Some Equivalent Characterizations of the Polynomial Numerical Hull of Degree \( k \)

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Abstract

The polynomial numerical hull of degree \( k \) for a square matrix or bounded linear operator \( A \) on a Hilbert space is defined as

\[
\mathcal{H}_k(A) = \{z \in \mathbb{C} : \|p(A)\| \geq |p(z)| \quad \forall p \in \mathcal{P}_k\},
\]

where \( \mathcal{P}_k \) denotes the set of polynomials of degree \( k \) or less. An equivalent definition is

\[
\mathcal{H}_k(A) = \{\zeta \in \mathbb{C} : \min_{c_1, \ldots, c_k} \|I - \sum_{j=1}^{k} c_j (A - \zeta I)^j\| = 1\}.
\]

In this paper we establish several equivalent characterizations, some of which provide a natural distinction between boundary points and interior points, and hence should prove useful for computations. Additionally, we use one of these equivalent definitions to prove that the polynomial numerical hull of any fixed degree \( k \) for a Toeplitz matrix whose symbol is piecewise continuous approaches all or most of that of the infinite-dimensional Toeplitz operator, as the matrix size goes to infinity.

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1 Introduction

The polynomial numerical hull of degree $k$ for a square matrix or bounded linear operator $A$ on a Banach space is defined as

$$\mathcal{H}_k(A) := \{ z \in \mathbb{C} : \| p(A) \| \geq |p(z)| \ \forall p \in \mathcal{P}_k \},$$

(1)

where $\mathcal{P}_k$ denotes the set of polynomials of degree $k$ or less [8, 9, 4]. The sets $\mathcal{H}_k(A)$ are nonempty and compact, and $\mathcal{H}_1(A)$ is equal to the closed convex hull of the field of values of $A$ [8]. In this paper, we will restrict attention to operators on a Hilbert space, where the field of values is a convex set, and we define $\mathcal{F}(A)$ to be the closure of that set:

$$\mathcal{F}(A) := \text{cl} (\{ \langle Aq, q \rangle : \| q \| = 1 \}).$$

(2)

In a finite-dimensional space, the field of values is closed, but this is not necessarily so for infinite-dimensional spaces.

An equivalent characterization of the polynomial numerical hull of degree $k$ is [4]

$$\mathcal{H}_k(A) = \{ \zeta \in \mathbb{C} : \min_{c_1, \ldots, c_k} \| I - \sum_{j=1}^{k} c_j (A - \zeta^j) \| = 1 \}.$$

(3)

In other words, $\zeta \in \mathcal{H}_k(A)$ if and only if a solution to the minimization problem $\min_{c_1, \ldots, c_k} \| I - \sum_{j=1}^{k} c_j (A - \zeta^j) \|$ is $c_1 = \ldots = c_k = 0$. In this paper we establish several equivalent characterizations, some of which provide a natural distinction between boundary points and interior points, and hence should prove useful for computations. Additionally, we use one of these equivalent definitions to prove that the polynomial numerical hull of any fixed degree $k$ for a Toeplitz matrix whose symbol is piecewise continuous approaches all or most of that of the infinite-dimensional Toeplitz operator, as the matrix size goes to infinity. More precisely, we show that the polynomial numerical hull of degree $k$ for the Toeplitz operator contains the uniform and partial limits as $N \to \infty$ of that of the $N$ by $N$ Toeplitz matrix, which in turn contain the closure of the interior of that of the Toeplitz operator.
2 Equivalent Characterizations

It follows from definition (3) and the identity \( \mathcal{H}_1(A) = \mathcal{F}(A) \), that for any bounded linear operator \( B \),

\[
\min_{c \in \mathbb{C}} \| I - cB \| < 1 \quad \text{if and only if} \quad 0 \notin \mathcal{F}(B).
\]

It also follows from (3) that a point \( \zeta \) lies outside \( \mathcal{H}_k(A) \) if and only if there exist scalars \( c_1, \ldots, c_k \) such that

\[
\| I - \sum_{j=1}^{k} c_j (A - \zeta I)^j \| < 1,
\]

or, equivalently, such that

\[
\min_{c \in \mathbb{C}} \| I - c(\sum_{j=1}^{k} c_j (A - \zeta I)^j) \| < 1.
\]

Hence we have

\[
\zeta \notin \mathcal{H}_k(A) \quad \text{if and only if} \quad
\exists \ c_1, \ldots, c_k \ \text{such that} \ 0 \notin \mathcal{F} \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right). \tag{4}
\]

An equivalent statement is

\[
\zeta \in \mathcal{H}_k(A) \quad \text{if and only if} \quad
\forall c_1, \ldots, c_k, \ 0 \in \mathcal{F} \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right). \tag{5}
\]

The following theorem gives a criterion for distinguishing boundary points of \( \mathcal{H}_k(A) \) from interior points.

**Theorem 1.** A point \( \zeta \) lies on the boundary of \( \mathcal{H}_k(A) \) if and only if: for all \( c_1, \ldots, c_k \)

\[
0 \in \mathcal{F} \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right), \tag{6}
\]
and for some \( \hat{c}_1, \ldots, \hat{c}_k \), not all zero,

\[
0 \in \partial \mathcal{F} \left( \sum_{j=1}^{k} \hat{c}_j (A - \zeta I)^j \right),
\]

(7)

where \( \partial \) denotes the boundary.

**Proof:** Condition (6) was already shown to be equivalent to the statement \( \zeta \in \mathcal{H}_k(A) \); hence it holds for \( \zeta \in \partial \mathcal{H}_k(A) \) since the set is closed.

To show that \( \zeta \in \partial \mathcal{H}_k(A) \) implies (7), let \( \zeta \) be a point on the boundary of \( \mathcal{H}_k(A) \), and let \( \{ \zeta_m \}_{m=0}^{\infty} \) be a sequence of points outside \( \mathcal{H}_k(A) \) and converging to \( \zeta \). For each point \( \zeta_m \), it follows from (4) that there exist coefficients \( c_{1,m}, \ldots, c_{k,m} \) such that

\[
0 \not\in \mathcal{F} \left( \sum_{j=1}^{k} c_{j,m} (A - \zeta_m I)^j \right).
\]

We can take these coefficients to satisfy, say, \( \sum_{j=1}^{k} |c_{j,m}|^2 = 1 \), since multiplying an operator by a nonzero scalar just multiplies its field of values by that scalar and does not affect whether or not the closure of the field of values contains the origin. Since the vectors \( (c_{1,m}, \ldots, c_{k,m}, \zeta_m) \) are bounded, there exists a convergent subsequence, \( \{ (c_{1,m}, \ldots, c_{k,m}, \zeta_m) \}_{m=1}^{\infty} \), converging to, say, \( (\hat{c}_1, \ldots, \hat{c}_k, \zeta) \). Since the field of values is a continuous function of the linear operator [6] and since

\[
0 \not\in \mathcal{F} \left( \sum_{j=1}^{k} c_{j,m} (A - \zeta_m I)^j \right) \rightarrow \mathcal{F} \left( \sum_{j=1}^{k} \hat{c}_j (A - \zeta I)^j \right) \ni 0 \quad \text{(by (5))},
\]

it follows that \( 0 \in \partial \mathcal{F} (\sum_{j=1}^{k} \hat{c}_j (A - \zeta I)^j) \).

Conversely, suppose \( \zeta \) is a point in the interior of \( \mathcal{H}_k(A) \) and suppose that

\[
0 \in \partial \mathcal{F} \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right),
\]

for certain scalars \( c_1, \ldots, c_k \). We will show that each \( c_j \) must be 0.

Since \( \mathcal{H}_k(A) \) contains a disk of radius, say, \( r > 0 \) about \( \zeta \), it follows from (5) that for any scalars \( d_1, \ldots, d_k \), any \( \delta \in [0, r) \), and any \( \theta \in [0, 2\pi) \),

\[
0 \in \mathcal{F} \left( \sum_{j=1}^{k} d_j (A - (\zeta - \delta e^{i\theta})I)^j \right).
\]

(8)
Expanding using the binomial formula, we find
\[
\sum_{j=1}^{k} d_j (A - (\zeta - \delta e^{i\theta})I)^j = \sum_{j=1}^{k} d_j \sum_{\ell=0}^{j} \binom{j}{\ell} \delta^\ell e^{i\theta} (A - \zeta I)^{j-\ell}
\]
\[
= \sum_{j=1}^{k} d_j (A - \zeta I)^j + \sum_{j=1}^{k} d_j \sum_{\ell=1}^{j-1} \binom{j}{\ell} \delta^\ell e^{i\theta} (A - \zeta I)^{j-\ell}
\]
\[
+ \left( \sum_{j=1}^{k} d_j \delta^j e^{i\theta} \right) I
\]
\[
= \sum_{j=1}^{k} \left( d_j + \sum_{\ell=1}^{k-j} d_{j+\ell} \binom{j+\ell}{\ell} \delta^\ell e^{i\theta} \right) (A - \zeta I)^j
\]
\[
+ \left( \sum_{j=1}^{k} d_j \delta^j e^{i\theta} \right) I.
\]
Suppose \(d_k = c_k\) and \(d_j, j = k - 1, \ldots, 1\) are determined by
\[
d_j = c_j - \sum_{\ell=1}^{k-j} d_{j+\ell} \binom{j+\ell}{\ell} \delta^\ell e^{i\theta}.
\] (9)

Then
\[
\sum_{j=1}^{k} d_j (A - (\zeta - \delta e^{i\theta})I)^j = \sum_{j=1}^{k} c_j (A - \zeta I)^j + \left( \sum_{j=1}^{k} d_j \delta^j e^{i\theta} \right) I,
\] (10)
and the field of values of this matrix is that of \(\sum_{j=1}^{k} c_j (A - \zeta I)^j\) shifted by \(\sum_{j=1}^{k} d_j \delta^j e^{i\theta}\). To express \(\sum_{j=1}^{k} d_j \delta^j e^{i\theta}\) in terms of the \(c_j\)’s, note that if \(\sum_{j=1}^{k} c_j (A - \zeta I)^j = \sum_{j=1}^{k} c_j (A - (\zeta - \delta e^{i\theta})I - \delta e^{i\theta}I)^j\) is expanded using the binomial formula, then the expansion involves powers from 1 to \(k\) of \(A - (\zeta - \delta e^{i\theta})I\) plus the term \(\sum_{j=1}^{k} c_j (-1)^j \delta^j e^{i\theta}I\). Comparing this with (10), it follows that
\[
\sum_{j=1}^{k} d_j \delta^j e^{i\theta} = \sum_{j=1}^{k} c_j (-1)^{j+1} \delta^j e^{i\theta}.
\] (11)

If the origin is on the boundary of \(\mathcal{F}(\sum_{j=1}^{k} c_j (A - \zeta I)^j)\), then there is a line through the origin that separates the field of values from a half-plane.
Suppose \( c_1 = \ldots = c_{t-1} = 0 \) but \( c_t \neq 0 \). Choose \( \theta \) so that \( e^{i\theta}(-1)^{t+1}c_t \) lies in the half-plane containing \( \mathcal{F}(\sum_{j=1}^{k} c_j (A - \zeta I)^j) \) and is orthogonal to the separating line. Then if the field of values is shifted in the direction of \( e^{i\theta}(-1)^{t+1}c_t \) its closure will exclude the origin. By choosing \( \delta > 0 \) sufficiently small, one can make the shift term in (11) arbitrarily close to \( c_t(-1)^{t+1}\delta^t e^{i\theta} \), and so one can exclude the origin from the closure of the field of values of \( \sum_{j=1}^{k} d_j (A - (\zeta - \delta e^{i\theta})I)^j \), but this contradicts (8). Therefore each coefficient \( c_j \) must be 0. \( \square \)

Theorem 1 and (5) imply also that

\[ \zeta \in \text{Int}(\mathcal{H}_k(A)) \text{ if and only if } \]

\[ \forall c_1, \ldots, c_k, \text{ not all zero, } 0 \in \text{Int} \left( \mathcal{F} \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right) \right), \]  

(12)

where \( \text{Int} \) denotes the interior.

If the origin lies outside the closure of the field of values of an operator \( B \), then \( \mathcal{F}(B) \) can be rotated into the right half-plane by multiplying \( B \) by a certain scalar \( e^{i\phi} \). Since the real part of the closure of the field of values, \( \text{Re} \mathcal{F}(e^{i\phi}B) := \{ \text{Re } z : z \in \mathcal{F}(e^{i\phi}B) \} \), is the convex hull of the spectrum of the Hermitian part of \( e^{i\phi}B \), \( H(e^{i\phi}B) := (e^{i\phi}B + e^{-i\phi}B^*)/2 \), [6, pp. 9,21], it follows that \( H(e^{i\phi}B) \) is positive definite. Therefore an equivalent statement of (4) is

\[ \zeta \notin \mathcal{H}_k(A) \text{ if and only if } \]

\[ \exists c_1, \ldots, c_k \text{ such that } H \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right) \text{ is positive definite, } \]  

(13)

and (5) becomes

\[ \zeta \in \mathcal{H}_k(A) \text{ if and only if } \]

\[ \forall c_1, \ldots, c_k, H \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right) \text{ has a nonpositive eigenvalue. } \]  

(14)

Theorem 1 can be stated as

\[ \zeta \in \partial \mathcal{H}_k(A) \text{ if and only if } \]

\[ \forall c_1, \ldots, c_k, \lambda_{\text{min}} \left( H \left( \sum_{j=1}^{k} c_j (A - \zeta I)^j \right) \right) \leq 0, \text{ and } \]

(15)
\[ \exists \ c_1, \ldots, c_k, \text{ not all 0, such that } \lambda_{\min} \left( H \sum_{j=1}^{k} c_j (A - \zeta I)^j \right) = 0, \quad (15) \]

where \( \lambda_{\min} \) denotes the smallest eigenvalue. In other words, if one chooses \( c_1, \ldots, c_k \) with, say, \( \sum_{j=1}^{k} |c_j|^2 = 1 \), to maximize \( \lambda_{\min} (H (\sum_{j=1}^{k} c_j (A - \zeta I)^j)) \), then this minimum eigenvalue will be positive if \( \zeta \notin \mathcal{H}_k(A) \), zero if \( \zeta \in \partial \mathcal{H}_k(A) \), and negative if \( \zeta \in \text{Int}(\mathcal{H}_k(A)) \).

We note one more equivalent definition. In [2], Davies introduces the set

\[ \text{Num}_k(A) := \{ z \in \mathbb{C} : p(z) \in \mathcal{F}(p(A)) \ \forall p \in \mathcal{P}_k \}. \]

From (5), and the fact that adding a scalar times the identity to an operator shifts its field of values by that scalar, we can write

\[ \zeta \in \mathcal{H}_k(A) \text{ if and only if} \]

\[ \forall \ c_0, c_1, \ldots, c_k, \quad c_0 \in \mathcal{F} \left( c_0 I + \sum_{j=1}^{k} c_j (A - \zeta I)^j \right). \quad (16) \]

Since any polynomial \( p \) of degree \( k \) or less in \( A \) can be written in the form \( c_0 I + \sum_{j=1}^{k} c_j (A - \zeta I)^j \) for certain coefficients \( c_0, c_1, \ldots, c_k \), and then \( p(\zeta) = c_0 \), equivalence (16) implies

\[ \zeta \in \mathcal{H}_k(A) \iff \forall p \in \mathcal{P}_k, \ p(\zeta) \in \mathcal{F}(p(A)); \]

i.e., \( \mathcal{H}_k(A) = \text{Num}_k(A) \). It is interesting to note that although \( \mathcal{H}_k(A) \) and \( \text{Num}_k(A) \) are the same for bounded operators, only the latter definition makes sense for unbounded operators.

### 3 Polynomial Numerical Hulls of Toeplitz Matrices and Operators

Let \( T \) denote an infinite-dimensional Toeplitz operator:

\[ T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (17) \]
The symbol of $T$ evaluated at a point $z$ is defined as
\[ a(z) := a_0 + \sum_{\ell=1}^{\infty} a_\ell z^\ell + \sum_{\ell=-1}^{\infty} a_{-\ell} z^{-\ell}. \] (18)

We will assume throughout that $a(z) \in L^\infty(U)$, where $U := \{ z : |z| = 1 \}$ denotes the unit circle, so that the matrix in (17) corresponds to a bounded linear operator on $\ell^2$ (see, e.g., [1]).

The polynomial numerical hull of degree 1 for $T$, i.e., the closure of the field of values, is known [7]:
\[ \mathcal{H}_1(T) = \text{co} (a(U)), \] (19)
where co(·) denotes the convex hull.

**Theorem 2.** The polynomial numerical hull of degree $k \geq 1$ for $T$ satisfies
\[ \text{co} (a(U)) \supset \mathcal{H}_k(T) \supset \text{pco}_k (a(U)). \] (20)

Here pco$_k$$(·)$ denotes the polynomially convex hull of degree $k$, defined for any compact set $S \subset \mathbb{C}$ as
\[ \text{pco}_k(S) := \{ z \in \mathbb{C} : |p(z)| \leq \max_{\zeta \in S} |p(\zeta)| \ \forall p \in \mathcal{P}_k \}. \]

**Proof:** The left inclusion follows from (19) and the fact that $\mathcal{H}_k(T) \subset \mathcal{H}_1(T)$. To establish the right inclusion, let $p(T)$ be any polynomial in $T$. By the spectral mapping theorem, $\sigma(p(T)) = p(\sigma(T))$, where $\sigma(·)$ denotes the spectrum. By the Hartman-Wintner theorem (see, e.g., [1, Theorem 1.25, p. 27]), the essential spectrum of $T$ contains the range of its symbol. Hence
\[ \| p(T) \| \geq \sup_{\zeta \in \sigma(p(T))} |\zeta| \geq \sup_{|\ell|=1} |p(a(z))|. \]
The result then follows from definition (1) of $\mathcal{H}_k(T)$. \qed

In words, the polynomial numerical hull of degree $k$ for an infinite-dimensional Toeplitz operator lies somewhere between the polynomially convex hull of degree $k$ and the ordinary convex hull of the image, under the symbol, of the unit circle.
Let $T_N$ be the $N$ by $N$ Toeplitz matrix consisting of the upper left block of the operator in (17):

$$T_N = \begin{pmatrix}
a_0 & a_{-1} & a_{-2} & \cdots & a_{-(N-1)} \\
a_1 & a_0 & a_{-1} & \cdots & a_{-(N-2)} \\
a_2 & a_1 & a_0 & \cdots & a_{-(N-3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0
\end{pmatrix} \quad (21)$$

It is known that if $T$ is banded or, more generally, if $a(z)$ is piecewise continuous on the unit circle, then the polynomial numerical hull of degree 1 (i.e., the field of values) of $T_N$ approaches that of $T$ as $N \to \infty$ [3, 5, 10]. Moreover, if $p(T_N)$ is any polynomial in $T_N$ and if $a(z)$ is piecewise continuous on the unit circle, then it follows from results in [10, Theorem 2] that

$$p\lim_{N \to \infty} \mathcal{F}(p(T_N)) = u\lim_{N \to \infty} \mathcal{F}(p(T_N)) = \mathcal{F}(p(T)). \quad (22)$$

Here $u\lim_{N \to \infty} S_N$, for a sequence of sets $S_N$, denotes the set of all limits of sequences of points $\{s_N \in S_N\}_{N=1}^\infty$, while $p\lim_{N \to \infty} S_N$ denotes all limits of subsequences $\{s_{N_1} \in S_{N_1}\}_{N_1=1}^\infty$, where $(N_1, N_2, \ldots)$ is a subsequence of $(1, 2, \ldots)$. Clearly $u\lim_{N \to \infty} S_N \subseteq p\lim_{N \to \infty} S_N$. Note that if $\zeta \in \text{Int}(\mathcal{F}(p(T)))$, then (22) implies that for $N$ large enough, $\zeta \in \text{Int}(\mathcal{F}(p(T_N)))$, since these sets are convex and they contain sequences of points approaching points on a circle about $\zeta$ in $\text{Int}(\mathcal{F}(p(T)))$. [See, for example, [11, Prop. 4.15].]

**Theorem 3.** Assume $a(z)$ in (18) is piecewise continuous on the unit circle. For any fixed degree $k \geq 1$,

$$\mathcal{H}_k(T) \supset p\lim_{N \to \infty} \mathcal{H}_k(T_N) \supset u\lim_{N \to \infty} \mathcal{H}_k(T_N) \supset \text{cl}(\text{Int}(\mathcal{H}_k(T))), \quad (23)$$

where $\text{cl}(\cdot)$ denotes the closure.

**Proof:** First suppose $\zeta_{N_\ell} \in \mathcal{H}_k(T_{N_\ell})$ is a convergent partial sequence with $\zeta_{N_\ell} \to \zeta$ as $\ell \to \infty$. Then it follows from (5) that for all $c_1, \ldots, c_k$, $0 \in \mathcal{F}(\sum_{j=1}^k c_j(T_{N_\ell} - \zeta I)^j)$. Since the field of values is a continuous function of the operator, it follows that for each $c_1, \ldots, c_k$, there is a sequence of points $s_{N_\ell} \in \mathcal{F}(\sum_{j=1}^k c_j(T_{N_\ell} - \zeta I)^j)$ such that $s_{N_\ell} \to 0$ as $\ell \to \infty$. By (22), this implies that $0 \in \mathcal{F}(\sum_{j=1}^k c_j(T - \zeta I)^j)$, and hence by (5), $\zeta \in \mathcal{H}_k(T)$. This
establishes the first inclusion in (23). The second inclusion is clear from the definitions of $u$-lim and $p$-lim.

Now let $\zeta$ be any point in Int$(H_k(T))$. Then we know from (12) that for all $c_1, \ldots, c_k$ with, say, $\sum_{j=1}^{k} |c_j|^2 = 1$, $0 \in \text{Int} (\mathcal{F} (\frac{1}{2} \sum_{j=1}^{k} c_j (T - \zeta I)^j))$. It follows from (22) that there exists $n \equiv n(c_1, \ldots, c_k)$ such that for all $N \geq n$, $0 \in \text{Int}(\mathcal{F}(\sum_{j=1}^{k} c_j(T_N - \zeta I)^j))$. Since the coefficients come from a compact set and since the field of values is a continuous function of $c_1, \ldots, c_k$, there exist coefficients $\hat{c}_1, \ldots, \hat{c}_k$ for which the required minimum $n$-value, $\hat{n} \equiv n(\hat{c}_1, \ldots, \hat{c}_k)$, is maximal and finite. Therefore, for $N \geq \hat{n}$ and for all $c_1, \ldots, c_k$ with $\sum_{j=1}^{k} |c_j|^2 = 1$, $0 \in \text{Int}(\mathcal{F}(\sum_{j=1}^{k} c_j(T_N - \zeta I)^j))$. Hence by (12), $\zeta \in \text{Int}(H_k(T_N))$. This shows that every interior point of $H_k(T)$ eventually lies in $H_k(T_N)$ and hence that every limit of interior points of $H_k(T)$ is a limit of points in $H_k(T_N)$. This proves the third inclusion in (23). \hfill \square

The preceding theorems leave open the possibility that $H_k(T)$ contains isolated points or curves between the region enclosed by $a(U)$ and $\text{co}(a(U))$, and, in this case, it is not known if there is a sequence or partial sequence of points in $H_k(T_N)$ converging to these points. In the simplest case, where the region enclosed by $a(U)$ is a convex set, Theorems 2 and 3 imply that

$$
p-\lim_{N \to \infty} H_k(T_N) = u-\lim_{N \to \infty} H_k(T_N) = H_k(T) = \text{co}(a(U)).\n$$

Finally, note that Theorem 3 makes no direct use of Toeplitz properties; it uses only relation (22) and properties of the polynomial numerical hull. Hence for any sequence of matrices or operators $A_N$ satisfying

$$
p-\lim_{N \to \infty} \mathcal{F}(p(A_N)) = u-\lim_{N \to \infty} \mathcal{F}(p(A_N)) = \mathcal{F}(p(A)),\n$$

for all polynomials $p$, the analogue of relation (23) will hold, namely,

$$
H_k(A) \supset p-\lim_{N \to \infty} H_k(A_N) \supset u-\lim_{N \to \infty} H_k(A_N) \supset \text{cl}(\text{Int}(H_k(A))),
$$

for any fixed $k \geq 1$.

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**References**


