version discontinuous Galerkin finite element methods for the numerical solution of quasilinear parabolic equations of the form $u' - \nabla \cdot \{a(x, t, |\nabla u|)\nabla u\} = f(x, t, u)$ on a bounded open set $\Omega \subset \mathbb{R}^d$, subject to mixed Dirichlet and Neumann boundary conditions on $\partial \Omega$. It is assumed that $a$ is a real-valued function which is Lipschitz-continuous and uniformly monotonic in its last argument, and $f$ is a real-valued function which is locally Lipschitz-continuous and satisfies a suitable growth condition in its last argument; both functions are measurable in the first and second arguments. For quasi-uniform $hp$-meshes, if $u \in H^1(0, T; H^k(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ with $k \geq \frac{3}{2}$, for discontinuous piecewise polynomials of degree not less than 1, the approximation error, measured in the broken $H^1$ norm, is proved to be the same as in the linear case: $O(h^{s-1}/p^{k-3/2})$ with $1 \leq s \leq \min\{p+1, k\}$.

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Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England  OX1 3QD
E-mail: andris.lasis@comlab.oxford.ac.uk

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1 Introduction

Let \( \Omega \) be a bounded open polyhedral domain in \( \mathbb{R}^d, \ d \geq 2 \), with Lipschitz-continuous boundary. We consider the quasilinear parabolic partial differential equation

\[
  u' - \nabla \cdot \{ a(x, t, |\nabla u|)\nabla u \} = f(x, t, u) \quad \text{in} \quad \Omega \times (0, T],
\]

where \( u' := \partial u/\partial t \) and \( T > 0 \).

We shall assume throughout that \( a \) and \( f \) are real–valued functions defined on \( \Omega \times (0, T] \times \mathbb{R} \) which satisfy the following assumptions.

(A) \( a(\cdot, \cdot, v): (x, t) \in \Omega \times (0, T] \mapsto a(x, t, v) \in \mathbb{R} \) is measurable in \( (x, t) \in \Omega \times (0, T] \) for all \( v \in \mathbb{R} \) and \( a(x, t, \cdot): v \in \mathbb{R}_+ \mapsto a(x, t, v) \in \mathbb{R} \) is Lipschitz-continuous and uniformly monotonic for a.e. \((x, t) \in \Omega \times (0, T] \), in the sense that there exist positive constants \( m_a \) and \( M_a \) such that

\[
m_a(w - v) \leq a(x, t, w)w - a(x, t, v)v \leq M_a(w - v) \quad \left\{ \begin{array}{l}
  w \geq v \geq 0, \\
  \text{a.e. } (x, t) \in \Omega \times (0, T].
\end{array} \right.
\]

(B) \( f(\cdot, \cdot, v): (x, t) \in \Omega \times (0, T] \mapsto f(x, t, v) \in \mathbb{R} \) is measurable in \( (x, t) \in \Omega \times (0, T] \) for all \( v \in \mathbb{R} \), with \( f(x, t, 0) = 0 \) for all \( (x, t) \in \Omega \times (0, T] \), and the mapping \( f(x, t, \cdot): v \in \mathbb{R} \mapsto f(x, t, v) \in \mathbb{R} \) is locally Lipschitz-continuous for a.e. \((x, t) \in \Omega \times (0, T] \), in the sense that there exist real numbers \( G_f > 0 \) and \( \gamma \geq 0 \) such that

\[
|f(x, t, w) - f(x, t, v)| \leq G_f (1 + |w| + |v|)^\gamma |w - v| \quad \left\{ \begin{array}{l}
  \forall w, v \in \mathbb{R}, \\
  \text{a.e. } (x, t) \in \Omega \times (0, T].
\end{array} \right.
\]

We shall suppose that \( 0 \leq \gamma < \infty \) when \( d = 2 \), and \( 0 \leq \gamma \leq 2/(d - 2) \) when \( d \geq 3 \). The trivial case of \( \gamma = 0 \) corresponds to assuming that the function \( f \) is globally Lipschitz-continuous in its third argument.

When the function \( a \) satisfies (1.2), it follows from [3], Lemma 2.1 (for the case \( d = 2 \), the case of \( d > 2 \) being analogous) that there exist positive constants \( C_1 \) and \( C_2 \) such that for all \( d \times d \) real symmetric matrices \( \mathbf{Y} \) and \( \mathbf{Z} \), and a.e. \((x, t) \in \Omega \times (0, T] \),

\[
|a(x, t, |\mathbf{Y}|)\mathbf{Y} - a(x, t, |\mathbf{Z}|)\mathbf{Z}| \leq C_1 |\mathbf{Y} - \mathbf{Z}|, \tag{1.4}
\]

\[
C_2 |\mathbf{Y} - \mathbf{Z}|^2 \leq (a(x, t, |\mathbf{Y}|)\mathbf{Y} - a(x, t, |\mathbf{Z}|)\mathbf{Z}) : (\mathbf{Y} - \mathbf{Z}). \tag{1.5}
\]

By choosing \( \mathbf{Y} = \text{diag}(y_1, \ldots, y_d) \) and \( \mathbf{Z} = \text{diag}(z_1, \ldots, z_d) \), (1.4) and (1.5) hold for \( \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^d \) with \( |\cdot| \) signifying the Euclidean norm on \( \mathbb{R}^d \).

Let \( \partial \Omega \) denote the union of all \((d - 1)\)-dimensional open faces of the polyhedron \( \Omega \). Upon decomposing \( \partial \Omega \) into two parts, \( \Gamma_D \) and \( \Gamma_N \), so that \( \Gamma_D \cup \Gamma_N = \partial \Omega \) and \( \Gamma_D \) has positive \((d - 1)\)-dimensional Hausdorff measure, and denoting by \( \nu = (\nu_1, \ldots, \nu_d)^\top \) the unit outward normal vector to \( \partial \Omega \), we impose Dirichlet and Neumann boundary conditions on \( \Gamma_D \) and \( \Gamma_N \), respectively:

\[
  u = g_D \quad \text{on} \quad \Gamma_D \times (0, T],
\]

\[
  \{a(x, t, |\nabla u|)\nabla u\} \cdot \nu = g_N \quad \text{on} \quad \Gamma_N \times (0, T]. \tag{1.6}
\]
with \( g_D \in H^{1/2}(\Gamma_D) \) and \( g_N \in L^2(\Gamma_N) \). Given a function \( u_0 \in L^2(\Omega) \), we supplement (1.1) and (1.6) with the initial condition

\[
\text{on } \Omega \times \{0\}.
\]

In what follows, for the sake of simplicity when defining the trace of the function \( \{a(x,t,|\nabla u|)\nabla u\} \cdot \nu \) on the Neumann boundary \( \Gamma_N \), we shall assume that \( a(\cdot,t,v) \) is piecewise continuous for a.e. \( t \in (0,T] \) and all \( v \in \mathbb{R} \).

**Remark 1.1** In the case of a linear differential operator \( \mathcal{L} u := \nabla \cdot \{a(x)\nabla u\} \), it is possible to generalise the results on existence of the trace operator \( \gamma_a u = \{a(x)\nabla u\} \cdot \nu \) to the case when \( a \in L^\infty(\Omega) \) and a solution of the boundary value problem belongs to the space \( H^1_L := \{w \in H^1(\Omega), \mathcal{L} w \in L^2(\Omega)\} \) (see [2]): then, there is no need to assume piecewise continuity of \( a \). The extension of this result to the case of a nonlinear operator \( \mathcal{L} \) is a nontrivial exercise: for example, \( H^1_L \) is no longer a linear space.

We shall label the problem described by (1.1), (1.6), and (1.7), with (P). As the solution to the problem (P) may exhibit blow-up in finite time, we shall assume that, for the potential blow-up time \( T^* \in (0,\infty) \), the time interval \([0,T]\) on which the problem is considered excludes the blow-up time, i.e., \( T < T^* \).

In this work we shall be concerned with the error analysis of a one-parameter family of \( hp \)-version discontinuous Galerkin finite element methods (DGFEMs), parametrised by \( \theta \in [-1,1] \), for the initial-boundary value problem (P). In particular, we focus on the spatial semidiscretisation of the problem.

This work is an extension of our previous analysis of DGFEMs for semilinear parabolic problems [9]; it also extends the recent work by Houston, Robson & Süli to quasilinear elliptic boundary value problems [8].

The paper is structured as follows. In Section 2 we state the broken weak formulation of the problem. The error analysis of the \( hp \)-DGFEM for quasilinear parabolic equations is discussed in Section 3. We begin by recalling from [9] the local Lipschitz-continuity of the mapping \( w \mapsto f(\cdot,t,w(\cdot,t)) : L^q(\Omega) \to L^2(\Omega) \), followed by results on the Lipschitz continuity and uniform monotonicity of the semilinear form \( B : S^p(\Omega,T_h\mathbf{F}) \times S^p(\Omega,T_h\mathbf{F}) \to \mathbb{R} \) appearing in the broken weak formulation of the initial-boundary value problem under consideration. Section 3.2 contains the error analysis of the \( hp \)-DGFEM: we prove an \emph{a priori} error bound in the broken \( H^1 \) norm that is \( h \)-optimal and \( p \)-suboptimal (by half an order of \( p \)). Full \( hp \)-optimality of the error bound in the broken \( H^1 \) norm can be easily restored by hypothesising piecewise regularity of the solution in augmented Sobolev spaces instead of classical Sobolev spaces, as was done in [7] in the elliptic case; for the sake of brevity we shall not pursue this line of study here since the necessary modifications are quite straightforward. Unlike the case when the function \( f \) is globally Lipschitz-continuous, corresponding to the particular choice of \( \gamma = 0 \) in (1.3), for \( \gamma > 0 \) a broken counterpart of the Sobolev–Poincaré inequality has to be used to complete the error analysis of the \( hp \)-DGFEM. The variant of the Sobolev–Poincaré inequality considered here is inspired by the work of Brenner [5]. As in [9], a further ingredient of our error analysis is an adaptation to discontinuous Galerkin methods of a continuity-argument due to Thomée and Wahlbin (cf. pp. 382–384 in [13]).
2 Broken weak formulation

Throughout the paper, we let $W^{s,q}(\Omega)$ signify the usual Sobolev space equipped with the norm $\| \cdot \|_{s,q,\Omega}$ and seminorm $| \cdot |_{s,q,\Omega}$. In the case when $q = 2$, we shall write $H^s(\Omega) := W^{s,2}(\Omega)$ and suppress the index $q$ in the notation of the norm and seminorm, writing $\| \cdot \|_{s,\Omega}$ and $| \cdot |_{s,\Omega}$, respectively. For a Banach space $X$ equipped with a norm $\| \cdot \|$, the space $L^q(0,T;X)$ consists of all strongly measurable functions $u : [0,T] \to X$ with the norm
\[
\|u\|_{L^q(0,T;X)} := \left( \int_0^T \|u(t)\|^q \, dt \right)^{1/q} < \infty \quad \text{for} \quad 1 \le q < \infty,
\]
and
\[
\|u\|_{L^\infty(0,T;X)} := \sup_{0 \le t \le T} \|u(t)\| < \infty \quad \text{for} \quad q = \infty.
\]
The Sobolev space $W^{1,q}(0,T;X)$ consists of all functions $u \in L^q(0,T;X)$ such that $\dot{u}$ exists in the weak sense and belongs to $L^q(0,T;X)$, with the associated norm
\[
\|u\|_{W^{1,q}(0,T;X)} := \left( \int_0^T \left( \|u(t)\|^q + \|\dot{u}(t)\|^q \right) \, dt \right)^{1/q} < \infty \quad \text{for} \quad 1 \le q < \infty,
\]
and
\[
\|u\|_{W^{1,\infty}(0,T;X)} := \sup_{0 \le t \le T} \|u(t)\| + \|\dot{u}(t)\|.
\]
In the context of the initial-boundary value problem in consideration, $u(t) = u(\cdot,t)$; with a slight abuse of notation, we shall simply write $u$ in place of $u$. Also, for the sake of brevity, we shall write $H^1(0,T;X) := W^{1,2}(0,T;X)$.

Let $\mathcal{T}_h$ be a subdivision of $\Omega$ into disjoint open elements $\kappa$ such that $\Omega = \cup_{\kappa \in \mathcal{T}_h} \hat{\kappa}$, where $\mathcal{T}_h$ is regular or 1–irregular, i.e., each face of $\kappa$ has at most one hanging node. We let $h_\kappa := \text{diam}(\hat{\kappa})$ and $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$; it will be assumed throughout that $h \le 1$. We assume that the family of subdivisions $\{\mathcal{T}_h\}$ is shape-regular (see pages 61, 113, and Remark 2.2 on page 114 in [4]), and require each $\kappa \in \mathcal{T}_h$ to be an affine image of a fixed master element $\hat{\kappa}$, i.e., $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in \mathcal{T}_h$, where $\hat{\kappa}$ is the open unit simplex or the open unit hypercube in $\mathbb{R}^d$. For a non-negative integer $p$, we denote by $P_p(\hat{\kappa})$ the set of polynomials of total degree $p$ on $\hat{\kappa}$; if $\hat{\kappa}$ is the open unit hypercube in $\mathbb{R}^d$ we also consider $Q_p(\hat{\kappa})$, the set of all tensor-product polynomials on $\hat{\kappa}$ of degree $p$ in each coordinate direction. To each $\kappa \in \mathcal{T}_h$ we assign a non-negative integer $p_\kappa$ (the local polynomial degree) and a non-negative integer $s_\kappa$ (the local Sobolev index), collect the $p_\kappa$, $s_\kappa$ and $F_\kappa$ into vectors $p = \{p_\kappa : \kappa \in \mathcal{T}_h\}$, $s = \{s_\kappa : \kappa \in \mathcal{T}_h\}$ and $F = \{F_\kappa : \kappa \in \mathcal{T}_h\}$, respectively, and consider the finite element space
\[
S^p(\Omega, \mathcal{T}_h, F) := \left\{ v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\hat{\kappa}), \kappa \in \mathcal{T}_h \right\}, \tag{2.1}
\]
where $\mathcal{R}$ is either $\mathcal{P}$ or $\mathcal{Q}$ if $\hat{\kappa}$ is the open unit hypercube, and $\mathcal{R}$ is $\mathcal{P}$ if $\hat{\kappa}$ is the open unit simplex in $\mathbb{R}^d$. We shall refer to the following number as the number of degrees of freedom, or dimension, of the finite element space:

$$\delta := \dim(S^p(\Omega, T_h, F)) = \sum_{\kappa \in T_h} (p_\kappa + 1)^d.$$ 

We shall assume that the polynomial degree vector $p$, with $p_\kappa \geq 1$ for each $\kappa \in T_h$, has bounded local variation, i.e. there exists a constant $\rho \geq 1$, independent of $h$, such that, for any pair of elements $\kappa$ and $\kappa'$ in $T_h$ which share a $(d-1)$-dimensional face,

$$\rho^{-1} \leq p_\kappa/p_{\kappa'} \leq \rho. \quad (2.2)$$

We assign to the subdivision $T_h$ the broken Sobolev space of composite order $s$,

$$W^{s,q}(\Omega, T_h) := \left\{ u \in L^q(\Omega) : u|_\kappa \in W^{s,q}(\kappa) \text{ for all } \kappa \in T_h \right\},$$

equipped with the broken Sobolev norm and seminorm, respectively,

$$\|u\|_{s,q,T_h} := \left( \sum_{\kappa \in T_h} \|u\|_{s,q,\kappa}^q \right)^{1/q}, \quad |u|_{s,q,T_h} := \left( \sum_{\kappa \in T_h} |u|_{s,q,\kappa}^q \right)^{1/q}.$$ 

When $s_\kappa = s$ for all $\kappa \in T_h$, we write $W^{s,q}(\Omega, T_h)$, $\|u\|_{s,q,T_h}$, $|u|_{s,q,T_h}$, and for $q = 2$ we let $H^s := W^{s,2}$, and omit the index $q$ in the notations of the norm and seminorm.

Let $\mathcal{E}$ denote the set of all open $(d-1)$-dimensional faces of the subdivision $T_h$, containing the smallest common $(d-1)$-dimensional interfaces $e$ of neighbouring elements. We denote by $\mathcal{E}_{\text{int}}$ the set of all faces in $\mathcal{E}$ that are contained in $\Omega$, and let $\Gamma_{\text{int}} := \{ x \in \Omega : x \in e \text{ for some } e \in \mathcal{E}_{\text{int}} \}$. Further, we denote by $\mathcal{E}_\partial$ the set of all $(d-1)$-dimensional boundary faces. Assuming that each $e \in \mathcal{E}_\partial$ belongs to the interior of exactly one of $\Gamma_D$ and $\Gamma_N$, we label the associated sets of faces by $\mathcal{E}_D$ and $\mathcal{E}_N$.

Given that $e \in \mathcal{E}_{\text{int}}$, there exist positive integers $i, j$ such that $i > j$ and $\kappa_i$ and $\kappa_j$ share the face $e$; we define the jump of $v \in W^{s,q}(\Omega, T_h)$, $s_\kappa > 1/q$, $\kappa \in T_h$, across $e$ and the mean value of $v$ on $e$ by

$$[v]_e := v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e} \quad \text{and} \quad \{v\}_e := \frac{1}{2} \left( v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e} \right),$$
respectively, with $\partial\kappa$ denoting the union of all open faces of the element $\kappa$. With each face $e$ we associate the unit normal vector $\nu$ pointing from the element $\kappa_i$ to $\kappa_j$ when $i > j$; when the face belongs to $\mathcal{E}_\partial$, we choose $\nu$ to be the unit outward normal vector.

With these notations, we introduce the semilinear form

$$B(w, v) := \sum_{\kappa \in T_h} \int_{\kappa} a(x, t, |\nabla w|) \nabla w \cdot \nabla v \, dx$$

$$- \int_{\Gamma_D} a(x, t, |\nabla w|) \frac{\partial w}{\partial \nu} v \, ds - \int_{\Gamma_{\text{int}}} \left\{ a(x, t, |\nabla w|) \frac{\partial w}{\partial \nu} \right\} [v] \, ds$$
$$+ \theta \int_{\Gamma_D} a(x, t, h^{-1} |w - g_D|) \frac{\partial v}{\partial \nu} (w - g_D) \, ds + \theta \int_{\Gamma_{\text{int}}} \left\{ a(x, t, h^{-1} |w|) \frac{\partial v}{\partial \nu} \right\} [w] \, ds$$
$$+ \int_{\Gamma_D} \sigma w v \, ds + \int_{\Gamma_{\text{int}}} \sigma [w][v] \, ds,$$ 

$$\quad (2.3)$$
and the linear functional $\ell(\cdot)$ by

$$\ell(v) := \int_{\Gamma_N} g_N v \, ds + \int_{\Gamma_D} \sigma g_D v \, ds.$$  

(2.4)

Here $\sigma$ is called the \textit{discontinuity-penalisation parameter} and is defined by

$$\sigma|_e = \sigma_e \quad \text{for} \quad e \in E_{\text{int}} \cup E_B,$$

where $\sigma_e$ is a non-negative constant on the face $e$:

$$\sigma|_e = \sigma_e = \alpha \frac{\{p^2\}_e}{h_e} \quad \text{for} \quad e \in \Gamma_{\text{int}} \cup \Gamma_D.$$  

(2.5)

If $e \subset \Gamma_D$, and thus for some $\kappa \in \mathcal{T}_h$ we have $e \subset \partial\kappa \cap \Gamma_D$, then $\{p^2\}_e = p^2$, i.e., for $e \in E_B$ the contribution from outside $\Omega$ in the definition of $\sigma_e$ is set to 0. Here $\alpha$ is a positive constant whose size will be fixed later. The subscript $e$ in these definitions will be suppressed when no confusion is likely to occur. The parameter $\theta$ takes its values in the interval $[-1, 1]$.

In (2.3) the role of integrals multiplied by $\theta$ is, respectively, to weakly and approximately impose the Dirichlet boundary condition $u = g_D$ on $\Gamma_D$ and the continuity condition $[u] = 0$ on $\Gamma_{\text{int}}$, which is satisfied by the exact solution $u$. The choice of factors $a(x, t, h^{-1}|w - g_D|)$ and $a(x, t, h^{-1}||w||)$ in the corresponding integrals has been guided by the following considerations: when the problem is linear, the scheme collapses to a standard $hp$–DGFEM; to make use of monotonicity condition (A), the arguments of $a$ in the two relevant terms should be multiples of $|w - g_D|$ and $||w||$, respectively; the factor $h^{-1}$ chosen for reasons of scaling so that the terms $a(x, t, h^{-1}|w - g_D|)$ and $a(x, t, h^{-1}||w||)$ resemble $a(x, t, |\nabla w|)$.

Then, the broken weak formulation of the problem (P) reads:

$$\text{find } u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; \mathfrak{A}) \text{ such that}$$

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} u'v \, dx + \mathcal{B}(u, v) - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} f(x, t, u)v \, dx = \ell(v), \quad \text{for all } v \in H^1(\Omega, \mathcal{T}_h),$$

(2.6)

$$u(0) = u_0,$$

where by $\mathfrak{A}$ we denote the function space

$$\mathfrak{A} = \{ w \in H^2(\Omega, \mathcal{T}_h): w, \{a(x, t, |\nabla w|)\nabla w\} \cdot \nu \text{ are continuous across each } e \in E_{\text{int}} \}. $$

Note that if $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ is a weak solution of (P) and $u \in L^2(0, T; \mathfrak{A})$, then it also solves (2.6); in the sequel we shall always assume that such a $u$ exists.

The $hp$–DGFEM approximation of the problem (P) is as follows:

$$\text{find } u_{DG} \in H^1(0, T; S^p(\Omega, \mathcal{T}_h, F)) \text{ such that}$$

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} u_{DG}'v \, dx + \mathcal{B}(u_{DG}, v) - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} f(x, t, u_{DG})v \, dx = \ell(v), \quad \text{for all } v \in S^p(\Omega, \mathcal{T}_h, F),$$

(2.7)

$$u_{DG}(0) = u_0^{DG}.$$
where $u_0^{DG}$ denotes an approximation of the function $u_0$ from the finite element space $SP(\Omega, T_h, \mathbf{F})$. The purpose of this paper is to quantify the size of the error between the solution $u$ to (2.6) and its $hp$–DGFEM approximation $u_{DG}$.

The equation (2.7) can be interpreted as a system of ordinary differential equations in $t$ for the coefficients in the expansion of $u_{DG}(\cdot, t)$ in terms of basis functions of the finite-dimensional space $SP(\Omega, T_h, \mathbf{F})$. Thus, (2.7) defines a nonautonomous system of ordinary differential equations with locally Lipschitz-continuous right-hand side, given that $a(x, t, \cdot)$ is Lipschitz-continuous and $f(x, t, \cdot)$ is locally Lipschitz-continuous, uniformly in $(x, t) \in \Omega \times (0, T]$, and the other terms are linear. By Carathéodory’s theorem (see Theorems II.4.1 and II.4.5 in [14]) this, in turn, implies the existence of a unique local solution to (2.7).

Since no pointwise continuity requirement is imposed on the elements of the finite element space, the approximation $u_{DG}$ in $SP(\Omega, T_h, \mathbf{F})$ to the solution $u$ will be, in general, discontinuous.

3 Error analysis

Before embarking on the analysis of (2.7), we state and prove some preliminary results.

3.1 Preliminary results

We begin by recalling from [9] the local Lipschitz-continuity of $w \mapsto f(\cdot, t, w(\cdot, t))$ as a mapping from $L^q(\Omega)$ to $L^2(\Omega)$.

Lemma 3.1 Suppose that $f$ satisfies (B). Then, there exists a positive constant $C_f = C_f(\gamma, G_f, |\Omega|)$ such that

$$
\|f(\cdot, t, w) - f(\cdot, t, v)\|_{0, \Omega} \leq C_f \|w - v\|_{0,2(\gamma+1), \Omega} \times \left(1 + \|w\|_{0,2(\gamma+1), \Omega}^\gamma + \|v\|_{0,2(\gamma+1), \Omega}^\gamma\right)
$$

(3.1)

for all $w, v \in L^2(\gamma+1)(\Omega)$ and a.e. $t \in (0, T]$.

We equip $H^1(\Omega, T_h)$ with the norm $\|\cdot\|_{1,h}$ defined by

$$
\|w\|_{1,h} := \left(|w|_{1,T_h}^2 + \int_{\Gamma_D} \sigma w^2 \, ds + \int_{\Gamma_{\text{int}}} \sigma [w]^2 \, ds\right)^{1/2},
$$

(3.2)

where $\sigma$ is the positive discontinuity-penalisation parameter introduced after (2.4).

Suppose that $e$ is a $(d-1)$-dimensional face of an element $\kappa \in T_h$. Then, the following inverse inequalities hold: there exists a positive constant $C_{\text{inv}}$, dependent only on the shape-regularity constant of $\{T_h\}$ (see Schwab [12], Theorem 4.76), such that

$$
\|w\|_{0,e}^2 \leq C_{\text{inv}} \frac{p_e^2}{h_e} \|w\|_{0,\kappa}^2, \quad \text{and} \quad \|\nabla w\|_{0,e}^2 \leq C_{\text{inv}} \frac{p_e^2}{h_e} \|\nabla w\|_{0,\kappa}^2,
$$

(3.3)
for all \( w \in S^p(\Omega, T_h, F) \).

The next lemma will play a key role.

**Lemma 3.2 (Broken Sobolev–Poincaré inequality)** There exists a positive constant \( C \) independent of \( h \), such that for any \( q \in [1, \infty) \) when \( d = 2 \) and any \( q \in [1, 2d/(d - 2)] \) when \( d \geq 3 \),

\[
\| w \|_{0,q,\Omega} \leq C \| w \|_{1,h} \quad \forall w \in H^1(\Omega, T_h). \tag{3.4}
\]

**Proof** Noting the definition of the parameter \( \sigma \) in (2.5), from Theorem 3.7 in [10], using the notation therein, we define \( \Psi \) as in Example 3.6 of that paper, with \( \psi \in L^2(\partial \Omega) \), \( \psi \equiv 0 \) on \( \Gamma_N \), and \( |\Psi(\xi)|^2 \leq C \sum_{e \in E_B} \sigma_e \int_{e} \xi^2 \, ds \), to obtain (3.4). \( \blacksquare \)

We recall the following polynomial approximation result by Babuška–Suri projection operator for the space \( S^p(\Omega, T_h, F) \) [6, 11].

**Lemma 3.3** Suppose that \( \kappa \in T_h \) with \( h_{\kappa} = \text{diam}(\kappa) \) and \( u|_{\kappa} \in H^{k_{\kappa}}(\kappa) \) for some \( k_{\kappa} \geq 0 \); then, there exists a sequence of algebraic polynomials \( z_{p_{\kappa}}(u) \in R_{p_{\kappa}}(\kappa), p_{\kappa} \geq 1 \), such that for any \( l, \) with \( 0 \leq l \leq s_{\kappa}, \)

\[
\| u - z_{p_{\kappa}}(u) \|_{l,\kappa} \leq C \frac{h_{\kappa}^{s_{\kappa} - l}}{p_{\kappa}^{s_{\kappa} - l}} \| u \|_{k_{\kappa},\kappa}, \tag{3.5}
\]

and when \( k_{\kappa} > 1/2 \), then

\[
\| u - z_{p_{\kappa}}(u) \|_{0,e_{\kappa}} \leq C \frac{h_{\kappa}^{s_{\kappa} - 1/2}}{p_{\kappa}^{k_{\kappa} - 1/2}} \| u \|_{k_{\kappa},\kappa}, \tag{3.6}
\]

further, if \( k_{\kappa} > 3/2 \), then

\[
\| \nabla(u - z_{p_{\kappa}}(u)) \|_{0,e_{\kappa}} \leq C \frac{h_{\kappa}^{s_{\kappa} - 3/2}}{p_{\kappa}^{k_{\kappa} - 3/2}} \| u \|_{k_{\kappa},\kappa}, \tag{3.7}
\]

where \( e_{\kappa} \) is any face \( e_{\kappa} \subset \partial \kappa \), \( s_{\kappa} = \min \{ p_{\kappa} + 1, k_{\kappa} \} \), and \( C \) is a constant independent of \( u, h_{\kappa}, \) and \( p_{\kappa} \), but dependent on \( k = \max_{\kappa \in T_h} k_{\kappa} \).

**Proof** For the proof of (3.5), see Lemma 4.5 in [1] for \( d = 2 \) (the argument being analogous when \( d > 2 \)). By using the multiplicative trace inequality

\[
\| u \|_{0,\partial \kappa} \leq C(d) \left( h_{\kappa}^{-1/2} \| u \|_{0,\kappa} + \| u \|_{0,\kappa}^{1/2} \| \nabla u \|_{0,\kappa}^{1/2} \right),
\]

we obtain (3.6) and (3.7) from (3.5). \( \blacksquare \)

**Remark 3.4** If the reference element \( \hat{\kappa} \) is the \( d \)-dimensional hypercube, instead of Babuška–Suri projector \( z_{p_{\kappa}}^{\hat{\kappa}} \), we can use the Jackson-type quasi-interpolation operator \( J_{p_{\kappa}}^{\hat{\kappa}} \) (for definition see Chapter 7 in [6]; the error bounds are presented in Theorem A.3 in [11]).
We continue by investigating the Lipschitz-continuity and uniform monotonicity of the semifilinear form $B(\cdot, \cdot)$ in the norm $\| \cdot \|_{1,h}$ over the space $S^p(\Omega, T_h, F) \times S^p(\Omega, T_h, F)$.

**Lemma 3.5** The semiflinear form $B(\cdot, \cdot)$ is Lipschitz-continuous in its first argument in the sense that

$$
|B(w_1, v) - B(w_2, v)| \leq L \|w_1 - w_2\|_{1,h} \|v\|_{1,h} \quad \forall w_1, w_2, v \in S^p(\Omega, T, F),
$$

(3.8)

where $L = \max \{C_1, 1\} + (1 + |\theta|)C_1(C_{inv}\alpha^{-1}c_d)^{\frac{1}{2}}$, $\theta \in [-1, 1]$, $\alpha > 0$, and $c_d = 2^d$.  

**Proof** (See also [8].) First, we note that, by the fact that $|\partial v/\partial \nu| = |\nabla v \cdot \nu| \leq |\nabla v|$

$$
|B(w_1, v) - B(w_2, v)| \leq \sum_{\kappa \in T_h} \int_{\kappa} |a(x, t, |\nabla w_1|)\nabla w_1 - a(x, t, |\nabla w_2|)\nabla w_2| |\nabla v| \, dx
$$

$$
+ \int_{\Gamma_D} |a(x, t, |\nabla w_1|)\nabla w_1 - a(x, t, |\nabla w_2|)\nabla w_2| |v| \, ds
$$

$$
+ \int_{\Gamma_{int}} |\{a(x, t, |\nabla w_1|)\nabla w_1 - a(x, t, |\nabla w_2|)\nabla w_2\}| |\nabla v| \, ds
$$

$$
+ |\theta| \int_{\Gamma_D} |a(x, t, h^{-1}|w_1 - g_D|)(w_1 - g_D) - a(x, t, h^{-1}|w_2 - g_D|)(w_2 - g_D)| |\nabla v| \, ds
$$

$$
+ |\theta| \int_{\Gamma_{int}} |\{a(x, t, h^{-1}|w_1|)\}w_1 - \{a(x, t, h^{-1}|w_2|)\}w_2\}| |\nabla v| \, ds
$$

$$
+ \int_{\Gamma_D} \sigma |w_1 - w_2| |v| \, ds + \int_{\Gamma_{int}} \sigma |\nabla w_1 - w_2| |v| \, ds
$$

$$
=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
$$

By (1.4) and the Cauchy–Schwarz inequality, we have

$$
I_1 \leq C_1 \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla(w_1 - w_2)|^2 \, dx \right)^{\frac{1}{2}} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{\frac{1}{2}}.
$$

By the same argument,

$$
I_2 \leq C_1 \int_{\Gamma_D} |\nabla(w_1 - w_2)| |v| \, ds
$$

$$
\leq C_1 \left( \int_{\Gamma_D} \sigma^{-1} |\nabla(w_1 - w_2)|^2 \, ds \right)^{\frac{1}{2}} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{\frac{1}{2}}.
$$

Using the second inverse inequality in (3.3) and recalling the definition of the parameter $\sigma_e$ on $e \subset \Gamma_D$, we obtain

$$
I_2 \leq C_1(C_{inv}\alpha^{-1})^{\frac{1}{2}} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla(w_1 - w_2)|^2 \, ds \right)^{\frac{1}{2}} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{\frac{1}{2}}.
$$
Analogously,
\[ I_3 \leq C_1 \left( \int_{\Gamma_{\text{int}}} \sigma^{-1} \{ |\nabla (w_1 - w_2)| \} \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} |v|^{2} \, ds \right)^{1/2}. \]

We note that
\[ \int_{\Gamma_{\text{int}}} \sigma^{-1} \{ |\nabla (w_1 - w_2)| \} \, ds = \sum_{e \in E_{\text{int}}} \sigma^{-1}_e \int_{e} \{ |\nabla (w_1 - w_2)| \} \, ds; \]
for \( e \in E_{\text{int}} \), let \( \kappa \) and \( \kappa' \) be two elements sharing the face \( e \). Then we have
\[ \int_{e} \{ |\nabla (w_1 - w_2)| \} \, ds \leq \frac{1}{2} \int_{e} |\nabla (w_1 - w_2)|_{\kappa}^2 \, ds + \frac{1}{2} \int_{e} |\nabla (w_1 - w_2)|_{\kappa'}^2 \, ds \]
\[ \leq C_{\text{inv}} \frac{p^2_{\kappa}}{2h_{\kappa}} \int_{\kappa} |\nabla (w_1 - w_2)|^2 \, dx + C_{\text{inv}} \frac{p^2_{\kappa'}}{2h_{\kappa'}} \int_{\kappa'} |\nabla (w_1 - w_2)|^2 \, dx \]
\[ \leq C_{\text{inv}} \frac{\{ p^2 \}^c_{e}}{h_{e}} \max \{ \int_{\kappa} |\nabla (w_1 - w_2)|^2 \, dx, \int_{\kappa'} |\nabla (w_1 - w_2)|^2 \, dx \}, \]
and, recalling the definition of \( \sigma \), we have that
\[ \sum_{e \in E_{\text{int}}} \sigma^{-1}_e \int_{e} \{ |\nabla (w_1 - w_2)| \} \, ds \leq C_{\text{inv}} \alpha^{-1} \sum_{e \in E_{\text{int}}} \max \{ h_{\kappa}, c_{\kappa} \} \int_{\kappa} |\nabla (w_1 - w_2)|^2 \, dx. \]

Thanks to our assumption that no face \( e \) of any element \( \kappa \in T_h \) contains more than one hanging node, it follows that no element \( \kappa \) can have more than \( 2d \cdot 2^{d-1} = 2^d d \) faces if \( \hat{\kappa} \) is the \( d \)-dimensional hypercube, or more than \( (d + 1)d \) faces if \( \hat{\kappa} \) is the \( d \)-dimensional simplex. Denoting \( c_d := \max \{ 2^dd, (d + 1)d \} = 2^d d \), we have that
\[ \sum_{e \in E_{\text{int}}} \sigma^{-1}_e \int_{e} \{ |\nabla (w_1 - w_2)| \} \, ds \leq C_{\text{inv}} \alpha^{-1} c_d \sum_{\kappa \in T_h} \int_{\kappa} |\nabla (w_1 - w_2)|^2 \, dx, \]
and hence
\[ I_3 \leq C_1 (C_{\text{inv}} \alpha^{-1} c_d)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla (w_1 - w_2)|^2 \, dx \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} |v|^{2} \, ds \right)^{1/2}. \]

For \( I_4 \), by the same argument as for \( I_2 \) (exchanging \( v \) and \( w_1 - w_2 \)), we have
\[ I_4 \leq |\theta| C_1 (C_{\text{inv}} \alpha^{-1})^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2} \left( \int_{\Gamma_{\text{D}}} |w_1 - w_2|^2 \, ds \right)^{1/2}. \]

For \( I_5 \), by the same argument as for \( I_3 \) (exchanging \( v \) and \( w_1 - w_2 \)), we have
\[ I_5 \leq |\theta| C_1 (C_{\text{inv}} \alpha^{-1} c_d)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2} \left( \int_{\Gamma_{\text{D}}} |w_1 - w_2|^2 \, ds \right)^{1/2}. \]
Finally,

\[ I_6 \leq \left( \int_{\Gamma_D} \sigma |w_1 - w_2|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}, \]

and

\[ I_7 \leq \left( \int_{\Gamma_{int}} \sigma |w_1 - w_2|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_{int}} \sigma |v|^2 \, ds \right)^{1/2}. \]

Substituting these bounds into (3.9) and collecting the constants, we obtain

\[ |B(w_1, v) - B(w_2, v)| \leq L \|w_1 - w_2\|_{1,h} \|v\|_{1,h} \quad \text{for all} \quad w_1, w_2, v \in S^p(\Omega, T_h, F), \]

with \( L = \max\{C_1, 1\} + (1 + |\theta|)C_1(C_{inv}^{-1}c_d)^{1/2}. \quad \blacklozenge \)

**Remark 3.6** We note that, if \( \alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_{inv} c_d \), as we shall assume from now on, then \( L \leq \max\{C_1, 1\} + C_1^{1/2} \), and thus we may redefine \( L = \max\{C_1, 1\} + C_1^{1/2}. \)

**Lemma 3.7** Suppose that \( \theta \in [-1, 1] \) and that \( \alpha \geq (1 + |\theta|)^2 C_1^2 C_2^{-1} C_{inv} c_d \); then, the semilinear form \( B(\cdot, \cdot) \) is uniformly monotone in the sense that

\[ B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq \frac{1}{2} \min\{C_2, 1\} \|w_1 - w_2\|_{1,h}^2 \quad (3.10) \]

for all \( w_1, w_2 \in S^p(\Omega, T_h, F). \)

**Proof** (See also [8].) Writing \( w = w_1 - w_2 \) and using (1.5), we have that

\[
B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq C_2 \sum_{\kappa \in T_h} \int_\kappa |\nabla w|^2 \, dx
\]

\[ - C_1 (1 + |\theta|) \left( \int_{\Gamma_D} \sigma^{-1} |\nabla w|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_D} \sigma |w|^2 \, ds \right)^{1/2}
\]

\[ - C_1 (1 + |\theta|) \left( \int_{\Gamma_{int}} \sigma^{-1} |\nabla w|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_{int}} \sigma |w|^2 \, ds \right)^{1/2}
\]

\[ + \int_{\Gamma_D} \sigma |w|^2 \, ds + \int_{\Gamma_{int}} \sigma |w|^2 \, ds.
\]

Proceeding in the same fashion as for \( I_2 \) and \( I_3 \) in the proof of Lemma 3.5, we obtain

\[ \left( \int_{\Gamma_D} \sigma^{-1} |\nabla w|^2 \, ds \right)^{1/2} \leq (C_{inv}^{-1})^{1/2} \left( \sum_{\kappa \in T_h} \int_\kappa |\nabla w|^2 \, dx \right)^{1/2}
\]

and

\[ \left( \int_{\Gamma_{int}} \sigma^{-1} |\nabla w|^2 \, ds \right)^{1/2} \leq (C_{inv}^{-1} c_d)^{1/2} \left( \sum_{\kappa \in T_h} \int_\kappa |\nabla w|^2 \, dx \right)^{1/2}. \]
Therefore,

\[
\mathcal{B}(w_1, w_1 - w_2) - \mathcal{B}(w_2, w_1 - w_2) \geq C_2 \sum_{\kappa \in T_h} \int_{\kappa} |\nabla w|^2 \, dx
\]

\[
- \left( (1 + |\theta|)^2 C^2 C_{\text{inv}} \alpha^{-1} \right)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{\Gamma_D} \sigma |w|^2 \, ds \right)^{1/2}
\]

\[
- \left( (1 + |\theta|)^2 C^2 C_{\text{inv}} \alpha^{-1} \right)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} \sigma |w|^2 \, ds \right)^{1/2}
\]

\[+ \int_{\Gamma_D} \sigma |w|^2 \, ds + \int_{\Gamma_{\text{int}}} \sigma |w|^2 \, ds.\]

Applying Cauchy’s inequality to the second and third terms on the right-hand side, we have

\[
\mathcal{B}(w_1, w_1 - w_2) - \mathcal{B}(w_2, w_1 - w_2) \geq C_2 \left( 1 - \frac{(1 + |\theta|)^2 C^2 C_{\text{inv}} \alpha}{2 C^2 \alpha} \right) \sum_{\kappa \in T_h} \int_{\kappa} |\nabla w|^2 \, dx
\]

\[+ \frac{1}{2} \int_{\Gamma_D} \sigma |w|^2 \, ds + \frac{1}{2} \int_{\Gamma_{\text{int}}} \sigma |w|^2 \, ds.\]

Selecting \( \alpha \geq (1 + |\theta|)^2 C^2 C_{\text{inv}} \alpha \) completes the proof. ■

In the error analysis, we shall need a result similar to the one obtained in Lemma 3.5, but for functions in infinite-dimensional function spaces.

**Lemma 3.8** Suppose that \( u|_{\kappa} \in H^{k_{\kappa}}(\kappa) \) for some Sobolev index \( k_{\kappa} \geq 2 \) and \( \kappa \in T_h \). Assume, further, that \( \theta \in [-1, 1] \) and \( \alpha \geq (1 + |\theta|)^2 C_{\text{inv}} \alpha^{-1} \); then, the semilinear form \( \mathcal{B}(\cdot, \cdot) \) satisfies the following bound:

\[
|\mathcal{B}(u, v) - \mathcal{B}(z_{p_{\kappa}}^{h_{\kappa}}(u), v)| \leq C \left( \sum_{\kappa \in T_h} \frac{h_{2s_{\kappa}-2}^{2s_{\kappa}-2}}{p_{2s_{\kappa}-3}} \|u\|_{k_{\kappa}, \kappa}^2 \right)^{1/2} \|v\|_{1, h}, \tag{3.11}
\]

for all \( v \in S^p(\Omega, T_h, F) \), where \( 1 \leq s_{\kappa} \leq \min \{p_{\kappa} + 1, k_{\kappa}\} \), \( p_{\kappa} \geq 1 \), \( \kappa \in T_h \), and \( C \) is a positive constant independent of \( u, v, \) and of the discretisation parameters.
Proof We shall proceed in a similar way as in Lemma 3.5. We have that,

\[
|\mathcal{B}(u, v) - \mathcal{B}(z_{p_h}^h(u), v)| \leq \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} a(x, t, |\nabla u|) \nabla u - a(x, t, |\nabla z_{p_h}^h(u)|) \nabla z_{p_h}^h(u) - |\nabla v| \, dx \\
+ \int_{\Gamma_D} \left| a(x, t, |\nabla u|) \nabla u - a(x, t, |\nabla z_{p_h}^h(u)|) \nabla z_{p_h}^h(u) \right| |v| \, ds \\
+ \int_{\Gamma_{int}} \left\{ \left| a(x, t, |\nabla u|) \nabla u - a(x, t, |\nabla z_{p_h}^h(u)|) \nabla z_{p_h}^h(u) \right| \right\} ||v|| \, ds \\
+ ||\int_{\Gamma_{int}} \left| a(x, t, h^{-1}|u - g_D|)(u - g_D) - a(x, t, h^{-1}|z_{p_h}^h(u) - g_D|(z_{p_h}^h(u) - g_D) \right| |\nabla v| \, ds \\
+ \int_{\Gamma_D} \sigma |u - z_{p_h}^h(u)| |v| \, ds + \int_{\Gamma_{int}} \sigma |u - z_{p_h}^h(u)| ||v|| \, ds \\
=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\]

By (1.4), the Cauchy–Schwarz inequality, and the results of Lemma 3.3, we have that

\[
I_1 \leq C_1 \left( \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} |\nabla (u - z_{p_h}^h(u))|^2 \, dx \right)^{1/2} \left( \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2}
\leq C \left( \sum_{\kappa \in \mathcal{T}_h} \frac{h_{\kappa}^{2s_\kappa - 2}}{p_{\kappa}^{2s_\kappa - 3}} \|u\|_{k_{\kappa}, \kappa}^2 \right)^{1/2} \left( \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2},
\]

for all \( v \in SP(\Omega, \mathcal{T}_h, F) \) and with \( 1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\} \), \( p_\kappa \geq 1 \), for \( \kappa \in \mathcal{T}_h \), and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.

By the same argument,

\[
I_2 \leq C_1 \int_{\Gamma_D} |\nabla (u - z_{p_h}^h(u))| |v| \, ds \\
\leq C_1 \left( \int_{\Gamma_D} \sigma^{-1} |\nabla (u - z_{p_h}^h(u))|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}.
\]

Again, using the results of Lemma 3.3 and recalling the definition of the parameter \( \sigma_\epsilon \) on \( e \subset \Gamma_D \), we obtain

\[
I_2 \leq C \left( \sum_{\kappa \in \mathcal{T}_h} \frac{h_{\kappa}^{2s_\kappa - 2}}{p_{\kappa}^{2s_\kappa - 3}} \|u\|_{k_{\kappa}, \kappa}^2 \right)^{1/2} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2},
\]

for all \( v \in SP(\Omega, \mathcal{T}_h, F) \) and with \( 1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\} \), \( p_\kappa \geq 1 \), for \( \kappa \in \mathcal{T}_h \), and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.

Analogously,

\[
I_3 \leq C_1 \int_{\Gamma_{int}} \left\{ \left| \nabla (u - z_{p_h}^h(u)) \right| \right\} ||v|| \, ds \\
\leq C_1 \left( \int_{\Gamma_{int}} \sigma^{-1} \left| \nabla (u - z_{p_h}^h(u)) \right|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_{int}} \sigma ||v||^2 \, ds \right)^{1/2}.
\]
We note that
\[
\int_{\Gamma_{\text{int}}} \sigma_e^{-1} \left\{ |\nabla (u - z_{p_h}^e(u))|^2 \right\} ds = \sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \left\{ |\nabla (u - z_{p_h}^e(u))|^2 \right\} ds;
\]
for \( e \in \mathcal{E}_{\text{int}} \), let \( \kappa \) and \( \kappa' \) be two elements sharing the face \( e \). Then we have
\[
\int_e \left\{ |\nabla (u - z_{p_h}^e(u))|^2 \right\} ds \leq \frac{1}{2} \int_e \left\{ \left| \nabla (u - z_{p_h}^e(u)) \right|^2 + \left| \nabla (u - z_{p_h}^e(u)) \right|_{\kappa'}^2 \right\} ds
\]
\[
\leq C \left( \frac{h_{2s_h - 3}}{p_{\kappa}} \| u \|_{k_{\kappa}, \kappa}^2 + \frac{h_{2s_{\kappa'} - 3}}{p_{\kappa'}^2} \| u \|_{k_{\kappa'}, \kappa'}^2 \right)
\]
\[
\leq C \frac{\{ p^2 \}}{h_e} \max \left\{ \frac{h_{2s_{\kappa} - 2}}{p_{\kappa}} \| u \|_{k_{\kappa}, \kappa}^2, \frac{h_{2s_{\kappa} - 2}}{p_{\kappa'}^2} \| u \|_{k_{\kappa'}, \kappa'}^2 \right\},
\]
and, recalling the definition of \( \sigma \), we have that
\[
\sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \left\{ |\nabla (u - z_{p_h}^e(u))|^2 \right\} ds \leq C \alpha^{-1} \sum_{e \in \mathcal{E}_{\text{int}}} \max_{\kappa \in T_h} \frac{h_{2s_{\kappa} - 2}}{p_{\kappa}} \| u \|_{k_{\kappa}, \kappa}^2.
\]
Denoting \( c_d := \max \{ 2^d, (d + 1) \} \) as in the proof of Lemma 3.5, we have that
\[
\sum_{e \in \mathcal{E}_{\text{int}}} \sigma_e^{-1} \int_e \left\{ |\nabla (u - z_{p_h}^e(u))|^2 \right\} ds \leq C \alpha^{-1} c_d \sum_{e \in \mathcal{E}_{\text{int}}} \max_{\kappa \in T_h} \frac{h_{2s_{\kappa} - 2}}{p_{\kappa}} \| u \|_{k_{\kappa}, \kappa}^2,
\]
and hence
\[
I_3 \leq C \left( \sum_{\kappa \in T_h} \frac{h_{2s_{\kappa} - 2}}{p_{\kappa}} \| u \|_{k_{\kappa}, \kappa}^2 \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} \sigma v^2 ds \right)^{1/2},
\]
for all \( v \in \text{SP}(\Omega, T_h, \mathbf{F}) \) and with \( 1 \leq s_{\kappa} \leq \min \{ p_{\kappa} + 1, k_{\kappa} \}, p_{\kappa} \geq 1 \), for \( \kappa \in T_h \), and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.

For \( I_4 \), by (1.4) and the Cauchy–Schwarz inequality, we have the bound
\[
I_4 \leq \left[ \theta \right] C_1 \int_{\Gamma_D} \left| u - z_{p_h}^e(u) \right| |\nabla v| ds
\]
\[
\leq \left[ \theta \right] C_1 \left( \int_{\Gamma_D} \sigma \left| u - z_{p_h}^e(u) \right| ds \right)^{1/2} \left( \int_{\Gamma_D} \sigma^{-1} |\nabla v| ds \right)^{1/2},
\]
and hence, using the second of inverse inequalities (3.3) in the second factor on the right-hand side, the results of Lemma 3.3, and recalling the definition of the penalty parameter \( \sigma_e \) on \( e \subset \Gamma_D \), we have that
\[
I_4 \leq \left[ \theta \right] C \left( \sum_{\kappa \in T_h} \frac{h_{2s_{\kappa} - 2}}{p_{\kappa} - 3} \| u \|_{k_{\kappa}, \kappa}^2 \right)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\Gamma_D} |\nabla v|^2 dx \right)^{1/2},
\]
for all \( v \in \text{SP}(\Omega, T_h, \mathbf{F}) \) and with \( 1 \leq s_{\kappa} \leq \min \{ p_{\kappa} + 1, k_{\kappa} \}, p_{\kappa} \geq 1 \), for \( \kappa \in T_h \), and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.
Analogously, we get that,

\[ I_5 \leq |\theta| C \left( \sum_{\kappa \in T_h} \frac{h_{2s}^{2s-2}}{p_{k_s}^{k_{s}}-3} \|u\|_{k_s,\kappa}^2 \right)^{1/2} \left( \sum_{\kappa \in T_h} \int_{\kappa} |\nabla v|^2 \, dx \right)^{1/2}, \]

for all \( v \in S^p(\Omega, T_h, F) \) and with \( 1 \leq s_{\kappa} \leq \min \{p_{\kappa} + 1, k_{\kappa}\}, p_{\kappa} \geq 1, \) for \( \kappa \in T_h, \) and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.

Finally,

\[ I_6 \leq \left( \int_{\Gamma_D} \sigma \left| u - z_{p_{\kappa}}^{h_{\kappa}}(u) \right|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2} \]

\[ \leq C \left( \sum_{\kappa \in T_h} \frac{h_{2s}^{2s-2}}{p_{k_s}^{k_{s}}-3} \|u\|_{k_s,\kappa}^2 \right)^{1/2} \left( \int_{\Gamma_D} \sigma |v|^2 \, ds \right)^{1/2}, \]

and

\[ I_7 \leq \left( \int_{\Gamma_{\text{int}}} \sigma \left| u - z_{p_{\kappa}}^{h_{\kappa}}(u) \right|^2 \, ds \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} \sigma |v|^2 \, ds \right)^{1/2} \]

\[ \leq C \left( \sum_{\kappa \in T_h} \frac{h_{2s}^{2s-2}}{p_{k_s}^{k_{s}}-3} \|u\|_{k_s,\kappa}^2 \right)^{1/2} \left( \int_{\Gamma_{\text{int}}} \sigma |v|^2 \, ds \right)^{1/2}, \]

for all \( v \in S^p(\Omega, T_h, F) \) and with \( 1 \leq s_{\kappa} \leq \min \{p_{\kappa} + 1, k_{\kappa}\}, p_{\kappa} \geq 1, \) for \( \kappa \in T_h, \) and \( C \) a positive constant independent of \( u, v \) and the discretisation parameters.

Substituting these bounds into (3.12), we obtain (3.11).

### 3.2 Error analysis in the H¹ norm

Our aim is to derive a bound on the H¹ norm of the error \( u - u_{\text{DG}} \); here \( u_{\text{DG}} \) is the hp-DGFEM approximation of the analytical solution \( u \). We decompose the error \( u - u_{\text{DG}} \) as \( u - u_{\text{DG}} = \eta + \xi \), where \( \eta := u - z_{p_{\kappa}}^{h_{\kappa}}(u) \) and \( \xi := z_{p_{\kappa}}^{h_{\kappa}}(u) - u_{\text{DG}}, \) with \( z_{p_{\kappa}}^{h_{\kappa}} \) denoting the Babuška–Suri projection operator on the element \( \kappa \), defined in Lemma 3.3. We assume for simplicity that the initial value is chosen as \( u_0^{\text{DG}} = z_{p_{\kappa}}^{h_{\kappa}}(u_0) \), and thus \( \xi(0) = 0 \).

**Lemma 3.9** Let \( \{T_h\} \) be a family of shape-regular subdivisions of \( \Omega \subset \mathbb{R}^d \), and let \( \sigma \) be as in (2.5) with \( \alpha \geq (1 + |\theta|)^2 C_2 C_3^{-1} C_{\text{inv}}, \) \( \theta \in [-1, 1] \). Assume that (A) and (B) hold and that \( u \in L^\infty(0, T; H^1(\Omega)) \). Suppose further that

a) \( p_{\kappa} \geq 2 \) and \( u|_{\kappa} \in H^1(0, T; H_{k_s}(\kappa)) \), \( k_{\kappa} \geq 3 \frac{1}{2} \) on each \( \kappa \in T_h; \)

b) the hp-mesh is quasi-uniform in the sense that there exists a positive constant \( C_0 \) such that

\[ \max_{\kappa \in T_h} \frac{h_{\kappa}}{p_{\kappa}^2} \leq C_0 \min_{\kappa \in T_h} \frac{h_{\kappa}}{p_{\kappa}^2}. \]
Then, there exists \( h_0 \in (0, 1] \) such that for all \( h \in (0, h_0] \), \( h = \max_{\kappa \in T_h} h_{\kappa} \), and for all \( t \in [0, T] \), the following inequality holds, with \( C \) a positive constant that depends only on the domain \( \Omega \), the quasi-uniformity constant \( C_0 \), on \( k = \max_{\kappa \in T_h} k_{\kappa} \), the final time \( T \), the exponent \( \gamma \) in the growth condition for the function \( f \), and on Lebesgue and Sobolev norms of \( u \) over the time interval \((0, T)\):

\[
\int_0^t \|(u - u_{DG})(s)\|_{1,h}^2 \, ds \leq C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 + \sum_{\kappa \in T_h} \frac{h_{2\kappa} - 2}{h_{k_{\kappa}}} \|u(s)\|_{k_{\kappa},\kappa}^2 \right\} \, ds, \quad \text{(3.13)}
\]

where \( 1 \leq s_{\kappa} \leq \min \{p_{\kappa} + 1, k_{\kappa}\} \).

**Proof** From the formulation of the \( hp\)-DGFEM (2.7), for all \( v \in S^p(\Omega, T_h, F) \), we have

\[
\sum_{\kappa \in T_h} \int_{\kappa} u'_{DG}(v) \, dx + B(u_{DG}, v) = \sum_{\kappa \in T_h} \int_{\kappa} f(x, t, v) \, dx + \ell(v). \quad \text{(3.14)}
\]

On the other hand, the broken weak formulation (2.6) implies that

\[
\sum_{\kappa \in T_h} \int_{\kappa} z_{p_{\kappa}}(u') \, dx + B(z_{p_{\kappa}}(u), v) = \sum_{\kappa \in T_h} \int_{\kappa} f(x, t, u) \, dx + \ell(v) = \sum_{\kappa \in T_h} \int_{\kappa} (z_{p_{\kappa}}(u') - u') \, dx + \left\{ B(z_{p_{\kappa}}(u), v) - B(u, v) \right\} \quad \text{(3.15)}
\]

for all \( v \in H^1(\Omega, T_h) \). Upon subtracting (3.14) from (3.15), taking \( v = \xi = z_{p_{\kappa}}(u) - u_{DG} \), and noting that

\[
\sum_{\kappa \in T_h} \int_{\kappa} \xi' \xi \, dx = \frac{1}{2} \frac{d}{dt} \sum_{\kappa \in T_h} \|\xi\|_{0,\kappa}^2 = \frac{1}{2} \frac{d}{dt} \|\xi\|_{0,\Omega}^2,
\]

we deduce from the above identity that

\[
\left\{ \frac{1}{2} \frac{d}{dt} \|\xi\|_{0,\Omega}^2 + \left\{ B(z_{p_{\kappa}}(u), \xi) - B(u_{DG}, \xi) \right\} \right\} \leq \sum_{\kappa \in T_h} \left\{ \int_{\kappa} \left\{ f(x, t, u) - f(x, t, z_{p_{\kappa}}(u)) \right\} \xi \, dx \right\} + \sum_{\kappa \in T_h} \left\{ \int_{\kappa} \left\{ f(x, t, z_{p_{\kappa}}(u)) - f(x, t, u_{DG}) \right\} \xi \, dx \right\} + \sum_{\kappa \in T_h} \int_{\kappa} \eta' \xi \, dx + \left| B(u, \xi) - B(z_{p_{\kappa}}(u), \xi) \right|. \quad \text{(3.16)}
\]

By the Cauchy–Schwarz inequality and Cauchy’s inequality, with \( \varepsilon_1 > 0 \), we have

\[
\left\| \sum_{\kappa \in T_h} \int_{\kappa} \eta' \xi \, dx \right\| \leq \left( \sum_{\kappa \in T_h} \|\eta'\|_{0,\kappa}^2 \right)^{1/2} \left( \sum_{\kappa \in T_h} \|\xi\|_{0,\kappa}^2 \right)^{1/2} \leq \frac{\varepsilon_1}{2} \|\eta'\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_1} \|\xi\|_{0,\Omega}^2.
\]
By the same argument, with \( \varepsilon_2, \varepsilon_3 > 0 \),

\[
\left| \sum_{\kappa \in T_h} \int_{\kappa} \left( f(x, t, u) - f(x, t, z_{p_n}^h(u)) \right) \xi \, dx \right| \leq \frac{\varepsilon_2}{2} \left\| f(\cdot, t, u) - f(\cdot, t, z_{p_n}^h(u)) \right\|_{0, \Omega}^2 + \frac{1}{2 \varepsilon_2} \left\| \xi \right\|_{0, \Omega}^2,
\]

\[
\left| \sum_{\kappa \in T_h} \int_{\kappa} \left( f(x, t, z_{p_n}^h(u)) - f(x, t, u_{DG}) \right) \xi \, dx \right| \leq \frac{\varepsilon_3}{2} \left\| f(\cdot, t, z_{p_n}^h(u)) - f(\cdot, t, u_{DG}) \right\|_{0, \Omega}^2 + \frac{1}{2 \varepsilon_3} \left\| \xi \right\|_{0, \Omega}^2.
\]

Further, by Lemma 3.7, with \( \alpha \geq C_1^2 C_2^{-1} C_{\text{inv}}(c_d + 1) \), we have that

\[
B(z_{p_n}^h(u), \xi) - B(u_{DG}, \xi) \geq \frac{1}{2} \min \{ C_2, 1 \} \left\| \xi \right\|_{1, h}^2,
\]

and, by Lemma 3.8, applying the Cauchy–Schwarz inequality and Cauchy’s inequality, we get

\[
\left| B(u, \xi) - B(z_{p_n}^h(u), \xi) \right| \leq \frac{C}{2} \sum_{\kappa \in T_h} \frac{h_{p_n}^{2s_n-2}}{p_{k_n}^{s_n-3}} \left\| u \right\|_{k_n, \kappa}^2 + \frac{1}{2} \left\| \eta \right\|_{1, h}^2,
\]

for all \( v \in S^p(\Omega, T_h, F) \) and with \( 1 \leq s_n \leq \min \{ p_n + 1, k_n \} \), \( k_n \geq 2 \), \( p_n \geq 1 \), for \( \kappa \in T_h \), and \( C \) a positive constant independent of \( u, v \), and the discretisation parameters.

By Lemma 3.1, upon absorbing all constants into \( C \) and using the broken Sobolev–Poincaré inequality, for a.e. \( t \in (0, T) \) we have that

\[
\left\| f(\cdot, t, u) - f(\cdot, t, z_{p_n}^h(u)) \right\|_{0, \Omega}^2 \leq C_f \left\| \eta \right\|_{0, 2(\gamma+1), \Omega}^2 \left( 1 + \left\| u \right\|_{0, 2(\gamma+1), \Omega}^\gamma \right) \left\| z_{p_n}^h(u) \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma}
\]

\[
\leq C \left\| \eta \right\|_{0, 2(\gamma+1), \Omega}^2 \left( 1 + \left\| u \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma} \right) \left\| z_{p_n}^h(u) - u \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma}
\]

\[
= C \left\| \eta \right\|_{0, 2(\gamma+1), \Omega}^2 \left( 1 + \left\| u \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma} + \left\| \eta \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma} \right)
\]

\[
\leq C \left\| \eta \right\|_{0, 2(\gamma+1), \Omega} \left( 1 + \left\| u \right\|_{0, 2(\gamma+1), \Omega}^{2\gamma} + \left\| \eta \right\|_{1, h}^{2\gamma} \right)
\]

where the constant \( C > 0 \) depends only the domain \( \Omega \), the exponent \( \gamma \) in the growth condition for the function \( f \), and on Lebesgue and Sobolev norms of \( u \) over the time interval \( (0, T) \). For ease of writing, the dependence of the norms of \( u, \eta \) and \( z_{p_n}^h(u) \) on \( t \) in the last chain of inequalities has been suppressed.
Applying these bounds to the right-hand side of (3.16) with \( \varepsilon_1 = \varepsilon_2 = 1 \) (the value of \( \varepsilon_3 \) will be fixed below) and absorbing all constants into \( K_1 \) and \( K_2 = K_2(\varepsilon_3) \), we obtain

\[
\frac{d}{dt}\|\xi(t)\|_{0,\Omega}^2 + \min\{C_2, 1\}\|\xi(t)\|_{1,h}^2 \\
\leq K_1 \left( \|\eta(t)\|_{1,h}^2 + \|\eta'(t)\|_{0,\Omega}^2 + \sum_{\kappa \in T_h} \frac{h^{2\kappa_s - 2}}{p\kappa_s - 3} \|u(t)\|_{k_s,\kappa}^2 \right) \\
+ K_2 \|\xi(t)\|_{0,\Omega}^2 + \varepsilon_3 \left\| f(\cdot, t, \xi^{h_{p\kappa}}(u)) - f(\cdot, t, u_{DG}) \right\|_{0,\Omega}^2,
\]

(3.17)

for a.e. \( t \in (0,T] \), where \( 1 \leq s_\kappa \leq \min\{p\kappa_s + 1, k_s\} \), \( k_s \geq 2 \), \( p_s \geq 1 \) on each \( \kappa \in T_h \), with the constant \( K_1 > 0 \) depending only on the domain \( \Omega \), the exponent \( \gamma \) in the growth condition for the function \( f \), on \( k = \max_{k \in T_h} k_s \), and on Lebesgue and Sobolev norms of \( u \) over the time interval \((0,T]\).

To bound \( \|f(\cdot, t, \xi^{h_{p\kappa}}(u)) - f(\cdot, t, u_{DG})\|_{0,\Omega}^2 \), we first note that, by a very similar argument as above, we have, for a.e. \( t \in (0,T] \),

\[
\left\| f(\cdot, t, \xi^{h_{p\kappa}}(u)) - f(\cdot, t, u_{DG}) \right\|_{0,\Omega}^2 \leq C\|\xi(t)\|_{0,\Omega}^2 \left( 1 + \|\xi(t)\|_{0,\Omega}^{2\gamma} \right),
\]

(3.18)

where the constant \( C > 0 \) depends only on the domain \( \Omega \), the exponent \( \gamma \) in the growth condition for the function \( f \), and on Lebesgue and Sobolev norms of \( u \) over the time interval \((0,T]\).

For \( T_h \) and the polynomial degree vector \( p \) fixed, let \( t_* = t_*(T_h, p) \in (0,T] \) be the largest time such that \( u_{DG} \) exists for all \( t \in [0,t_*) \) and \( \|\xi(t)\|_{1,h} \leq 1 \) for all \( t \in [0,t_*] \); the existence of such \( t_* \) is guaranteed by Carathéodory’s theorem (see Theorems II.4.1 and II.4.5 in [14]) together with the fact that \( t \mapsto \|\xi(t)\|_{1,h}^2 \) is continuous in the neighbourhood of \( t = 0 \), and \( \|\xi(0)\|_{1,h}^2 = 0 \). Our aim is to show that \( t_* = T \) for all \( t_* \), sufficiently small. We have that

\[
\|\xi(t)\|_{0,\Omega}^2 \leq Const. \|\xi(t)\|_{1,h}^2 \quad \text{for all} \quad t \in [0,t_*]
\]

by the broken Sobolev–Poincaré inequality (see Lemma 3.2); here \( Const. \) is a constant that is independent of the discretisation parameters and \( t \). This and (3.18), together with the fact that \( \|\xi(t)\|_{1,h} \leq 1 \) for all \( t \in [0,t_*] \) imply that, for a.e. \( t \in (0,t_*] \),

\[
\left\| f(\cdot, t, \xi^{h_{p\kappa}}(u)) - f(\cdot, t, u_{DG}) \right\|_{0,\Omega}^2 \leq \tilde{C} \|\xi(t)\|_{1,h}^2,
\]

where the constant \( \tilde{C} > 0 \) depends only on the domain \( \Omega \), the exponent \( \gamma \) in the growth condition for the function \( f \), and on Lebesgue and Sobolev norms of \( u \) over the time interval \((0,t_*] \).

On choosing \( \varepsilon_3 \tilde{C} < 1 \), after integration from 0 to \( t \leq t_* \) and noting that \( \xi(0) = 0 \), the inequality (3.17) yields that

\[
\|\xi(t)\|_{0,\Omega}^2 + \int_0^t \|\xi(s)\|_{1,h}^2 \, ds \leq K_1 \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 \\
+ \sum_{\kappa \in T_h} \frac{h^{2\kappa_s - 2}}{p\kappa_s - 3} \|u(s)\|_{k_s,\kappa}^2 \right\} \, ds + K_2 \int_0^t \|\xi(s)\|_{0,\Omega}^2 \, ds,
\]

(3.19)
with $1 \le s_\kappa \le \min \{p_\kappa + 1, k_\kappa\}$, and the constant $K_1 > 0$ depending only on the domain $\Omega$, the exponent $\gamma$ in the growth condition for the function $f$, on $k = \max_{k \in T_h} k_\kappa$, and on Lebesgue and Sobolev norms of $u$ over the time interval $(0, t_*)$.

We can make the first integral on the right-hand side of (3.19) as small as we like (for example, by fixing the local polynomial degree $p_\kappa$ on each element $\kappa \in T_h$ and reducing $h = \max_{\kappa \in T_h} h_\kappa$). In particular, let us take $h_0 \in (0, 1]$ so small that, for all $h \le h_0$ and $t \in [0, t_*)$, the following inequality holds:

$$K_1 \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 + \sum_{\kappa \in T_h} \frac{h_{2k_\kappa - 2}}{p_{k_\kappa} - 3} \|u(s)\|_{k_{\kappa},h_\kappa}^2 \right\} ds < \frac{1}{1 + T} e^{-K_2 T} \times K_{\text{inv}}^{-1} C_0^{-2} \left(\max_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^2,$$

where $K_{\text{inv}}$ is the constant from the inverse inequality

$$\|\xi(t)\|_{1,h}^2 \le K_{\text{inv}} \left(\max_{\kappa \in T_h} \frac{p_\kappa}{h_\kappa}\right)^2 \|\xi(t)\|_{0,\Omega}^2 \quad \text{for all} \quad t \in [0, t_*]. \quad (3.20)$$

We note in passing that, in order to be able to extract the factor $(\max_{\kappa \in T_h} (h_\kappa/p_\kappa)^2)^2$ from $\|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2$ with strict inequality (by using (3.4) and (3.5)), we need hypothesis a) in the statement of the Lemma. Hence (3.19) becomes

$$\|\xi(t)\|_{0,\Omega}^2 + \int_0^t \|\eta(s)\|_{1,h}^2 ds < \frac{e^{-K_2 T}}{1 + T} x K_{\text{inv}}^{-1} C_0^{-2} \left(\max_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^2 + K_2 \int_0^t \|\xi(s)\|_{0,\Omega}^2 ds,$$

which, by the Gronwall–Bellman inequality, implies that

$$\|\xi(t)\|_{0,\Omega}^2 < K_{\text{inv}}^{-1} C_0^{-2} \left(\max_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^2 \quad \text{for all} \quad t \in [0, t_*].$$

By the inverse inequality (3.20) we have that,

$$\|\xi(t)\|_{1,h}^2 \le C_0^{-2} \left(\max_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^2 \left(\max_{\kappa \in T_h} \frac{p_\kappa}{h_\kappa}\right)^2 = C_0^{-2} \left(\max_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^2 \left(\min_{\kappa \in T_h} \frac{h_\kappa}{p_\kappa}\right)^{-2},$$

for all $t \in [0, t_*]$, which, by the quasi-uniformity hypothesis b) above, is $\le 1$. Thus, for $h \le h_0$, we have $\|\xi(t)\|_{1,h}^2 \le 1$ for all $t \in [0, t_*)$. By continuity of the mapping $t \mapsto \|\xi(t)\|_{1,h}$ on $[0, t_*)$ it follows that $t_* = T$, provided that $h \in (0, h_0]$ (for, else $t_*$ would not be the largest real number in $(0, T]$ such that $\|\xi(t)\|_{1,h}^2 \le 1$ for all $t \in [0, t_*]$).

From (3.19) by the Gronwall–Bellman inequality we then obtain

$$\|\xi(t)\|_{0,\Omega}^2 + \int_0^t \|\xi(s)\|_{1,h}^2 ds \le C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 + \sum_{\kappa \in T_h} \frac{h_{2k_\kappa - 2}}{p_{k_\kappa} - 3} \|u(s)\|_{k_{\kappa},h_\kappa}^2 \right\} ds$$

for all $t \in [0, T]$, and hence

$$\int_0^t \|\xi(s)\|_{1,h}^2 ds \le C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 + \sum_{\kappa \in T_h} \frac{h_{2k_\kappa - 2}}{p_{k_\kappa} - 3} \|u(s)\|_{k_{\kappa},h_\kappa}^2 \right\} ds.$$
for all $t \in [0, T]$, with the constant $C > 0$ depending only on the domain $\Omega$, the quasi-uniformity constant $C_0$, on $k = \max_{k \in \mathcal{T}_h} k_\kappa$, the final time $T$, the exponent $\gamma$ in the growth condition for the function $f$, and on Lebesgue and Sobolev norms of $u$ over the time interval $(0, T)$.

Employing the triangle inequality $\|u - u_{\text{DG}}\|_{1,h} \leq \|\eta\|_{1,h} + \|\xi\|_{1,h}$, we deduce that

$$\int_0^t \|(u - u_{\text{DG}})(s)\|_{1,h}^2\, ds \leq C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 + \sum_{\kappa \in \mathcal{T}_h} \frac{h_{2s_\kappa}^{2s_{\kappa} - 2}}{p_{k_\kappa}^{2s_{\kappa} - 3}} \|u(s)\|_{k_\kappa,\kappa}^2 \right\} ds,$$

for all $t \in [0, T]$, and hence (3.13).

Lemma 3.9 yields the following error bound for the $hp$--DGFEM (2.7).

**Theorem 3.10** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded polyhedral domain with Lipschitz-continuous boundary, let $\{\mathcal{T}_h\}$ be a family of shape-regular and $hp$-quasi-uniform subdivisions of $\Omega$ (cf. b) in Lemma 3.9), and suppose that $p$ is a polynomial degree vector of bounded local variation. Let each face $e \in \mathcal{E}_{\text{int}} \cup \mathcal{E}_D$ be assigned a positive real number $\sigma_e$, defined as in (2.5), with $\sigma = (1 + |\theta|)^2 C^2 C_{\text{int}}^{-1} C_{\text{inv}} c_d$, $\theta \in [-1, 1]$. Suppose that (A) and (B) hold and $u \in L^\infty(0, T; H^1(\Omega))$. Then, if $p_\kappa \geq 2$ and $u|_\kappa \in H^1(0, T; H^\kappa(\kappa))$ with $k_\kappa \geq 3^{\frac{1}{2}}$ on each $\kappa \in \mathcal{T}_h$, there exists $h_0 \in (0, 1]$ such that for all $h \in (0, h_0]$, $h = \max_{\kappa \in \mathcal{T}_h} h_{\kappa}$, and all $t \in [0, T]$, the solution $u_{DG}(\cdot, t) \in \mathcal{S}^p(\Omega, \mathcal{T}_h, \mathbf{F})$ of the DGFEM (2.7) satisfies the following error bound:

$$\|u - u_{\text{DG}}\|_{L^2(0, T; H^1(\Omega, \mathcal{T}_h))} \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_{2s_\kappa}^{2s_\kappa - 3}}{p_{k_\kappa}^{2s_\kappa - 3}} \|u\|_{H^1(0, T; H^\kappa(\kappa))}^2; \quad (3.21)$$

with $1 \leq s_\kappa \leq \min \{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 2$ on each $\kappa \in \mathcal{T}_h$, where $C$ is a positive constant depending only on the domain $\Omega$, the shape-regularity and quasi-uniformity constants of $\mathcal{T}_h$, the parameter $\rho$ in (2.2), on $k = \max_{\kappa \in \mathcal{T}_h} k_\kappa$, the final time $T$, the exponent $\gamma$ in the growth condition for the function $f$, and on Lebesgue and Sobolev norms of $u$ over the time interval $(0, T)$.

**Proof** From Lemma 3.3 we have the estimates

$$\|\eta\|_{0,0,\kappa}^2 \leq C \frac{h_{2s_\kappa}^{2s_\kappa - 1}}{p_{k_\kappa}} \|u\|_{k_\kappa,\kappa}^2; \quad \|\nabla \eta\|_{0,0,\kappa}^2 \leq C \frac{h_{2s_\kappa}^{2s_\kappa - 3}}{p_{k_\kappa}} \|u\|_{k_\kappa,\kappa}^2;$$

$$\|\eta\|_{1,\kappa}^2 \leq C \frac{h_{2s_\kappa}^{2s_\kappa - 2}}{p_{k_\kappa}} \|u\|_{k_\kappa,\kappa}^2; \quad \|\eta\|_{0,0,\kappa}^2 \leq C \frac{h_{2s_\kappa}^{2s_\kappa - 3}}{p_{k_\kappa}} \|u\|_{k_\kappa,\kappa}^2.$$

Upon noting the definition of $\sigma$ (2.5) and the bounded local variation condition (2.2) to relate $p_\kappa$ to $p_\kappa$, applying the above estimates to $\|\eta(s)\|_{1,h}^2$ yields

$$\|\eta(s)\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_{2s_\kappa}^{2s_\kappa - 2}}{p_{k_\kappa}} \|u(s)\|_{k_\kappa,\kappa}^2; \quad \forall s \in [0, T],$$

with $1 \leq s_\kappa \leq \min \{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$ on each $\kappa \in \mathcal{T}_h$. 
By differentiating $\eta(t) = u(t) - z^h_{p_\kappa}(u(t))$ with respect to $t$ we deduce that $\eta'(t) = u'(t) - z^h_{p_\kappa}(u'(t))$ for all $u \in H^1(0, T; W^{k_\kappa, q}(\kappa))$ and all $\kappa \in \mathcal{T}_h$, $t \in [0, T]$. Hence, by applying Lemma 3.3 to $u'(t)$, we obtain that

$$
\|\eta'(t)\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u'(t)\|_{k_\kappa, p_\kappa}^2 \quad \text{for all} \quad \forall t \in [0, T],
$$

and with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$ on each $\kappa \in \mathcal{T}_h$. Therefore, by the broken Sobolev–Poincaré inequality (3.4), an identical bound holds for the norm $\|\eta'(t)\|_{0, \Omega}$, for all $t \in [0, T]$.

Applying these bounds in the right-hand side of (3.13) for $t \in [0, T]$, we obtain the desired bound, with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 2$ on each $\kappa \in \mathcal{T}_h$, where $C$ is a positive constant depending only on the domain $\Omega$, the shape-regularity and quasi-uniformity constants of $\mathcal{T}_h$, the final time $T$, the exponent $\gamma$ in the growth condition for the function $f$, the parameter $\rho$ in (2.2), on $k = \max_{\kappa \in \mathcal{T}_h} k_\kappa$, and on Lebesgue and Sobolev norms of $u$ over the time interval $(0, T)$. ■

When $f$ is globally Lipschitz-continuous the hypotheses a) and b) stated in Lemma 3.9 are redundant as (3.19) holds automatically for all $t \in [0, T]$, and it is not necessary to separately prove that $\|\xi(t)\|_{1,h} \leq 1$ for all $t \in [0, T]$ and all $h$ sufficiently small.

4 Conclusions

In this paper, we have been concerned with the error analysis of the spatial discretisation of quasilinear parabolic initial-boundary value problems with mixed Dirichlet and Neumann boundary conditions by a one-parameter family of $hp$–DGFEMs. We derived the broken weak formulation of the quasilinear parabolic problem, and showed that, provided the nonlinearities satisfy certain monotonicity and growth conditions, the associated semilinear form is Lipschitz-continuous and uniformly monotonic. We also developed the techniques of handling the locally Lipschitz-continuous nonlinearities in the error analysis, which allowed us to perform our proofs on the entire time interval of existence of the solution. We showed that the presence of the nonlinearities, satisfying certain monotonicity and growth conditions, does not degrade the convergence rates observed in the case of linear parabolic PDE.

The resulting error bound is optimal in $h$ and slightly suboptimal in $p$. As we have noted in the Introduction, full $hp$–optimality of the error bound can be restored by hypothesising piecewise regularity of the solution in augmented Sobolev spaces instead of classical Sobolev spaces, as was done in [7] in the case of linear elliptic equations. To the best of our knowledge, the error bound presented here is the first of this kind for quasilinear parabolic equations with nonlinearities of the specified type.

The extension of the analysis of our semidiscrete scheme to simple fully discrete schemes, such as a backward Euler time-discretisation of our semidiscrete scheme, would proceed along very similar lines and is, therefore, not considered here.
References


