EXTENDING CONSTRAINT PRECONDITIONERS FOR SADDLE POINT PROBLEMS

H. S. DOLLAR∗†

Abstract. The problem of finding good preconditioners for the numerical solution of a certain important class of indefinite linear systems is considered. These systems are of a saddle point structure

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
c \\
d
\end{bmatrix},
\]

where \(A \in \mathbb{R}^{n \times n}\), \(C \in \mathbb{R}^{m \times m}\) are symmetric and \(B \in \mathbb{R}^{m \times n}\).

In Constraint preconditioning for indefinite linear systems, SIAM J. Matrix Anal. Appl., 21 (2000), Keller, Gould and Wathen introduced the idea of using constraint preconditioners that have a specific 2 by 2 block structure for the case of \(C\) being zero. We shall extend this idea by allowing the (2,2) block to be non-zero. Results concerning the spectrum and form of the eigenvectors are presented, as are numerical results to validate our conclusions.

Key words. preconditioning, indefinite linear systems, Krylov subspace methods

1. Introduction. Recently, a large amount of work has been devoted to the problem of solving large linear systems in saddle point form. Such systems arise in a wide variety of technical and scientific applications. For example, interior point methods in both linear and nonlinear optimization require the solution of a sequence of systems in saddle point form [20]. Another popular field, which is a major source of saddle point problems, is that of mixed finite element methods in engineering fields, see [7] and [16, Chapters 7,9].

We wish to find solutions of block 2 \(\times\) 2 linear systems of the form

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
c \\
d
\end{bmatrix},
\]

(1.1)

where \(A \in \mathbb{R}^{n \times n}\), \(C \in \mathbb{R}^{m \times m}\) are symmetric and \(B \in \mathbb{R}^{m \times n}\). We shall assume that \(m \leq n\) and \(B\) is of full rank. If \(A\) and \(C\) are positive definite, then the matrix \(A\) is a permuted quasidefinite matrix, [19]. Vanderbei has shown that quasidefinite matrices are strongly factorizable, i.e., a Cholesky-like factorization \(LDL^T\) exists for any symmetric row and column permutation of the quasidefinite matrix, [19]. The diagonal matrix has \(n\) positive and \(m\) negative pivots. However, we shall not confine ourselves to quasidefinite matrices.

It may be attractive to use iterative methods to solve systems such as (1.1), particularly for large \(m\) and \(n\). In particular, Krylov subspace methods might be used. It is often advantageous to use a preconditioner, \(P\), with such iterative methods. The preconditioner should reduce the number of iterations required for convergence but not significantly increase the amount of computation required at each iteration,[18, Chapter 13].

In Section 2 we shall review the well known spectral properties of constraint preconditioning when \(C = 0\), [14]. Recently Dollar, Gould and Wathen [5] analyzed

∗Oxford University Computing Laboratory, Numerical Analysis Group, Wolfson Building, Parks Road, Oxford, OX1 3QD, U.K. (hsd@comlab.ox.ac.uk).
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what happens if we reproduce particular parts of the matrix $A$ with the constraint preconditioner. We shall also summarize these results in Section 2.

It is restrictive to assume that the matrix $C$ in saddle point systems is always a zero matrix: a number of situations arise in which $C \neq 0$ [1, 15, 17]. In all these cases, $C$ is positive semi-definite, hence, we shall consider the idea of extending constraint preconditioners to the case of $C$ being positive semidefinite. We shall extend the constraint preconditioning idea of Keller, Gould and Wathen [14] in which the (constraint) blocks $B, B^T$ are exactly reproduced in the preconditioner by also reproducing the $C$ matrix in our preconditioner, along with $B$ and $B^T$; this is considered in Section 3. In Section 4 we shall report numerical results where our preconditioners have been used to solve various test problems.

2. Constraint preconditioners. Let us initially assume that $C = 0$. Keller, Gould and Wathen [14] investigated the use of a preconditioner of the form

$$P = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

(2.1)

where $G$ approximates but is not the same as $A$. They were able to prove various results about the eigenvalues and eigenvectors for the preconditioned systems $P^{-1}A$, where $A$ and $P$ are defined in (1.1) and (2.1) respectively. $P$ is called a constraint preconditioner. Proof of the following theorem can be found in [14].

**Theorem 2.1.** Let $A \in \mathbb{R}^{(n+m)\times(n+m)}$ be a symmetric and indefinite matrix of the form

$$A = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and $B \in \mathbb{R}^{m\times n}$ is of full rank. Assume $Z$ is an $n \times (n-m)$ basis for the nullspace of $B$. Preconditioning $A$ by a matrix of the form

$$P = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

where $G \in \mathbb{R}^{n\times n}$ is symmetric, $G \neq A$, and $B \in \mathbb{R}^{m\times n}$ is as above, implies that the matrix $P^{-1}A$ has

1. an eigenvalue at 1 with multiplicity $2m$;
2. $n - m$ eigenvalues $\lambda$ which are defined by the generalized eigenvalue problem $Z^T AZ x_z = \lambda Z^T G Z x_z$.

Assume, in addition, that $Z^T G Z$ is positive definite. Then $P^{-1}A$ has the following $m + i + j$ linearly independent eigenvectors:

1. $m$ eigenvectors of the form $[0^T, y^T]^T$ corresponding to the eigenvalue 1 of $P^{-1}A$;
2. $i$ ($0 \leq i \leq n$) eigenvectors of the form $[w^T, y^T]^T$ corresponding to the eigenvalue 1 of $P^{-1}A$, were the components $w$ arise from the generalized eigenvalue problem $A w = G w$;
3. $j$ ($0 \leq j \leq n - m$) eigenvectors of the form $[x_z^T, 0^T, y^T]^T$ corresponding to the eigenvalues of $P^{-1}A$ not equal to 1, where the components $x_z$ arise from the generalized eigenvalue problem $Z^T AZ x_z = \lambda Z^T G Z x_z$ with $\lambda \neq 1$.

If either $Z^T AZ$ or $Z^T G Z$ are positive definite, then the indefinite preconditioner $P$ applied to the indefinite saddle point matrix $A$ with $C = 0$ yields a preconditioned
matrix $\mathcal{P}^{-1}A$ which has real eigenvalues, [14]. If both $Z^T AZ$ and $Z^T GZ$ are positive definite, then we can use a projected preconditioned conjugate gradient method to find $x$ and $y$, see [11].

If $Z^T GZ$ is positive definite, then for each initial residual $r_0$, the Krylov subspace $K(\mathcal{P}^{-1}A, b)$ has dimension of at most $n - m + 2$, [14, Theorem 3.5]. In many applications, $n - m + 2$ is quite large, so that total of $n - m + 2$ iterations may be prohibitive. Clearly, the better $G$ approximates $A$, the more the non-unit eigenvalues will be clustered around 1 in which case iterative convergence might be expected to take many fewer iterations. Dollar, Gould and Wathen investigated how different choices of $G$ will effect the eigenvalues; in particular, how the bound on the number of non-unitary eigenvalues changes. Suppose that we partition $A$ and $G$ so that

$$A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} G_{11} & G_{21}^T \\ G_{21} & G_{22} \end{bmatrix}, \quad (2.2)$$

where $A_{11}$ and $G_{11}$ are (respectively) the leading $m$ by $m$ sub-matrices of $A$ and $G$.

Similarly, we shall partition the columns of $B$ such that

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

where $B_1$ and its transpose are easily invertible. We shall be particularly concerned with the reduced-space basis matrix

$$Z = \begin{bmatrix} R \\ I \end{bmatrix}, \quad \text{where} \quad R = -B_1^{-1}B_2. \quad (2.3)$$

Equations (2.2) and (2.3) give

$$Z^T AZ = A_{22} + R^T A_{21}^T + A_{21}R + R^T A_{11}R$$

and

$$Z^T GZ = G_{22} + R^T G_{21}^T + G_{21}R + R^T G_{11}R.$$

Proofs of the following theorems can be found in [5].

**Theorem 2.2.** Suppose that $A$ and $G$ are as in (2.2) and $G_{22} = A_{22}$, but $G_{11} = 0$ and $G_{21} = 0$ holds. Suppose furthermore that $A_{22}$ is positive definite, and let

$$\rho = \min \{ \text{rank} (B_2), \text{rank} (A_{21}) \} + \min \{ \text{rank} (B_2), \text{rank} (A_{21}) + \min (\text{rank} (B_2), \text{rank} (A_{11})) \}.$$  

Then $\mathcal{P}^{-1}A$ has at most

$$\text{rank} \left( R^T A_{21}^T + A_{21}R + R^T A_{11}R \right) + 1 \leq \min (\rho, n - m) + 1 \leq \min (2m, n - m) + 1$$

distinct eigenvalues.

**Theorem 2.3.** Suppose that $A$ and $G$ are as in (2.2) and that $G_{22} = A_{22}$, $G_{11} = A_{11}$, but $G_{21} = 0$ holds. Suppose furthermore that $A_{22} + R^T A_{11}^T R$ is positive definite, and that

$$\nu = 2 \min \{ \text{rank} (B_2), \text{rank} (A_{21}) \}.$$  

Then $\mathcal{P}^{-1}A$ has at most

$$\text{rank} \left( R^T A_{21}^T + A_{21}R \right) + 1 \leq \min (\nu, n - m) + 1 \leq \min (2m, n - m) + 1$$
distinct eigenvalues.

**Theorem 2.4.** Suppose that $A$ and $G$ are as in (2.2) and that $G_{22} = A_{22}$, $G_{21} = A_{21}$, but $G_{11} = 0$ holds. Suppose furthermore that $A_{22} + R^T A_{21}^T + A_{21} R$ is positive definite, and that

$$
\mu = \min \{ \text{rank} (B_2), \text{rank} (A_{11}) \}.
$$

Then $P^{-1}A$ has at most

$$\text{rank} (R^T A_{11} R) + 1 \leq \mu + 1 \leq \min(m, n - m) + 1$$

distinct eigenvalues.

In [5], Dollar, Gould and Wathen illustrate some of the results from these theorems by considering the complete set of linear and quadratic programming examples from the Netlib [9] and CUTEr [12] test sets. They observed that in some cases there are useful gains to be made from trying to reproduce $A_{22}$ and, less often, $A_{21}$. Moreover, the upper bounds on the rank obtained in Theorems 2.2 and 2.4 can be significantly larger than even the estimates $\rho + 1$ and $\mu + 1$ of the number of distinct eigenvalues.

3. Constraint preconditioners for the case $C \neq 0$. As we have already mentioned, it is restrictive to assume that $C = 0$ in (1.1). In the following sections we consider the case of $C$ being positive semi-definite. The term constraint preconditioner was introduced in [14] because the (1,2) and (2,1) matrix blocks of the preconditioner are exact representations of those in $A$, where these blocks represent constraints. However, we also observe that the (2,2) matrix block is also an exact representation when $C = 0$. This motivates us to generalize the constraint preconditioner to take the form

$$
P = \begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix},
$$

where $G \in \mathbb{R}^{n \times n}$ approximates, but is not the same as $A$.

For symmetric (and in general normal) matrix systems, the convergence of an applicable iterative method is determined by the distribution of the eigenvalues of the coefficient matrix. It is often desirable for the number of distinct eigenvalues to be small so that the rate of convergence is rapid. For non-normal systems the convergence is not so readily described, see [13, page 6].

**Theorem 3.1.** Let $A \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form

$$
A = \begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix},
$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank. We shall assume that $C$ has rank $p > 0$. Let $Z \in \mathbb{R}^{n \times (n-m)}$ be a basis for the nullspace of $B$. Preconditioning $A$ by a matrix of the form

$$
P = \begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix},
$$

where $G \in \mathbb{R}^{n \times n}$ is symmetric, $G \neq A$, and $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times m}$ are as above, implies that the matrix $P^{-1}A$ has at most $i + j + 1$ distinct eigenvalues which satisfy
• at least \( m \) eigenvalues at 1,
• \( i \) \((0 \leq i \leq n - p)\) non-unit eigenvalues that satisfy
\[
\lambda(P^{-1}A) = \frac{x_Z^T Z^T A Z x_Z}{x_Z^T Z^T G Z x_Z}
\]
for some \( x_Z \neq 0 \),
• \( j \) \((0 \leq j \leq p)\) non-unit eigenvalues that satisfy
\[
\lambda(P^{-1}A) = \frac{x^T A x + y^T C y}{x^T G x + y^T C y},
\]
for some \( y \neq 0 \), and \( B x = C y \).

If \( C \) is non-singular, then the \( j \) \((0 \leq j \leq m)\) non-unit eigenvalues also satisfy
\[
\lambda(P^{-1}A) = \frac{x^T (A + B^T C^{-1} B) x}{x^T (G + B^T C^{-1} B) x}
\]
for some \( x \neq 0 \).

**Proof.** The eigenvalues of the preconditioned coefficient matrix \( P^{-1}A \) may be derived by considering the generalized eigenvalue problem
\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \lambda
\begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
\]
Expanding this out we obtain
\[
A x + B^T y = \lambda G x + \lambda B^T y,
\]
and
\[
B x - C y = \lambda B x - \lambda C y.
\]
Suppose that \( \lambda = 1 \). Then (3.4) will trivially hold, and (3.3) implies that
\[
A x = G x.
\]
Hence, for any \( y \in \mathbb{R}^m \) and \( x = 0 \), Equations (3.3) and (3.4) will hold when \( \lambda = 1 \). Therefore, there are \( m \) linearly independent eigenvectors of the form \([0^T, y^T]^T\) associated with the eigenvalue at 1. This implies that there are at least \( m \) eigenvalues at 1.

If \( \lambda \neq 1 \) and \( C y = 0 \), then (3.4) implies that \( B x = 0 \). Hence, \( x \in \text{Null}(B) \), so we can write \( x = Z x_Z \) for some \( x_Z \in \mathbb{R}^{n-m} \). Premultiplying (3.3) by \( x \) and substituting in \( x = Z x_Z \) gives
\[
\lambda(P^{-1}A) = \frac{x_Z^T Z^T A Z x_Z}{x_Z^T Z^T G Z x_Z}.
\]
Therefore, if \( C y = 0 \), then there are at most \((n-m) + (m-p)\) linearly independent eigenvectors of the form \([x_Z^T Z^T, y^T]^T\) associated with the non-unit eigenvalues.

If \( \lambda \neq 1 \) and \( C y \neq 0 \), then (3.4) implies that \( B x \neq 0 \). We can write \( x = B^T x_B + Z x_Z \), where \( x_B \in \mathbb{R}^m \) and \( x_Z \in \mathbb{R}^{n-m} \), since \( B \) is of full rank. Equation 3.4
implies that $x_B = (BB^T)^{-1}Cy$ and $x = B^T(BB^T)^{-1}Cy + Zx_Z$. Premultiplying (3.3) by $x^T$ and substituting in $Bx = Cy$ gives

$$\lambda(P^{-1}A) = \frac{x^TAx + y^TCy}{x^TGx + y^TCy}. \quad (3.6)$$

There are at most $p$ linearly independent eigenvectors of the form $[x_B^T B + x_Z^T Z^T, y^T]^T$, where $Cy \neq 0$, associated with these eigenvalues.

If, in addition, $C$ is non-singular, then $p = m$. Equation (3.4) along with the non-singularity of $C$ implies that $y = C^{-1}Bx$. Substituting this into (3.6) gives the required result.

Note: In the case of symmetric positive definite matrices $C$ and $G$, Theorem 3.1 is not new, see [2] and [3].

We shall follow a similar method to Keller, Gould and Wathen [14] to find the number of linearly independent eigenvectors of $P^{-1}A$ for the case of general symmetric and positive semi-definite $C$. In the following theorem we assume that $C$ is of rank $p$, and that we can find a decomposition $C = EDE^T$, where $E$ is $m$ by $p$ with orthogonal columns and $D$ is $p$ by $p$ and invertible. A singular value decomposition is suitable for this (but is also clearly not unique).

**Theorem 3.2.** Let $A \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form

$$A = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank. We shall assume that $C$ has rank $p$ and is factored as $EDE^T$, where $E \in \mathbb{R}^{m \times p}$ has orthogonal columns and $D \in \mathbb{R}^{p \times p}$ is non-singular, $F \in \mathbb{R}^{m \times (m-p)}$ is a basis for the nullspace of $E^T$, $Z \in \mathbb{R}^{n \times (n-m)}$ is a basis for the nullspace of $B$ and $Y \in \mathbb{R}^{n \times m}$ is such that $[Y \ Z]$ spans $\mathbb{R}^n$. Suppose that $Z^TGZ$ and $G + B^TED^{-1}E^TB$ are positive definite. Preconditioning $A$ by a matrix of the form

$$P = \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix},$$

where $G \in \mathbb{R}^{n \times n}$ is symmetric, $G \neq A$, and $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times m}$ are as above, implies that the matrix $P^{-1}A$ has $n + m$ eigenvalues as defined in Theorem 3.1 and $m + i + j + k$ linearly independent eigenvectors. There are

1. $m$ eigenvectors of the form $[0^T \ 0^T \ y_f^T \ y_e^T]^T$ that correspond to the case $\lambda = 1$;
2. $i$ ($0 \leq i \leq n$) eigenvectors of the form $[x_z^T \ x_y^T \ y_f^T \ y_e^T]^T$ arising from $Aw = \sigma Gw$ with $w = [x_y^T \ x_z^T]^T$ linearly independent, $\sigma = 1$, and $\lambda = 1$;
3. $j$ ($0 \leq j \leq n - m$) eigenvectors of the form $[x_z^T \ 0^T \ y_f^T \ 0^T]^T$ that correspond to the case $\lambda \neq 1$ and $Cy = 0$ with $y = Fy_f + Ey_e$, where the components $x_z$ arise from the generalized eigenvalue problem $Z^TAZx_z = \lambda Z^TGZx_z$;
4. $k$ ($0 \leq k \leq p$) eigenvectors of the form $[x_z^T \ x_y^T \ 0^T \ y_e^T]^T$ that correspond to the case $\lambda \neq 1$ and $Cy \neq 0$ with $y = Fy_f + Ey_e$, where the components $w = [x_y^T \ x_z^T]^T$ arise from the generalized eigenvalue problem $(A + B^TED^{-1}E^TB)w = \lambda(G + B^TED^{-1}E^TB)w$. 
Proof. If \( p = 0 \), then we have Theorem 2.1. Suppose that \( p > 0 \).

Let \( QR = [Y \ Z] \begin{bmatrix} R^T & 0^T \end{bmatrix}^T \) be an orthogonal factorization of \( B^T \), where \( R \in \mathbb{R}^{m \times m} \) is upper triangular, \( Y \in \mathbb{R}^{n \times m} \), and \( Z \in \mathbb{R}^{n \times (n-m)} \) is a basis for the nullspace of \( B \). We can therefore write any \( x \in \mathbb{R}^n \) as \( x = Zx_z + Yx_y \), where \( x_z \in \mathbb{R}^{n-m}, x_y \in \mathbb{R}^m \) are unique vectors. Similarly, by our assumptions, any \( y \in \mathbb{R}^m \) can be written as \( y = Fy_f + Ey_e \) where \( y_f \in \mathbb{R}^{m-p}, y_e \in \mathbb{R}^p \) are unique vectors. Premultiplying (3.2) by the nonsingular and square matrix

\[
\begin{bmatrix}
Z^T & 0 \\
y^T & 0 \\
0 & F^T \\
0 & E^T
\end{bmatrix}
\]

and post multiplying by its transpose gives

\[
\begin{bmatrix}
Z^T A Z & Z^T A Y & 0 & 0 \\
Y^T A Z & Y^T A Y & RF & RE \\
0 & F^T R^T & 0 & 0 \\
0 & E^T R^T & 0 & -D
\end{bmatrix}
\begin{bmatrix}
x_z \\
x_y \\
y_f \\
y_e
\end{bmatrix}
= \lambda
\begin{bmatrix}
Z^T G Z & Z^T G Y & 0 & 0 \\
Y^T G Z & Y^T G Y & RF & RE \\
0 & F^T R^T & 0 & 0 \\
0 & E^T R^T & 0 & -D
\end{bmatrix}
\begin{bmatrix}
x_z \\
x_y \\
y_f \\
y_e
\end{bmatrix},
\]

(3.7)

where we made use of the the equalities \( BZ = 0, C = EDE^T, CF = 0, E^T E = I \), and \( R = (BY)^T \). Expanding out the general eigenvalue problem (3.7) yields

\[
\begin{align*}
Z^T A Z x_z + Z^T A Y x_y &= \lambda [Z^T G Z x_z + Z^T G Y x_y], \\
Y^T A Z x_z + Y^T A Y x_y + RF y_f + RE y_e &= \lambda [Y^T A Z x_z + Y^T A Y x_y + RF y_f + RE y_e], \\
F^T R^T x_y &= \lambda F^T R^T x_y, \\
E^T R^T x_y - D y_e &= \lambda [E^T R^T x_y - D y_e].
\end{align*}
\]

From (3.10), it may be deduced that either \( \lambda = 1 \) or \( R^T x_y \in \text{Null}(F^T) \). In the former case, (3.11) holds trivially, whilst (3.8) and (3.9) simplify to

\[
\begin{align*}
Z^T A Z x_z + Z^T A Y x_y &= Z^T G Z x_z + Z^T G Y x_y, \\
Y^T A Z x_z + Y^T A Y x_y &= Y^T G Z x_z + Y^T G Y x_y,
\end{align*}
\]

which can consequently be written as

\[
Q^T A Q w = Q^T G Q w,
\]

(3.12)

where \( Q = [Y \ Z] \) and \( w = [x_y^T \ x_z^T]^T \). Since \( Q \) is orthogonal, the general eigenvalue problem (3.12) is equivalent to considering

\[
A w = \sigma G w,
\]

(3.13)

where \( w \neq 0 \) if and only if \( \sigma = 1 \). There are \( m \) linearly independent eigenvectors \( [0^T \ 0^T \ y_f^T \ y_e^T]^T \) corresponding to \( w = 0 \), and a further \( i \) \((1 \leq i \leq n)\) linearly independent eigenvectors (corresponding to eigenvalues \( \sigma = 1 \) of (3.13)).

Now, suppose that \( \lambda \neq 1 \), in which case \( R^T x_y \in \text{Null}(F^T) \). Equation (3.11) also implies that

\[
E^T R^T x_y = D y_e.
\]

(3.14)
We have two cases, either $C_y = 0$ or $C_y \neq 0$. If the former holds, then $y_e = 0$ and (3.14) implies that $R^T x_y \in \text{Null}(E^T)$, as well as $R^T x_y \in \text{Null}(F^T)$. Hence, $x_y = 0$.

From (3.8) and (3.9) we obtain

$$Z^T A Z x_z = \lambda Z^T G Z x_z,$$

$$Y^T A Z x_z + R F y_f = \lambda [Y^T G Z x_z + R F y_f].$$

(3.15)

(3.16)

The generalized eigenvalue problem (3.15) defines $n - m$ eigenvalues, where $j (1 \leq j \leq n - m)$ of these are not equal to 1 and for which two cases have to be distinguished.

If $x_z \neq 0$, $y_f$ must satisfy

$$[Y^T A Z - \lambda Y^T G Z] x_z = (\lambda - 1) R F y_f,$$

(3.17)

which follows that the corresponding eigenvectors are defined by $[x_z^T \ 0^T \ y_f^T \ 0^T]^T$.

If $x_z = 0$, then from Equation (3.16) we deduce that $y_f = 0$ since $\lambda \neq 1$. As $[x_z^T \ x_y^T \ y_f^T \ y_e^T]^T = 0$ in this case, no extra eigenvalues arise.

Suppose that $C_y \neq 0$, then any $y$ satisfying this can be written as $y = F y_f + E y_e$, where $y_e \neq 0$. The fact that the matrix $D$ is non-singular along with (3.14) implies that

$$y_e = D^{-1} E^T R^T x_y.$$

(3.17)

Equations 3.8 and 3.9, along with $\lambda \neq 1$ imply that $y_f = 0$. Substituting this and (3.17) into (3.9) gives

$$\begin{bmatrix} Y^T(A + B^T E D^{-1} E^T B) Y & Y^T A Z \\ Z^T A Y & Z^T A Z \end{bmatrix} \begin{bmatrix} x_y \\ x_z \end{bmatrix} = \lambda \begin{bmatrix} Y^T(G + B^T E D^{-1} E^T B) Y & Y^T G Z \\ Z^T G Y & Z^T G Z \end{bmatrix} \begin{bmatrix} x_y \\ x_z \end{bmatrix}.$$  

(3.18)

Now, it is straightforward to show that

$$Q^T B^T E D^{-1} E^T B Q = \begin{bmatrix} Y^T B^T E D^{-1} E^T B Y & 0 \\ 0 & 0 \end{bmatrix},$$

(3.19)

where $Q = [Y \ Z]$. The generalized eigenvalue problem (3.18) becomes

$$Q^T(A + B^T E D^{-1} E^T B) Q w = \lambda Q^T(G + B^T E D^{-1} E^T B) Q w,$$

(3.20)

where $w = [x_y^T \ x_z^T]^T$. The orthogonality of $Q$ implies that the generalized eigenvalue problem (3.18) is equivalent to considering

$$(A + B^T E D^{-1} E^T B) w = \lambda(G + B^T E D^{-1} E^T B) w.$$  

(3.21)

The generalized eigenvalue problem (3.21) defines at most $p$ eigenvalues for which $R^T x_y \in \text{Null}(F^T)$, and $C_y \neq 0$ are also satisfied. Of these, $k (0 \leq k \leq p)$ are not equal to 1. It follows that the corresponding eigenvectors are defined by $[x_z^T \ x_y^T \ 0^T \ y_e^T]^T$. 


It remains for us to prove that the \(m + i + j + k\) eigenvectors of \(\mathcal{P}^{-1}\mathcal{A}\) are linearly independent. We need to show that

\[
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
y_1^{(1)} & \cdots & y_m^{(1)} \\
y_1^{(1)} & \cdots & y_m^{(1)}
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{(1)} \\
\vdots \\
\alpha_m^{(1)} \\
\lambda_1 \\
\vdots \\
\lambda_i \\
\vdots \\
\lambda_j \\
\vdots \\
\lambda_k \\
\vdots \\
\lambda_l \\
\vdots \\
\lambda_m
\end{bmatrix}
+ \begin{bmatrix}
x_1^{(2)} & \cdots & x_{j+1}^{(2)} \\
x_1^{(2)} & \cdots & x_{j+1}^{(2)} \\
y_1^{(2)} & \cdots & y_m^{(2)} \\
y_1^{(2)} & \cdots & y_m^{(2)}
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{(2)} \\
\vdots \\
\alpha_m^{(2)} \\
\lambda_1 \\
\vdots \\
\lambda_i \\
\vdots \\
\lambda_j \\
\vdots \\
\lambda_k \\
\vdots \\
\lambda_l \\
\vdots \\
\lambda_m
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\tag{3.22}
\]

implies that the vectors \(\alpha_l^{(l)}\) \((l = 1, \ldots, 4)\) are zero vectors. Multiplying (3.22) by \(\mathcal{A}\) and \(\mathcal{P}^{-1}\), and recalling that in the previous equation the first matrix arises from \(\lambda_l = 1\) \((l = 1, \ldots, m)\), the second matrix from the case that \(\lambda_l = 1\) \((l = 1, \ldots, i)\), the third matrix from \(\lambda_j \neq 1\) \((l = 1, \ldots, j)\) and \(CY = 0\), and the last matrix from \(\lambda_l \neq 1\) \((l = 1, \ldots, k)\) and \(CY \neq 0\), gives

\[
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
y_1^{(1)} & \cdots & y_m^{(1)} \\
y_1^{(1)} & \cdots & y_m^{(1)}
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{(1)} \\
\vdots \\
\alpha_m^{(1)} \\
\lambda_1 \\
\vdots \\
\lambda_j \\
\vdots \\
\lambda_k \\
\vdots \\
\lambda_m
\end{bmatrix}
+ \begin{bmatrix}
x_1^{(2)} & \cdots & x_{j+1}^{(2)} \\
x_1^{(2)} & \cdots & x_{j+1}^{(2)} \\
y_1^{(2)} & \cdots & y_m^{(2)} \\
y_1^{(2)} & \cdots & y_m^{(2)}
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{(2)} \\
\vdots \\
\alpha_m^{(2)} \\
\lambda_1 \\
\vdots \\
\lambda_i \\
\vdots \\
\lambda_j \\
\vdots \\
\lambda_k \\
\vdots \\
\lambda_l \\
\vdots \\
\lambda_m
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\tag{3.23}
\]

Subtracting (3.22) from (3.23) gives

\[
\begin{bmatrix}
x_1^{(3)} & \cdots & x_{j+2}^{(3)} \\
x_1^{(3)} & \cdots & x_{j+2}^{(3)} \\
y_1^{(3)} & \cdots & y_m^{(3)} \\
y_1^{(3)} & \cdots & y_m^{(3)}
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{(3)} a_1^{(3)} \\
\vdots \\
\lambda_j^{(3)} a_j^{(3)} \\
\vdots \\
\lambda_k^{(3)} a_k^{(3)} \\
\vdots \\
\lambda_m^{(3)}
\end{bmatrix}
+ \begin{bmatrix}
x_1^{(4)} & \cdots & x_{j+2}^{(4)} \\
x_1^{(4)} & \cdots & x_{j+2}^{(4)} \\
y_1^{(4)} & \cdots & y_m^{(4)} \\
y_1^{(4)} & \cdots & y_m^{(4)}
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{(4)} a_1^{(4)} \\
\vdots \\
\lambda_j^{(4)} a_j^{(4)} \\
\vdots \\
\lambda_k^{(4)} a_k^{(4)} \\
\vdots \\
\lambda_m^{(4)}
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\tag{3.24}
\]

The assumption that \(CY \neq 0\) and \(G + B^T E D^{-1} E^T B\) is positive definite, implies that \(y_l^{(l)}\) \((l = 1, \ldots, k)\) in (3.18) are linearly independent, giving rise to \((\lambda_j^{(4)} - 1) a_j^{(3)} = 0\) \((l = 1, \ldots, k)\). The eigenvalues \(\lambda_l^{(4)}\) \((l = 1, \ldots, j)\) are non-unit which implies that \(a_l^{(4)} = 0\) \((l = 1, \ldots, j)\). Similarly, the assumption that \(Z^T G Z\) is positive definite implies that \(x_l^{(3)}\) \((l = 1, \ldots, j)\) in (3.24) are linearly independent, and hence \(a_l^{(3)} = 0\) \((l = 1, \ldots, j)\).

We also have linear independence of \([x^{(2)}_l y^{(2)}_l]^T\) \((l = 1, \ldots, i)\), and thus \(a_l^{(2)} = 0\) \((l = 1, \ldots, i)\). Equation 3.22 simplifies to

\[
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
y_1^{(1)} & \cdots & y_m^{(1)} \\
y_1^{(1)} & \cdots & y_m^{(1)}
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{(1)} \\
\vdots \\
\alpha_m^{(1)}
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}.
\]
implies that the matrix $P$ definite will be hard to verify for general symmetric matrices $C$ and $G$. However, $C$ has been assumed to be semi-definite in this section, so $B^2ED^{-1}ETB$ is also semi-definite. Many simple choices of $G$ will be positive definite, for example, $G = I$. It is then trivial to show that $Z^TG$ and $G + B^2ED^{-1}ETB$ are positive.

To show that both the lower and upper bounds can be attained we need only consider Examples 2.5 and 2.6 from [14]. We shall slightly adapt the second example such that $C \neq 0$.

**Example 3.2** (minimum bound). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so that $m = 1$, $n = 2$ and $p = 0$. The preconditioned matrix $P^{-1}A$ has an eigenvalue at 1 with multiplicity 3, but only one eigenvector arising from case (1) of Theorem 3.2. This eigenvector may be taken to be $[0 \ 0 \ 1]^T$.

**Example 3.3** (maximum bound). Let $A \in \mathbb{R}^{3 \times 3}$ be as defined in Example 3.2, but its (3,3) component is now -1, i.e. $C = [1]$ and $p = 1$. Assume that $P = A$. The preconditioned matrix $P^{-1}A$ has an eigenvalue at 1 with multiplicity 3 and clearly a complete set of eigenvectors. These may be taken to be the columns of the identity matrix.

**Corollary 3.3.** Let $A \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form

$$A = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank. We shall assume that $C$ has rank $p$, and is factored as $EDET$, where $E \in \mathbb{R}^{m \times p}$ has orthogonal columns and $D \in \mathbb{R}^{p \times p}$ is non-singular, $F \in \mathbb{R}^{m \times (m-p)}$ is a basis for the nullspace of $C$, $Z \in \mathbb{R}^{n \times (n-m)}$ is a basis for the nullspace of $B$ and $Y \in \mathbb{R}^{n \times m}$ is such that $[Y \ Z]$ spans $\mathbb{R}^n$. Preconditioning $A$ by a matrix of the form

$$P = \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix},$$

where $G \in \mathbb{R}^{n \times n}$ is symmetric, $G \neq A$, and $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times m}$ are as above, implies that the matrix $P^{-1}A$ has

- at least $2m - p$ eigenvalues at 1,
- $n - m$ eigenvalues defined by the generalized eigenvalue problem
  $$Z^T A Z x_z = \lambda Z^T G Z x_z,$$

  of which $i$ ($0 \leq i \leq n - m$) are non-unit,
- at most $p$ eigenvalues defined by the generalized eigenvalue problem (3.21),

  where $w = \begin{bmatrix} x_y^T \\ x_z^T \end{bmatrix}$ and $E^T BY x_y \neq 0$. Of these, $j$ ($0 \leq j \leq p$) are not equal to 1.
Example 3.4. The CUTEr test set [12] provides a set of quadratic programming problems. We shall use the problem CVXQP1_S in the following examples. This problem is very small with \( n = 100 \) and \( m = 50 \). We shall set \( G = \text{diag}(A) \), \( C = \text{diag}(0, \ldots, 0, 1, \ldots, 1) \) and vary the number of zeros on the diagonal of \( C \) so as to change its rank.

In Figure 3.1, we illustrate the change in the eigenvalues of the preconditioned system \( P^{-1}A \) for three different choices of \( C \). The eigenvalues are sorted so that

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+m}.
\]

When \( C = 0 \), we expect there to be at least \( 2m \) unit eigenvalues [14]. We observe that our example has exactly \( 2m \) eigenvalues at 1. From Corollary 3.3, when \( C = I \), there will be at least \( m \) unit eigenvalues. Our example has exactly \( m \) unit eigenvalues, Figure 3.1.

When \( C \) has rank \( \frac{m}{2} \), then the preconditioned system \( P^{-1}A \) has at least \( \frac{3m}{2} \) unit eigenvalues, Corollary 3.3. Once again the number of unit eigenvalues for our example is exactly the lower bound given by the corollary.

3.1. Improved eigenvalue bounds. Dollar, Gould and Wathen considered how the bound on the number of distinct eigenvalues changes with three specific choices of \( G \) when \( C = 0 \), [5]. These results are summarized in Section 2. We shall now consider what happens to the bounds when \( C \) is positive semi-definite.

Using the block 2 by 2 forms of \( A \) and \( G \) as described in (2.2), and \( B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \), where \( B_1 \) and its transpose are easily invertible, we can express the
generalized eigenvalue problem (3.21) as
\[
\begin{bmatrix}
  A_{11} + B_1^T E D^{-1} E^T B_1 & A_{21}^T + B_2^T E D^{-1} E^T B_2 \\
  A_{21} + B_1^T E D^{-1} E^T B_1 & A_{22} + B_2^T E D^{-1} E^T B_2
\end{bmatrix}
\begin{bmatrix}
  x_y \\
  x_z
\end{bmatrix}
\]
\[
= \lambda
\begin{bmatrix}
  G_{11} + B_1^T E D^{-1} E^T B_1 & G_{21}^T + B_2^T E D^{-1} E^T B_2 \\
  G_{21} + B_1^T E D^{-1} E^T B_1 & G_{22} + B_2^T E D^{-1} E^T B_2
\end{bmatrix}
\begin{bmatrix}
  x_y \\
  x_z
\end{bmatrix}.
\]
(3.25)

This expansion will be used in the proofs of the following theorems.

**Theorem 3.4.** Suppose that \( A \) and \( G \) are as in (2.2) and \( G_{22} = A_{22} \), but \( G_{11} = 0 \) and \( G_{21} = 0 \). Suppose furthermore that \( A_{22} \) and \( G + B^T E D^{-1} E^T B \) are positive definite, and let
\[
\rho = \min \left[ \text{rank} (B_2), \text{rank} (A_{21}) \right] + \min \left[ \text{rank} (B_2), \text{rank} (A_{21}) + \min \text{rank} (B_2), \text{rank} (A_{11}) \right],
\]
and
\[
\varrho = \text{rank} (A_{11}) + 2 \text{rank} (A_{21}).
\]
Then \( P^{-1} A \) has at most
\[
\text{rank} \left( R^T A_{21}^T + A_{21} R + R^T A_{11} R \right) + \text{rank} \left( \begin{bmatrix} A_{11} & A_{21} \\ A_{21}^T & 0 \end{bmatrix} \right) + 1
\leq \min (\rho, n - m) + \min (\varrho, p) + 1
\leq \min (2m, n - m) + p + 1
\]
distinct eigenvalues.

**Proof.** Using Corollary 3.3, the non-unit eigenvalues of \( P^{-1} A \) are defined by either the non-unit eigenvalues of the generalized eigenvalue problem
\[
Z^T A Z x_z = \lambda Z^T G Z x_z,
\]
or the non-unit eigenvalues of (3.21) for which \( C_y \neq 0 \).

From Theorem 2.2 we know that there are at most \( \text{rank} \left( R^T A_{21}^T + A_{21} R + R^T A_{11} R \right) \) non-unit eigenvalues defined by the generalized eigenvalue problem
\[
Z^T A Z x_z = \lambda Z^T G Z x_z.
\]

Elementary bounds involving the products and sums of matrices show that the difference
\[
\tilde{A} - \tilde{G} = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & 0 \end{bmatrix}
\]
is a matrix of rank at most \( \min (\varrho, n) \). Since \( \tilde{G} \) is, by assumption, positive definite, we may write \( \tilde{G} = W^T W \) for some non-singular \( W \). Thus
\[
W^{-T} \tilde{A} W^{-1} = I + W^{-T} \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & 0 \end{bmatrix} W^{-1}
\]
Theorem 3.5. Suppose that $A$ and $G$ are as in (2.2), $G_{11} = A_{11}$ and $G_{22} = A_{22}$, but $G_{21} = 0$ holds. Suppose furthermore that $A_{22} + R^T A_{21}^T R$ and $G + B^T E D^{-1} E^T B$ are positive definite, and let

$$\nu = 2 \min \{ \rank(B_2), \rank(A_{21}) \}.$$ 

Then $P^{-1} A$ has at most

$$\rank \left( R^T A_{21}^T + A_{21} R \right) + \min(2 \rank(A_{21}), p) + 1$$

$$\leq \min(\nu, n-m) + \min(2 \rank(A_{21}), p) + 1$$

$$\leq \min(2m, n-m) + \min(2(n-m), p) + 1$$

distinct eigenvalues.

Proof. The proof is similar to that of Theorem 3.4. The difference

$$\tilde{A} - \tilde{G} = \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix}$$

differs from the identity matrix by a matrix of rank at most $\min(\nu, n-m)$, and hence the generalized eigenvalue problem (3.21) has at most $\min(\nu, p)$ non-unit eigenvalues where $Cy \neq 0$.

Theorem 3.6. Suppose that $A$ and $G$ are as in (2.2), $G_{21} = A_{21}$ and $G_{22} = A_{22}$, but $G_{11} = 0$ holds. Suppose furthermore that $A_{22} + R^T A_{21}^T R$ and $G + B^T E D^{-1} E^T B$ are positive definite, and let

$$\mu = \min \{ \rank(B_2), \rank(A_{11}) \} \text{ and } \nu = \min(\rank(A_{11}), p).$$

Then $P^{-1} A$ has at most

$$\rank \left( R^T A_{11}^T R \right) + \min(\rank(A_{11}), p) + 1 \leq \mu + \nu + 1 \leq \min(m, n-m) + p + 1$$

distinct eigenvalues.

Proof. The proof is similar to that of Theorem 3.4. The difference

$$\tilde{A} - \tilde{G} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

is a matrix of rank $\rank(A_{11})$. Since $\tilde{G}$ is, by assumption, positive definite, we may write $\tilde{G} = W^T W$ for some non-singular $W$. Thus

$$W^{-T} \tilde{A} W^{-1} = I + W^{-T} \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}$$
differs from the identity matrix by a matrix of rank $2\text{rank } (A_{21})$, and hence the generalized eigenvalue problem (3.21) has rank $(A_{11})$ non-unit eigenvalues. Since $Cy \neq 0$ must also be satisfied, there are at most $\min(\text{rank } (A_{11}), p)$ non-unit eigenvalues of (3.21) for which $Cy \neq 0$.

4. Numerical results. In this section we indicate that, in some cases, the extended constraint preconditioners of Section 3 can be very effective. We consider a subset of the quadratic programming examples from the CUTEr test set to obtain the matrices $A$ and $B$. $C$ is a diagonal matrix consisting of ones and zeros. We shall change the rank of $C$ to see how this effects the number of iterations required by GMRES, as well as considering the different choices of $G$ from Section 3.1.

The matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is the full row rank matrix of linear constraints. $A$ is positive definite in the nullspace of $B$. A permuted lu factorization of $B^T$ is used to find a permutation matrix, $M$, such that the first $m$ columns of $BM$ are linearly independent. [10, Chapter 3]. “Barrier” penalty terms (in this case 1.1) are added to the diagonal of $A$ to simulate systems that might arise during and iteration of an interior-point method for such problems. We then set $B \leftarrow BM$ and $A \leftarrow AM$. The right hand side of the equation is set to be the sum of the columns of $A$. The assumption about the nonsingularity of $B_1$ will now hold.

All tests were performed on a dual Intel Xeon 3.20GHz machine with hyper-threading and 2GiB of RAM. It was running Fedora Core 2 (Linux kernel 2.6.8) with MATLAB® 7.0. The linear systems were solved using the Simplified Quasi-Minimal Residual Algorithm (SQMR) [8] – MATLAB® code for SQMR can be obtained from the MATLAB® Central File Exchange at http://www.mathworks.fr/matlabcentral/. We terminate the iteration when the value of residual is reduced by at least a factor of $10^{-8}$.

In Tables 4.1, 4.2 and 4.3 we compare four preconditioning strategies for (approximately) solving the problem (1.1). In particular, we compare the number of iterations performed by SQMR. The first preconditioning strategy uses the simple choice of $G = I$. The results for this strategy are given in columns headed “$G = I$.” Our second strategy is to let $G$ defined as in Theorem 3.4. Columns headed “$G_{22}$” contain the results for this preconditioner. The third and fourth preconditioning strategies are such that $G$ is the same as in Theorems 3.5 and 3.6 respectively.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>$m$</th>
<th>$G = I$</th>
<th>$G_{22}$</th>
<th>$G_{22}$ and $G_{11}$</th>
<th>$G_{22}$ and $G_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVXQP1_M</td>
<td>1000</td>
<td>500</td>
<td>314</td>
<td>592</td>
<td>62</td>
<td>609</td>
</tr>
<tr>
<td>CVXQP2_M</td>
<td>1000</td>
<td>250</td>
<td>654</td>
<td>181</td>
<td>34</td>
<td>245</td>
</tr>
<tr>
<td>CVXQP3_M</td>
<td>1000</td>
<td>750</td>
<td>118</td>
<td>163</td>
<td>32</td>
<td>241</td>
</tr>
<tr>
<td>KSIP</td>
<td>1021</td>
<td>1001</td>
<td>6</td>
<td>9</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>MOSARQP1</td>
<td>3200</td>
<td>700</td>
<td>37</td>
<td>18</td>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>PRIMAL1</td>
<td>410</td>
<td>85</td>
<td>8</td>
<td>55</td>
<td>1</td>
<td>55</td>
</tr>
<tr>
<td>PRIMAL2</td>
<td>745</td>
<td>96</td>
<td>7</td>
<td>75</td>
<td>1</td>
<td>75</td>
</tr>
<tr>
<td>PRIMAL3</td>
<td>856</td>
<td>111</td>
<td>6</td>
<td>93</td>
<td>1</td>
<td>93</td>
</tr>
<tr>
<td>PRIMAL4</td>
<td>1564</td>
<td>75</td>
<td>6</td>
<td>76</td>
<td>1</td>
<td>76</td>
</tr>
<tr>
<td>SOSQP1</td>
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<td>2501</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>STCQP2</td>
<td>4097</td>
<td>2052</td>
<td>98</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 4.2
Number of iterations required by SQMR when $C = \text{diag}(0, \ldots, 0, 1, \ldots, 1)$, $p = \lceil \frac{m}{2} \rceil$ and different preconditioning strategies are used.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>$m$</th>
<th>$G = I$</th>
<th>$G_{22}$</th>
<th>$G_{22}$ and $G_{11}$</th>
<th>$G_{22}$ and $G_{21}$</th>
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<tbody>
<tr>
<td>CVXQP1,M</td>
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<td>500</td>
<td>631</td>
<td>1500*</td>
<td>112</td>
<td>1500*</td>
</tr>
<tr>
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<td>1000</td>
<td>250</td>
<td>634</td>
<td>936</td>
<td>41</td>
<td>1119</td>
</tr>
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<td>CVXQP3,M</td>
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<td>750</td>
<td>513</td>
<td>812</td>
<td>151</td>
<td>1750*</td>
</tr>
<tr>
<td>KSIP</td>
<td>1021</td>
<td>1001</td>
<td>6</td>
<td>12</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>MOSARQP1</td>
<td>3200</td>
<td>700</td>
<td>41</td>
<td>19</td>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>PRIMAL1</td>
<td>410</td>
<td>85</td>
<td>9</td>
<td>61</td>
<td>1</td>
<td>61</td>
</tr>
<tr>
<td>PRIMAL2</td>
<td>745</td>
<td>96</td>
<td>8</td>
<td>79</td>
<td>1</td>
<td>79</td>
</tr>
<tr>
<td>PRIMAL3</td>
<td>856</td>
<td>111</td>
<td>7</td>
<td>94</td>
<td>1</td>
<td>94</td>
</tr>
<tr>
<td>PRIMAL4</td>
<td>1564</td>
<td>75</td>
<td>7</td>
<td>76</td>
<td>1</td>
<td>76</td>
</tr>
<tr>
<td>SOSQP1</td>
<td>5000</td>
<td>2501</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>STCQP2</td>
<td>4097</td>
<td>2052</td>
<td>148</td>
<td>40</td>
<td>11</td>
<td>266</td>
</tr>
</tbody>
</table>

Table 4.3
Number of iterations required by SQMR when $C = I$ and different preconditioning strategies are used.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>$m$</th>
<th>$G = I$</th>
<th>$G_{22}$</th>
<th>$G_{22}$ and $G_{11}$</th>
<th>$G_{22}$ and $G_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVXQP1,M</td>
<td>1000</td>
<td>500</td>
<td>474</td>
<td>1500*</td>
<td>137</td>
<td>1500*</td>
</tr>
<tr>
<td>CVXQP2,M</td>
<td>1000</td>
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<td>495</td>
<td>1250*</td>
<td>55</td>
<td>1250*</td>
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<tr>
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<td>492</td>
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<td>179</td>
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<tr>
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<td>1001</td>
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<td>2052</td>
<td>159</td>
<td>389</td>
<td>26</td>
<td>4430</td>
</tr>
</tbody>
</table>

columns are headed “$G_{22}$ and $G_{11}$” and “$G_{22}$ and $G_{21}$” respectively. If a results has a superscript *, then the iterative process failed to converge to the required tolerance.

The problems in Table 4.1 have $C = 0$. For general $G$ we would expect no more than $n - m + 2$ iterations, see Section 2. It would be hoped that there would be lower iteration counts when $G$ is set to satisfy Theorem 3.4, 3.5 or 3.6. However, the choice $G_{11} = A_{11}$, $G_{22} = A_{22}$ and $G_{21} = 0$ is the only one to consistently produce reductions in the iteration count when compared to using $G = I$. We must point out that the KSIP and PRIMAL problems all have $A_{21} = 0$, so the third choice of preconditioner is equivalent to setting $\mathcal{P} = \mathcal{A}$, whilst the second and fourth strategies are identical.

The case of $C = \text{diag}(0, \ldots, 0, 1, \ldots, 1)$ and $p = \lceil \frac{m}{2} \rceil$ is considered in Table 4.2. By increasing the size of $p$ we expect the number of distinct non-unit eigenvalues to increase. It is reasonable to assume that this will produce an increase in the number of iterations performed by the SQMR algorithm. Indeed, comparing Tables 4.1 and 4.2 we observe that there is, in general, an increase in the number of iterations. Again, we find the choice $G_{11} = A_{11}$, $G_{22} = A_{22}$ and $G_{21} = 0$ to be preferable in terms of the number of iterations performed. In Table 4.3 the test problems had the matrix $C$ set to be the $m$ by $m$ identity matrix. We also see similar behaviour for this case.

We have not compared the execution times for the different choice of $G$. Instead
of building $\mathcal{P}$ and then factoring it, as has been done in these tests, we suggest the possible use of implicit-factorization constraint preconditioners which only require small factorizations to be carried out [4, 5, 6].

We must admit to being slightly disappointed that the choices of $G$ in Theorems 3.4 and 3.6 seem to show little advantage over using $G = I$. However, setting $G_{11} = A_{11}, G_{22} = A_{22}$ and $G_{21} = 0$ appears to be advantageous for all the problems, including those where $G_{21} \neq 0$.

5. Conclusions. In this paper, we investigated a new class of preconditioner for indefinite linear systems that incorporate the (1,2), (2,1) and (2,2) blocks of the original matrix. We showed that the inclusion of these blocks in the preconditioner clusters at least $2m - p$ eigenvalues at 1, regardless of the structure of $G$. However, the standard convergence theory for Krylov subspace methods if not readily applicable because, in general, $\mathcal{P}^{-1}A$ does not have a complete set of linearly independent eigenvectors.

We also considered how three different choices of $G$, that reproduce different parts of $A$ in their structures, affect the upper bound on the number of distinct eigenvalues. We hoped that the improved bounds would also imply an in improvement in the number of iterations performed by SQMR to reach a set tolerance when compared to using the simple preconditioner with $G = I$. We found that the choice $G_{11} = A_{11}, G_{22} = A_{22}$ and $G_{21} = 0$ gave, in general, a significant decrease in the number of iterations.

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REFERENCES


