A–Posteriori Existence in Adaptive Computations

Christoph Ortner

This short note demonstrates that it is not necessary to assume the existence of exact solutions in an a–posteriori error analysis. If the residual of a stable numerical solution is sufficiently small there exists a nearby exact solution for which an a–posteriori error estimate holds.

We first develop the idea in an abstract Banach space setting and then demonstrate some further practical details at the nonlinear Laplace equation.

Key words and phrases: a–posteriori error analysis, adaptivity, existence, local uniqueness

Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England OX1 3QD

July, 2006

*The author acknowledges the financial support received from the European research project HPRN-CT-2002-00284: New Materials, Adaptive Systems and their Nonlinearities. Modelling, Control and Numerical Simulation, and the kind hospitality of Carlo Lovadina (University of Pavia).
1 Introduction

The mathematical theory of adaptive procedures for the solution of partial differential equations has advanced considerably over the past decade. Major milestones were proofs of convergence [5, 9] and later proofs of optimality [3, 12]. But even if one is unable to prove convergence or optimality of an adaptive algorithm, excellent convergence properties are usually observed in practise.

The present paper concerns a simple observation that seems to have gone unnoticed so far: one can verify the convergence of an adaptive algorithm a–posteriori even when it is not known a–priori whether solutions exist. Suppose we are solving the nonlinear equation \( F(u) = 0 \), where \( F: X \to Y^* \). A point \( u \) of the domain of definition of \( F \) is called regular if \( F'(u)^{-1} \) exists and is bounded (cf. [13, Proposition 2.1]). A–posteriori error estimates for nonlinear problems are typically formulated in the following way (cf. [13 Proposition 2.1] or [1, Lemma 9.5]):

If \( u \) is a regular solution to \( F(u) = 0 \) and \( U \) is a numerical solution which is sufficiently close to \( u \) then \( \| u - U \|_X \lesssim \| F(U) \|_{Y^*} \).

The idea that is pursued in the following, is to exchange the role of numerical and exact solution. Obviously, the approximation \( U \) solves the equation \( F(u) - F(U) = 0 \). An exact solution with \( F(u) = 0 \) would then be considered the approximate solution to the new problem and its residual is again \( \| F(U) \|_{Y^*} \). Now, we only need to assume that \( U \) is regular rather than a nearby exact solution \( u \) which we usually do not know to exist a–priori.

More precisely, we shall prove that

if a regular point \( U \in X \) has a sufficiently small residual \( \| F(U) \|_{Y^*} \) then
there exists a nearby exact solution \( u \) to \( F(u) = 0 \) such that
\( \| u - U \|_X \lesssim \| F(U) \|_{Y^*} \| F(U)^{-1} \|_{L(Y^*,X)} \).

In an abstract Banach space setting, we give two general results. First, Theorem \( \text{T} \) provides an asymptotically optimal strategy based solely on Lipschitz continuity of \( F' \). Second, Lemma \( \text{L} \) is a slightly more general result based on a continuation argument, which is intended to outline a form for more specific strategies that may be available when a lot of analytical information about the problem is available.

In §3 we use the semilinear Laplace equation
\( -\Delta u + f(x, u) = 0, \quad u|_{\partial \Omega} = 0 \),
to demonstrate further details of the abstract idea which are required for its practical implementation.

A further application of a–posteriori existence is given in [11] where an adaptive numerical optimization method for the quasicontinuum method [10], a numerical coarse graining technique for atomistic models, is developed. The structure of atomistic energies necessitates the use of \( W^{1,\infty} \)-like norms in the analysis which makes the analysis of local stability particularly challenging.
For evolution equations where the uniqueness of the solution is usually guaranteed, a similar procedure can give improved a–posteriori error bounds based on local stability properties rather than global ones. We refer to [2] where this idea is used for Ginzburg–Landau type equations.

2 Abstract Result

Let \( X \) be a Banach space, and let \( Y \) be a Banach space with topological dual \( Y^* \). We denote the duality pairing between \( Y \) and \( Y^* \) by \( \langle \cdot, \cdot \rangle \). For \( v \in X \) and \( R > 0 \), we use \( B(v, R) \) to denote the closed ball with centre \( v \) and radius \( R \) in \( X \). Let \( \mathcal{A} \) be an open subset of \( X \) and let \( F : \mathcal{A} \to Y^* \). For example, the operator \( y \mapsto -\text{div} \sigma(\nabla y) \) with Dirichlet boundary conditions could be understood in the sense

\[
\langle F(y), v \rangle = \int_\Omega \sigma(\nabla y) : \nabla v \, dx \quad \forall v \in Y,
\]

where \( X \) and \( Y \) are appropriately chosen function spaces (cf. §3 for further detail).

We say that \( F \) is differentiable at a point \( u \in \mathcal{A} \) if there exists a bounded linear operator \( F'(u) \in L(X, Y^*) \) such that, for \( v \in \mathcal{A} \), we have

\[
F(v) = F(u) + F'(u)(v - u) + o(\|v - u\|_X).
\]

To avoid a cluttered notation, we shall always use \( \|T\| \) to denote the operator norm for linear, bounded operators between Banach spaces. It will always be clear from the context which spaces are meant. For example, \( \|F'(u)\| \) denotes the operator norm in \( L(Y^*, X) \).

We say that a point \( u \in \mathcal{A} \) is regular, if \( F \) is differentiable at \( u \) and \( F'(u) \) is an isomorphism, i.e., \( F'(u) \) has a bounded inverse. If \( Y \) is reflexive, then we can equivalently use the infsup condition,

\[
\|F'(u)^{-1}\|^{-1} = \inf_{v \in X} \sup_{\varphi \in Y} \langle F'(u)v, \varphi \rangle.
\]

In fact, all our results and proofs can be translated in this way, without major changes, even when \( Y \) is not reflexive.

Theorem 1 provides a strategy for computing a radius \( R \) and the stability constant \( \sup_{v \in B(U, R)} \|F'(v)^{-1}\| \), using only Lipschitz continuity of \( F' \). It is formulated in such a way that maximising \( \rho \) subject to the condition \( (2.1) \) yields an asymptotically optimal strategy for choosing \( R \).

**Theorem 1** Suppose that \( F \) is differentiable in \( \mathcal{A} \) and that \( U \in \mathcal{A} \) is regular. For \( \rho \in [1/2, 1] \) set \( R = R(\rho) = \|F(U)\|_{Y^*} \|F'(U)^{-1}\|/\rho \) and let \( L = L(\rho) \) be the Lipschitz constant of \( F' \) in \( B(U, R) \cap \mathcal{A} \). If \( R < \text{dist}(U, \partial \mathcal{A}) \) and if

\[
\|F(U)\|_{Y^*} \|F'(U)^{-1}\|^2 \leq \rho(1 - \rho)/L \tag{2.1}
\]
then there exists a unique \( u \in B(U, R) \) such that \( F(u) = 0 \). Furthermore, we have the error estimate

\[
\|u - U\|_X \leq \rho^{-1}\|F(U)\|_{X^*}\|F'(U)^{-1}\|.
\] (2.2)

**Proof** First, we determine a radius \( R \) and a stability constant for the set \( B(U, R) \). To this end, we construct an optimal fraction \( \rho \in (0, 1] \), letting \( R \) depend on \( \rho \), such that

\[
\sup_{v \in B(U, R)} \|F'(v)^{-1}\| \leq \rho^{-1}\|F'(U)^{-1}\|.
\]

This follows from the expansion of \( F'(v)^{-1}F'(U) \) in a power series of \( F'(v) - F'(U) \) (cf. [7, Section 7.3]). We estimate

\[
\|(F'(v) - F'(U))F'(U)^{-1}\| \leq LR\|F'(U)^{-1}\| = \rho^{-1}L\|F(U)\|_{X^*}\|F'(U)^{-1}\|^2
\]

to see that it is sufficient to assume

\[
\|F(U)\|_{X^*}\|F'(U)^{-1}\|^2 < \rho/L,
\]

which follows from (2.1) for any \( \rho \in (0, 1] \). Thus, in order to have \( \|F'(v)^{-1}\| \leq \rho^{-1}\|F'(U)^{-1}\| \)

we require

\[
\frac{\|F'(U)^{-1}\|}{1 - LR\|F'(U)^{-1}\|} \leq \rho^{-1}\|F'(U)^{-1}\|,
\]

which is equivalent to (2.1). If (2.1) is satisfied then every point \( v \in B(U, R) \) is regular and \( \|F'(v)^{-1}\| \leq \rho^{-1}\|F'(U)^{-1}\| \).

While \( L \) is, to some extent, dependent on the choice of \( \rho \) it is reasonable to assume that it remains roughly constant, particularly when the residual (and hence \( R \)) is small. Hence, the value \( \rho = 1/2 \) is roughly optimal in that (2.1) is most easily satisfied in this case. For smaller \( \rho \) the resulting error estimate deteriorates and (2.1) becomes more difficult to satisfy as well — hence the assumption that \( \rho \geq 1/2 \).

The remainder of the proof involves a careful tracking of the constants in the proof of the Implicit Function Theorem by Banach’s fixed point theorem.

We wish to prove the existence of a solution to \( F(u) = 0 \) in \( B(U, R) \). Since \( F' \) is continuous in \( B(U, R) \), we can use Taylor’s theorem to obtain

\[
F(v) - F(U) = \int_0^1 F'(U + \tau(v - U)) \, d\tau \cdot (v - U) =: F'_U(v - U) \quad \forall v \in B(U, R).
\] (2.3)
Assume, for the moment, that \( u \in B(U, R) \) satisfies \( \mathcal{F}(u) = 0 \) so that \( \mathcal{F}(u) - \mathcal{F}(U) = -\mathcal{F}(U) \).

Applying (2.3) to the right-hand side, we find that \( \mathcal{F}(u) = 0 \) is equivalent to

\[
\mathcal{F}_U(u) = -\mathcal{F}(U).
\]

We now define the map \( \mathcal{L}: B(U, R) \to \mathcal{X} \) by

\[
\mathcal{F}_U(v) - U) = -\mathcal{F}(U).
\] (2.4)

To show that the map is well-defined, we first use the Integral Mean Value Theorem to infer the existence of \( \theta_v \in \text{conv}\{U, v\} \) such that \( \mathcal{F}_U = \mathcal{F}(\theta_v) \). Since \( \theta_v \in B(U, R) \), it follows that \( \mathcal{F}_U(v) \) is an isomorphism satisfying \( \|\mathcal{F}_U(v)^{-1}\| \leq \rho^{-1}\|\mathcal{F}(U)^{-1}\| \), and in particular that (2.4) has a unique solution. In summary, an element \( u \in B(U, R) \) satisfies \( \mathcal{F}(u) = 0 \) if, and only if, \( u \) is a fixed point of \( \mathcal{L} \).

To show that \( \mathcal{L} \) maps \( B(U, R) \) into itself, we multiply \( (\mathcal{F}_U,v)^{-1} \) by (2.4) to infer

\[
\|\mathcal{L}(v) - U\| \leq \|\mathcal{F}_U(v)^{-1}\|\|\mathcal{F}(U)\|\mathcal{F}_U = R.
\]

To show that \( \mathcal{L} \) is a contraction of \( B(U, R) \) let \( v_1, v_2 \in B(U, R) \); then

\[
\mathcal{F}_U(v_i) - U) = -\mathcal{F}(U), \quad i = 1, 2.
\]

Subtracting these two equations, we obtain

\[
\mathcal{F}_U(v_1) - \mathcal{L}(v_2) = -\mathcal{F}_U(v_2) - U + \mathcal{F}_U(v_2) - U = -\mathcal{F}_U(v_2) - \mathcal{L}(v_2) - U.
\]

Multiplying by \((\mathcal{F}_U,v_1)^{-1}\) and using

\[
\left\| \int_0^1 \left[ \mathcal{F}(U + \tau(v_1 - U)) - \mathcal{F}(U + \tau(v_2 - U)) \right] d\tau \right\|
\leq \int_0^1 \|\mathcal{F}(U + \tau(v_1 - U)) - \mathcal{F}(U + \tau(v_2 - U))\| d\tau
\leq \int_0^1 \tau L\|v_1 - v_2\| d\tau \leq \frac{1}{2} L\|v_1 - v_2\|
\]

we obtain

\[
\|\mathcal{L}(v_1) - \mathcal{L}(v_2)\| \leq \frac{1}{2} L\|v_1 - v_2\| \leq \frac{1}{2} L\|v_1 - v_2\|\left\| (\mathcal{F}_U,v_1)^{-1}\right\| R
\leq \frac{1}{2} L\|v_1 - v_2\| \leq \frac{1}{2} \rho^{-1}||\mathcal{F}(U)||\mathcal{F}(U)^{-1}||\rho^{-1}||\mathcal{F}(U)||\mathcal{F}(U)^{-1}||
\]

where we also used (2.1) in the last estimate.

It follows that, if \( \frac{1}{2}(1 - \rho)/\rho < 1 \) which is true whenever \( \rho > 1/3 \), then \( \mathcal{L} \) is a contraction of \( B(U, R) \) with contractivity \( (1 - \rho)/(2\rho) \) and must therefore have a unique fixed point \( u \) in \( B(U, R) \) which is the unique solution of \( \mathcal{F}(u) = 0 \) in \( B(U, R) \).

Given our definition of \( R \), the error estimate follows immediately from the fact that \( u \in B(U, R) \).
It seems that the a–posteriori existence idea can be generalised to entirely different techniques for proving existence of nonlinear equations. Essentially all that is required is a method to prove the existence of a solution to a nonlinear equation when a solution to a perturbed equation is known.

In particular, the strategy outlined in Theorem 1 is only the most obvious general idea to compute a stability region $B(U, R)$ and a bound for the corresponding stability constant. Since knowledge of the infsup constant is so crucial to the success of the a–posteriori existence idea, any information that is available in a particular problem should be used to improve the estimate. For example, in [11] the radius $R$ and a corresponding uniform infsup constant $c_0$ (which can be used to replace the stability constant $\sup_{v \in B(U, R)} \| F'(v)^{-1} \|$) are computed by a search algorithm and sharp estimates that are tailored to specific classes of solutions to the problem. The result used there is of a more general form than Theorem 1, similar to Lemma 2 below, which is a continuation principle for the Implicit Function Theorem (cf. Sections 4.7 and 6.6 in [15]). It provides a slightly more general platform for a–posteriori existence than Theorem 1.

Lemma 2 Let $F$ be continuously differentiable in $\mathcal{A}$ and let $U \in \mathcal{A}$. If there exists an $R > 0$ such that
\[
\| F(U) \|_{\mathcal{Y}} \leq R \left( \sup_{v \in B(U, R)} \| F'(v)^{-1} \| \right)^{-1}
\]and $R' > R$ such that $F$ and $F'$ are uniformly continuous in the ball $B(U, R')$ then there exists a unique $u \in B(U, R)$ such that $F(u) = 0$. Furthermore, we have the estimate
\[
\| u - U \|_{\mathcal{X}} \leq \sup_{v \in B(U, R)} \| F'(v)^{-1} \| \| F(U) \|_{\mathcal{Y}}.
\]

Proof The result is in fact a specialisation of [15, Proposition 6.10]. For the reader’s convenience, we include a sketch of the proof.

Set $c = \sup_{v \in B(U, R)} \| F'(v)^{-1} \|$ and note that, unless $F(U) = 0$ which we exclude without loss of generality, we have implicitly assumed that $c < \infty$. For $t \in [0, 1]$, let $f_t = (1 - t)F(U)$. Suppose that $u_t \in B(U, R)$ is a solution to $F(u_t) = f_t$. By Taylor’s theorem there exists $\theta_t \in \text{conv}\{U, u_t\}$ such that $F(u_t) - F(U) = F'(\theta_t)(u_t - U)$ and hence, using (2.5),
\[
\| u_t - U \|_{\mathcal{X}} \leq \| F'(\theta_t)^{-1} \| \| f_t - F(U) \|_{\mathcal{Y}} \leq tc \| F(U) \| \leq tR
\]
which implies $u_t \in B(U, tR)$.

By tracking the constants in the proof of the Implicit Function Theorem (cf. [15, Theorem 4.B]), similarly as in the proof of Theorem 1 we can infer that there exist an $r > 0$, independent of $(t, u_t)$ (depending only on $c_0$ and on the uniform continuity conditions) such that $r < R' - R$ and, for $t \leq s < t + r$, there exists a solution of the problem $F(u_s) = f_s$. Since $U = u_0$ is a solution for $t = 0$, we can successively construct solutions $u_t$ for increasing $t$ until we obtain $u_1 = u$. The error estimate (2.6) follows from (2.7). $\blacksquare$
3 A Hilbert Space Example

We apply the idea of a–posteriori existence to the semilinear Laplace equation,

\[-\Delta u + f(x, u) = 0, \quad u|_{\partial \Omega} = 0.\]

Let \(d \leq 6\) and let \(\Omega\) be a domain in \(\mathbb{R}^d\). We set \(\mathcal{X} = \mathcal{Y} = H^1_0(\Omega)\) equipped with the norm \(|u|_{H^1} = \|\nabla u\|_{L^2}\). The operator \(\mathcal{F} : H^1_0(\Omega) \to H^{-1}(\Omega)\) is defined by

\[
\langle \mathcal{F}(u), \varphi \rangle = \int_{\Omega} \left[ \nabla u \cdot \nabla \varphi + f(x, u) \varphi \right] \, dx \quad \forall \varphi \in H^1_0(\Omega). \tag{3.1}
\]

For simplicity, we assume that \(f\) is differentiable with respect to \(u\) and that \(f_u\) is globally Lipschitz–continuous in the sense that

\[
|f_u(x, u_1) - f_u(x, u_2)| \leq L_{f_u} |u_1 - u_2| \quad \text{for a.e. } x \in \overline{\Omega} \quad \forall u_1, u_2 \in \mathbb{R}.
\]

**Proposition 3** The operator \(\mathcal{F}\) is differentiable in \(H^1_0\) with derivative

\[
\langle \mathcal{F}'(u)v, \varphi \rangle = \int_{\Omega} \left[ \nabla v \nabla \varphi + f_u(x, u)v \varphi \right] \, dx. \tag{3.2}
\]

\(\mathcal{F}'\) is globally Lipschitz continuous with Lipschitz constant \(L_{\mathcal{F}'} = L_{f_u} C^3_S\), where \(C^3_S\) is the Sobolev embedding constant of \(H^1_0(\Omega)\) in \(L^3(\Omega)\).

**Proof** The proof is a straightforward application of Taylor’s theorem, the generalised Hölder inequality and Sobolev’s embedding of \(H^1_0(\Omega)\) in \(L^3(\Omega)\). ■

To discretize the equation \(\mathcal{F}(u) = 0\) by a Galerkin finite element method, let \(T\) be a regular partition (cf. [4]) of \(\Omega\) into \(d\)–simplices \(\kappa\) with diameter \(h_\kappa\), and define

\(S^p_0(T) = \{ V \in H^1_0(\Omega) : V|_\kappa\) is a polynomial of degree \(p, \forall \kappa \in T \}\).

Furthermore, let \(E\) be the set of interior \((d-1)\)–dimensional faces in \(T\) and, for \(e \in E\), denote \(h_e = \text{diam}(e)\) and \(\nu_e\) any unit normal vector to \(e\). The finite element method (for simplicity we do not consider quadrature approximations) is then defined by

\[
\int_{\Omega} \left[ \nabla U \nabla \Phi + f(x, U) \Phi \right] \, dx = 0 \quad \forall \Phi \in S^p_0(T). \tag{3.3}
\]

Following Verfürth [13, Chapter 1], we obtain the following result. For a detailed discussion of the constant \(C(\Omega, T)\), which depends only on the quality of the mesh, see [14].

**Proposition 4** Let \(U \in S^p_0(T)\) satisfy (3.3); then

\[
\|\mathcal{F}(U)\|_{H^{-1}} \leq C(\Omega, T) \left\{ \sum_{e \in E} \eta^2_e + \sum_{\kappa \in T} \eta^2_\kappa \right\}^{1/2} =: \eta(U),
\]

where

\[
\eta^2_e = h_e \int_e \left| \frac{\partial U}{\partial \nu_e} \right|^2 \, ds, \quad \text{and}
\]

\[
\eta^2_\kappa = h_\kappa^2 \int_\kappa \left| -\Delta U + f(x, U) \right|^2 \, dx.
\]
To avoid an unnecessarily technical discussion of the stability constant, we shall view (3.1) as a minimisation problem, i.e., to find $u \in H^1_0$ (locally) minimising the functional

$$I(u) = \int_\Omega \left[ \frac{1}{2} |\nabla u|^2 + g(x, u) \right] \, dx,$$

where $f = g_u$. Thus, we are looking for solutions $U$ to the Galerkin method (3.3) for which $\mathcal{F}'(U)$ is positive. In this case, $\mathcal{F}'(U)$ is an isomorphism and it can be easily seen that

$$\|\mathcal{F}'(U)^{-1}\|^{-1} = \inf_{\varphi \in H^1_0(\Omega)} \frac{\langle \mathcal{F}'(U)\varphi, \varphi \rangle}{\|\nabla \varphi\|_{L^2}} = \inf_{\varphi \in H^1_0(\Omega)} \int_\Omega \left[ |\nabla \varphi|^2 + f_u(x, U)\varphi^2 \right] \, dx.$$

Thus, computing $\|\mathcal{F}'(U)^{-1}\|$ amounts to finding the smallest $H^1_0$-eigenvalue of $\mathcal{F}'(U)$, i.e., the smallest $\lambda \in \mathbb{R}$ for which there exists $v \in H^1_0$ such that the pair $(\lambda, v)$ is a solution of the eigenvalue–problem

$$\int_\Omega \left[ \nabla v \nabla \varphi + f_u(x, U)v\varphi \right] \, dx = \lambda \int_\Omega \nabla v \nabla \varphi \, dx \quad \forall \varphi \in H^1_0(T). \quad (3.4)$$

The strong form of (3.4) is

$$-\Delta v + f_u(x, U)v = -\lambda \Delta v.$$

The Galerkin finite element approximation of (3.4) is to find $(\Lambda, V) \in \mathbb{R} \times S^p_0(T)$ such that

$$\int_\Omega \left[ \nabla V \nabla \Phi + f_u(x, U)V\Phi \right] \, dx = \Lambda \int_\Omega \nabla V \nabla \Phi \, dx \quad \forall \Phi \in S^p_0(T), \quad (3.5)$$

and such that $\Lambda$ is minimal. The following discussion is based on ideas in [8], to which we also refer for further references on the adaptive solution of ($L^2$-) eigenvalue problems.

Set $c(x) = f_u(x, U(x))$. Obviously, we have $\lambda \leq \Lambda$. Let $v$ be the elliptic projection of $V$ onto the eigenspace of $\lambda$; then

$$\int_\Omega \left[ \nabla v \nabla + cv - \lambda \nabla v \nabla \right] \, dx = 0.$$

We add and subtract $\Lambda(\nabla v, \nabla V)$ to obtain

$$\int_\Omega \left[ \nabla v \nabla + cv - \Lambda \nabla v \nabla \right] \, dx + (\Lambda - \lambda)(\nabla v, \nabla V) = 0.$$

Using the fact that $v$ is the elliptic projection of $V$, we can rearrange this equality to yield

$$(\Lambda - \lambda)\|\nabla v\|_{L^2}^2 = -\int_\Omega \left[ \nabla v \nabla V + cv - \Lambda \nabla v \nabla V \right] \, dx.$$
At this point, we need to assume that $\|\nabla v\|_{L^2}^2 \geq 1 - \delta$ for some $\delta \in [0, 1)$. Since $v$ and $v - V$ are orthogonal, this is equivalent to $\|v - V\|_{L^2}^2 \leq \delta < 1$. In this case, we have

$$\Lambda - \lambda \leq \frac{-1}{1-\delta} \int_{\Omega} \left[ \nabla v \nabla V + c v V - \Lambda \nabla v \nabla V \right] \ dx.$$

The estimation of the residual on the right-hand side, using the usual procedure of Galerkin orthogonality, integration by parts and a Clément–type interpolation error estimate (cf. [13, Chapter 1]), gives the following result.

**Proposition 5** Let $\pi V$ be the elliptic projection of $V$ onto the eigenspace of $\lambda$. If $\delta = \|\nabla (\pi V - V)\|_{L^2}^2 < 1$ then

$$\Lambda - \lambda \leq \frac{C(\Omega, T)}{1-\delta} \left\{ |1 - \Lambda| \sum_{e \in E} \theta_e^2 + \sum_{\kappa \in T} \theta_\kappa^2 \right\} = \theta(\Lambda, V),$$

where

$$\theta_e^2 = h_e \int_e \left[ \frac{\partial V}{\partial \nu_e} \right]^2 \ ds, \quad \text{and}$$

$$\theta_\kappa^2 = h_\kappa^2 \int_\kappa \left[ - (1 - \Lambda) \Delta V + f_u(x, U) V \right]^2 \ dx.$$

Depending on the regularity of $f_u(x, U)$, the condition $\|\nabla (\pi V - V)\|_{L^2}^2 < 1$ should not pose a problem in practise. The factor $1/(1 - \delta)$ should then behave like a higher order term in the error estimate (cf. for example [6]).

Thus, having obtained a solution $U$ to the Galerkin method (3.3), we compute the smallest generalised eigenvalue of $\mathcal{F}'(U)$ in $S_0^p(T)$, using (3.5) and define the local ellipticity constant

$$c_0(U) = \Lambda - \theta(\Lambda, V).$$

If $c_0(U) > 0$ then $\mathcal{F}'(U)$ is positive and satisfies $\|\mathcal{F}'(U)^{-1}\| \leq c_0(U)^{-1}$.

Combined with the findings of this section, Theorem 1 with $\rho = 1/2$ gives the following a-posteriori existence result.

**Theorem 6** Let $U \in S_0^p(T)$ be a solution of (3.3). If $c_0(U) > 0$ and

$$\frac{\eta(U)}{c_0(U)^2} \leq \left( 4L f_u C_3^2 \right)^{-1},$$

then there exists a solution $u \in H_0^1(\Omega)$ of (3.1) (a strict local minimum of the energy $I$) which satisfies

$$\|\nabla (u - U)\|_{L^2} \leq 2 \frac{\eta(U)}{c_0(U)}.$$
References


