A NEW APPROACH TO YAKUBOVICH’S S-LEMMA

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Abstract. Subject to regularity assumptions, Yakubovich’s s-Lemma characterizes the quadratic functions \( f(x) \) over \( \mathbb{R}^n \) which for a given quadratic function \( q(x) \) satisfy the implication \( q(x) \geq 0 \Rightarrow f(x) \geq 0 \). This result has far-reaching consequences in optimization and control theory. Several approaches to its proof are known, some of which generalize to Hilbert spaces. In this paper we explore a new geometric approach to the proof of this classical result.


Key words. s-Lemma, robust optimization, control theory.

1. Introduction. Yakubovich’s s-Lemma [8], a well-known result from robust control theory, characterizes all quadratic functions that are copositive with a given other quadratic function. A function \( f \) is called copositive with \( q \) if \( q(x) \geq 0 \) implies \( f(x) \geq 0 \).

Theorem 1.1 (s-Lemma, [8]). Let \( f, q : \mathbb{R}^n \to \mathbb{R} \) be quadratic functions such that \( q(\overline{x}) > 0 \) for some \( \overline{x} \in \mathbb{R}^n \). Then \( f \) is copositive with \( q \) if and only if there exists \( \xi \geq 0 \) such that \( f(x) - \xi q(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

Note that \( f \) and \( q \) are neither assumed convex nor homogeneous. The s-Lemma arose as a generalization of earlier results by Finsler [3], Hestenes-McShane [4] and Dines [2]. Megretsky-Treil [5] later extended the result to infinite-dimensional spaces. Theorem 1.1 has surprisingly powerful consequences in robust optimization and control theory, as this result allows to replace certain nonconvex optimization problems by convex polynomial time solvable ones. For an excellent overview of the history of the s-Lemma and its applications, see the review article of Pölk-Terlaky [6]. In the same review article, the known approaches to proving Theorem 1.1 and the connections of the ideas underlying these proofs to other results in optimization are discussed comprehensively. The three known distinct approaches to proving Theorem 1.1 are due to Yakubovich [8], Ben-Tal and Nemirovskii [1] and Sturm and Zhang [7], and Yuan [9].

In this paper we present a new proof of Theorem 1.1 that does not seem to be related to any of these approaches. Our proof is based on geometric ideas that extend immediately to the infinite-dimensional case and also shed light on the limitations in extending the result to characterising copositivity over arbitrary convex domains.

The following notational conventions apply throughout the paper. We denote the set of real symmetric \( n \times n \) matrices by \( \mathcal{S}_n \). For any subset \( C \) of a topological space we denote its closure by \( \text{clo}(C) \). If \( C \) is a subset of a real vector space \( V \) then we write

\[
\text{cone}(C) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, \ x_i \in C, \ (i = 1, \ldots, n), \ n \in \mathbb{N} \right\}
\]
for the convex conic hull of $C$. A linear functional on $V$ is a (homogeneous) linear map $V \to \mathbb{R}$, i.e., an element in the dual space $V^\sharp$. The condition

$$x_1 \sim x_2 \iff \exists \lambda > 0 \text{ s.t. } x_2 = \lambda x_1$$

defines an equivalence relation on $V \setminus \{0\}$. From now on let $V$ be a real Hilbert space, and let

$$q : V \setminus \{0\} \to S(V) := (V \setminus \{0\}) / \sim$$

$$x \mapsto \{z \in V \setminus \{0\} : z \sim x\}$$

be the quotient map for the above defined equivalence relation. When $S(V)$ is equipped with the quotient-space topology under which a subset $\mathcal{C} \subseteq S(V)$ is open if and only if $q^{-1}(\mathcal{C})$ is open in $V$, this space is topologically equivalent to the unit sphere in $V$. For any $x_1, x_2 \in V \setminus \{0\}$ we write

$$[x_1, x_2] = \{\xi x_2 + (1 - \xi)x_1 : \xi \in [0, 1]\}$$

for the piece of straight-line between $x_1$ and $x_2$. For $y_1, y_2 \in S(V)$, we write $[y_1, y_2] := q([x_1, x_2])$, where $x_1 \in q^{-1}(y_1)$ and $x_2 \in q^{-1}(y_2)$. It is easy to check that the definition of $[y_1, y_2]$ does not depend on the specific choice of $x_1$ and $x_2$. Note that the quotient map $q$ is not defined at the origin of $V$. By abuse of language, if $C \subseteq V$ we write $q(C)$ for $q(C \setminus \{0\})$.

2. Spherical Convexity and a Farkas-Type Theorem. Our approach to proving Theorem 1.1 hinges on a notion of spherical convexity which we will now define.

**Definition 2.1.** We say that a subset $\mathcal{C} \subseteq S(V)$ is spherically convex if $[y_1, y_2] \in \mathcal{C}$ for all $y_1, y_2 \in \mathcal{C}$.

We introduce the following simple lemma for later convenience.

**Lemma 2.2.** Let $C \subset V$ be such that $q([x_1, x_2]) \subset \text{clo}(q(C))$ for all $x_1, x_2 \in C$. Then $\text{clo}(q(C))$ is spherically convex.

**Proof.** Let $y^1, y^2 \in \text{clo}(q(C))$, and let $x^i$ be the unique unit vector in $q^{-1}(y^i)$, $(i = 1, 2)$. If $x^1 = \pm x^2$, then $[y^1, y^2] = \{y^1, y^2\} \subset \text{clo}(q(C))$. Otherwise, if $x^1$ and $x^2$ are not colinear, then there exist sequences

$$(x^i_n) \subset \{\xi x : \xi > 0, x \in C \setminus \{0\}\} \cap \{x : \|x\| = 1\}$$

such that $x^i_n \to x^i$, $(i = 1, 2)$, and then $\lambda x^1_n + (1 - \lambda)x^2_n \to \lambda x^1 + (1 - \lambda)x^2$ for all $\lambda \in [0, 1]$. Since $q$ is continuous, we have

$$q(\lambda x^1_n + (1 - \lambda)x^2_n) \to q(\lambda x^1 + (1 - \lambda)x^2),$$

and furthermore, by assumption on $C$, $q(\lambda x^1_n + (1 - \lambda)x^2_n) \in \text{clo}(q(C))$. Finally, since $\text{clo}(q(C))$ is closed, we find $q(\lambda x^1 + (1 - \lambda)x^2) \in \text{clo}(q(C))$, and since $\lambda$ was arbitrary in $[0, 1]$, this proves the claim. □

It is also trivial to see that if $C \subset V$ is convex then $q(C)$ is spherically convex. However, convexity of $C$ is not a necessary condition for $q(C)$ to be spherically convex, see the Figure 2.1. In the sequel we will be interested in subsets $C \subset \mathbb{R}^2$ for which $\text{clo}(q(C))$ is spherically convex in $S(\mathbb{R}^2)$. Figure 2.1 shows some typical examples of
Let \( a(t) = a_0 + ta_1 + t^2a_2 \) and \( b(t) = b_0 + tb_1 + t^2b_2 \) where \( a_2 \) and \( b_2 \) are not both zero, so that the set
\[
C := \left\{ \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} : t \in \mathbb{R} \right\}
\]
consists of the points on a parabola. Then \( \text{clo}(q(C)) \) is spherically convex in \( S(\mathbb{R}^2) \).

Proof. Let \( x(t) := [a(t), b(t)]^T \), and let \( x(t_1), x(t_2) \) be two arbitrary points on \( C \) with \( t_1 < t_2 \). Two convex open domains \( D_1 \) and \( D_2 \) are bounded by the curve \( C \) and
the interval \([x(t_1), x(t_2)]\): the boundary of the infinite domain \(D_1\) is given by
\[
\partial D_1 = \{ x(t) : t \in (-\infty, t_1] \cup [x(t_1), x(t_2)] \cup \{ x(t) : t \in [t_2, +\infty) \},
\]
while the boundary of the finite domain \(D_2\) is given by
\[
\partial D_2 = \{ x(t) : t \in [t_1, t_2] \} \cup [x(t_2), x(t_1)].
\]
We now distinguish three cases depending on the location of the origin.

Firstly, if 0 lies on the (infinite) line passing through \(x(t_1)\) and \(x(t_2)\), then \(x(t_1)\) and \(x(t_2)\) are colinear and \([q(x(t_1)), q(x(t_2))] = \{ q(x(t_1)), q(x(t_2)) \} \subset q(C)\).

Secondly, if \(0 \in \text{clo}(D_1) \setminus [x(t_1), x(t_2)]\), then for any \(x \in [x(t_1), x(t_2)]\) the ray \(\{ \tau x : \tau > 0 \}\) intersects the boundary of the convex domain \(D_2\) in two points (which coincide when \(x = x(t_1)\) or \(x = x(t_2)\)), one of which is \(x\). Since \(x_1\) and \(x_2\) are not colinear, the other point must lie on the piece of curve \(\{ x(t) : t \in [t_1, t_2]\} \subset C\). Therefore, \(q(x) \in q(C)\) and \([q(x(t_1)), q(x(t_2))] \subset q(C)\).

Thirdly, if \(0 \notin D_1\) and \(x(t_1), x(t_2)\) are not colinear, then for any \(x \in [x(t_1), x(t_2)]\) it is either the case that \(x = \tau [a_2, b_2]^T\) for some \(\tau > 0\), in which case \(q(x) = \lim_{\tau \to \infty} q(x(t)) \in \text{clo}(q(C))\), or else the ray \(\{ \tau x : \tau > 0 \}\) intersects the boundary of the convex domain \(D_1\) in two points (which can coincide when \(x = x(t_1)\) or \(x = x(t_2)\)). Since \(x_1\) and \(x_2\) are not colinear the one of the two points must lie on \(\{ x(t) : t \in (-\infty, t_1] \} \cup \{ x(t) : t \in [t_2, +\infty) \} \subset C\). Therefore, \(q(x) \in q(C)\). Thus, \([q(x(t_1)), q(x(t_2))] \subset \text{clo}(q(C))\).

The claim now follows from Lemma 2.2. \(\square\)

The following property of convex conic hulls in \(\mathbb{R}^2\) is central to our approach of the s-Lemma.

![Diagram](image-url)
Lemma 2.4. Let $C \subseteq \mathbb{R}^2$ be such that $\text{clo}(q(C))$ is spherically convex in $\mathcal{S}((\mathbb{R}^2))$. Then for any linear functional $h : \mathbb{R}^2 \to \mathbb{R}$ it is true that

$$\text{clo}\left(\{x \in C : h(x) \geq 0\}\right) = \text{clo}\left(\{x \in \text{cone}(C) : h(x) \geq 0\}\right).$$

Proof. We only need to prove the inclusion

$$\text{clo}(\{x \in C : h(x) \geq 0\}) \supseteq \text{clo}(\{x \in \text{cone}(C) : h(x) \geq 0\}), \tag{2.1}$$

since the opposite inclusion is trivial. Let $x \in \text{cone}(C)$ be such that $h(x) \geq 0$. There exist $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$ such that $x = \lambda_1 x_1 + \lambda_2 x_2$. If either $\lambda_1$ or $\lambda_2$ is zero or if $h(x_1) \geq 0$ and $h(x_2) \geq 0$, then it is trivially true that $x \in \text{cone}(\{z \in C : h(z) \geq 0\})$.

Without loss of generality, we may thus assume that $h(x_1) > 0 > h(x_2)$ and $\lambda_1, \lambda_2 > 0$.

If $x_1$ and $x_2$ are colinear, then $x_2 = -\tau x_1$ for some $\tau > 0$, and in that case we have $x = (\lambda_1 - \tau \lambda_2) x_1$ and $\lambda_1 - \tau \lambda_2 \in [0, 1)$ since $h(x) \in [0, h(x_1))$. It then follows that

$$x \in \text{cone}(\{x \in C : h(x) \geq 0\}).$$

If on the other hand $x_1$ and $x_2$ are not colinear, then $y := \xi x_1 + (1 - \xi)x_2 \neq 0$, where $\xi = h(x_2)/(h(x_2) - h(x_1))$, and furthermore, $h(y) = 0$.

By assumption on $C$, $q(y) \in [q(x_1), q(x_2)] \subseteq \text{clo}(q(C))$. Hence, there exists a sequence $(y_n)_n \subseteq C$ such that $h(y_n) > 0$ and $q(y_n) \rightarrow q(y)$. Let us now set $\lambda := \lambda_1/(\lambda_1 + \lambda_2)$ and $\check{x} = \lambda x_1 + (1 - \lambda)x_2$, so that $x = (\lambda_1 + \lambda_2)\check{x}$ and $h(\check{x}) = (\lambda_1 + \lambda_2)^{-1}h(x) \geq 0$.

Since $h(y) = 0$, we have $\lambda \geq \xi$, so that $\eta := (\lambda - \xi)/(1 - \xi) \in [0, 1)$. Furthermore, it is easily checked that $\check{x} = \eta x_1 + (1 - \eta)y$. This shows that

$$x = (\lambda_1 + \lambda_2)\check{x} \in \text{cone}\{x_1, y\} \subseteq \text{clo}(\text{cone}(\{x_1\} \cup \{y_n : n \in \mathbb{N}\})) \subseteq \text{clo}(\text{cone}(\{z \in C : h(z) \geq 0\})).$$

In summary, we have established that

$$\text{clo}(\text{cone}(\{x \in C : h(x) \geq 0\})) \supseteq \{x \in \text{cone}(C) : h(x) \geq 0\}. \tag{2.2}$$

The claimed inclusion (2.1) now follows by taking closures on both sides of (2.2).

3. A Farkas Type Theorem. The following result gives a Farkas type condition for two-dimensional spherically convex sets.

Lemma 3.1. Let $C \subseteq \mathbb{R}^2$ be such that $\text{clo}(q(C))$ is spherically convex in $\mathcal{S}((\mathbb{R}^2))$, let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be linear functionals, and let $h$ be such that

$$\text{clo}\left(\text{cone}(C) \cap \{z : h(z) \geq 0\}\right) = \text{clo}\left(\text{cone}(C)\right) \cap \{z : h(z) \geq 0\} \tag{3.1}$$

holds true. Then the following conditions are equivalent,

i) $g(x) \geq 0$ for all $x \in C$ such that $h(x) \geq 0$,

ii) there exists $\xi \geq 0$ such that $g(x) - \xi h(x) \geq 0$ for all $x \in C$.

Proof. It is trivial to check that ii) implies i). Let us thus assume that i) holds, a condition which is easily seen to be equivalent to

$$g(x) \geq 0 \quad \forall x \in \text{clo}(\text{cone}(\{z \in C : h(z) \geq 0\})).$$
By Lemma 2.4, this is further equivalent to
\[ g(x) \geq 0 \quad \forall x \in \text{clo} (\text{cone}(C) \cap \{ z : h(z) \geq 0 \}) , \]
and finally, by condition (3.1) this is equivalent to
\[ g(x) \geq 0 \quad \forall x \in \text{clo} (\text{cone}(C)) \cap \{ z : h(z) \geq 0 \} . \quad (3.2) \]
Identifying the dual space \((\mathbb{R}^2)^*\) with \(\mathbb{R}^2\) via the canonical inner product, we can consider \(g, h\) as elements of \(\mathbb{R}^2\) and \(x\) as an element of \((\mathbb{R}^2)^*\), so that condition (3.2) reads
\[ g^* \geq \text{clo} (\text{cone}(C)) \cap h^* , \quad (3.3) \]
where \(g^*\) is defined as \(\{ x : g(x) \geq 0 \}\). Taking duals on both sides of (3.3), we find
\[ \text{cone}(g) \subseteq C^* + \text{cone}(h) , \]
which is the same as condition ii). \(\Box\)

Next, we give a convenient sufficient condition for (3.1) to hold.

**Lemma 3.2.** Let \(C\) be a subset of \(\mathbb{R}^2\) and \(h\) a linear functional \(\mathbb{R}^2 \rightarrow \mathbb{R}\). If there exists \(\overline{\pi} \in C\) such that \(h(\overline{\pi}) > 0\), then (3.1) holds.

**Proof.** We only need to prove the inclusion \(\supseteq\), the reverse relation being trivial. Let thus \((x_n)_n \subset \text{cone}(C)\) be such that \(x_n \rightarrow x \in \mathbb{R}^2\) and \(h(x) \geq 0\). Then for every \(\varepsilon > 0\) we have \(h(x_n + \varepsilon \overline{x}) \geq 0\) for all \(n \geq n_\varepsilon\) when \(n_\varepsilon\) is chosen large enough. Furthermore, \(x_n + \varepsilon \overline{x} \in \text{cone}(C)\). Therefore, \(x + \varepsilon \overline{x} \in \text{clo}[\text{cone}(C) \cap \{ z : h(z) \geq 0 \}]\). Since this holds for all \(\varepsilon > 0\), we have \(x \in \text{clo}[\text{cone}(C) \cap \{ z : h(z) \geq 0 \}]\). \(\Box\)

It is now trivial to lift Lemma 3.1 into higher dimensional spaces, resulting in the following theorem.

**Theorem 3.3.** Let \((V, \langle \cdot, \cdot \rangle)\) be a real Hilbert space, \(g, h : V \rightarrow \mathbb{R}\) continuous linear functionals, \(W := (\ker(h) \cap \ker(g))^\perp\) and \(\pi_W\) is the orthogonal projection of \(V\) onto \(W\) along \(\ker(h) \cap \ker(g)\). Let \(C\) be subset of \(V\) such that \(h(\overline{\pi}) > 0\) for some \(\overline{\pi} \in C\) and such that \(\text{clo}(q(\pi_W C))\) is spherically convex. Then the following conditions are equivalent,

i) \(g(x) \geq 0\) for all \(x \in C\) such that \(h(x) \geq 0\),

ii) there exists \(\xi \geq 0\) such that \(g(x) - \xi h(x) \geq 0\) for all \(x \in C\).

**Proof.** Applying Lemma 3.1 and the sufficient criterion of Lemma 3.2 to \(h|_W, g|_W\) and \(\pi_W C\) on the two-dimensional subspace \(W\), we find that i) \(\Leftrightarrow g|_W(x) \geq 0\) for all \(x \in \pi_W C\) such that \(h|_W(x) \geq 0\) \(\Leftrightarrow \exists \xi \geq 0\) such that \(g|_W(x) - \xi h|_W(x) \geq 0\) for all \(x \in \pi_W C\). \(\Box\)

Note that in Theorem 3.3 the set \(C\) was not assumed to be convex, but if it is then \(\pi_W C\) is also convex and \(\text{clo}(q(\pi_W C))\) is spherically convex. In this case Theorem 3.3 becomes a special case of the Farkas Theorem; see Theorem 2.2 in [6]. We further remark that \(V\) can of course be a finite-dimensional Euclidean space. Finally, despite the Farkas flavour of its formulation, Theorem 3.3 can in fact be seen as a generalisation of the s-Lemma, as we will now show.
4. A New Proof of the s-Lemma. Combining the tools developed above, we now obtain a proof of Theorem 1.1 – restated here for convenience – as a special case of Theorem 3.3.

**Theorem 4.1.** Let \( f, q : \mathbb{R}^n \to \mathbb{R} \) be quadratic functions such that \( q(\bar{x}) > 0 \) for some \( \bar{x} \in \mathbb{R}^n \). Then \( f \) is copositive with \( q \) if and only if there exists \( \xi \geq 0 \) such that \( f(x) - \xi q(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

**Proof.** Let \( f \) be given by \( f(x) = x^T Q x + 2 \ell^T x + c \), where \( Q \in \mathcal{S}_n \), \( \ell \in \mathbb{R}^n \) and \( c \in \mathbb{R} \). Then

\[
\left( x_1 \right) = \left( A, [\bar{x}] [\bar{x}]^T \right),
\]

where \( \langle A, X \rangle = \text{tr}(A^T X) \) is the trace inner product defined on the space \( \mathcal{S}_{n+1} \) of symmetric \((n+1) \times (n+1)\) matrices, and where

\[
A = \begin{bmatrix} Q & \ell \\ \ell^T & c \end{bmatrix}.
\]

Likewise, there exists \( B \in \mathcal{S}_{n+1} \) such that

\[
q(x) = \left( B, [\bar{x}] [\bar{x}]^T \right).
\]

Let \( C \subset \mathcal{S}_{n+1} \) be defined by

\[
C = \{ zz^T : z = [\bar{x}], x \in \mathbb{R}^n \}.
\]

Written in the notation just introduced, the claim of Theorem 1.1 is that the following two conditions are equivalent,

i) \( \langle A, X \rangle \geq 0 \) for all \( X \in C \) such that \( \langle B, X \rangle \geq 0 \),

ii) there exists \( \xi \geq 0 \) such that \( \langle A - \xi B, X \rangle \geq 0 \) for all \( X \in C \).

We note that \( g(X) = \langle A, X \rangle \) and \( h(X) = \langle B, X \rangle \) are linear functionals on \( \mathcal{S}_{n+1} \). Furthermore, if

\[
\overline{X} = [\bar{x}] [\bar{x}]^T,
\]

then \( \overline{X} \in C \) and \( h(\overline{X}) > 0 \). Thus, the equivalence of i) and ii) follows from Theorem 3.3 if it can be established that \( \text{clo}(q(\pi_W C)) \) is spherically convex when \( \pi_W \) is the orthogonal projection of \( (\mathcal{S}_{n+1}, \langle , \rangle) \) onto \( W := (\ker(h) \cap \ker(g))^\perp = \text{span}\{A, B\} \).

Let \( X_1, X_2 \in C \). Then

\[
X_i = [x_i] [x_i]^T
\]

for some \( x_i \in \mathbb{R}^n \), \( i = 1, 2 \). Let \( x(t) = x_1 + (1 - t)(x_2 - x_1) \) \( (t \in \mathbb{R}) \) and

\[
X(t) := [x(t)] [x(t)]^T
\]

\[
= [x_1] [x_1]^T + t \left( [x_1] [x_2 - x_1]^T + [x_2 - x_1] [x_1]^T \right) + t^2 [x_2 - x_1] [x_2 - x_1]^T
\]

\[
= G_0 + tG_1 + t^2 G_2.
\]
Let $\tilde{A}, \tilde{B} \in \mathcal{S}_{n+1}$ be an orthonormal basis of $W$. Then $\pi_W X(t) = a(t)\tilde{A} + b(t)\tilde{B}$, where
\begin{align*}
a(t) &= \langle G_0, \tilde{A} \rangle + t \langle G_1, \tilde{A} \rangle + t^2 \langle G_2, \tilde{A} \rangle, \\
b(t) &= \langle G_0, \tilde{B} \rangle + t \langle G_1, \tilde{B} \rangle + t^2 \langle G_2, \tilde{B} \rangle.
\end{align*}
Thus, the set
\[ T := \left\{ \left[ \begin{array}{c} a(t) \\ b(t) \end{array} \right] : t \in \mathbb{R} \right\} \]
is either a parabola (when either $\langle G_2, \tilde{A} \rangle \neq 0$ or $\langle G_2, \tilde{B} \rangle \neq 0$), or a straight line (when $\langle G_2, \tilde{A} \rangle = 0 = \langle G_2, \tilde{B} \rangle$ and either $\langle G_1, \tilde{A} \rangle \neq 0$ or $\langle G_1, \tilde{B} \rangle \neq 0$) or a singleton (when $\langle G_2, \tilde{A} \rangle = 0 = \langle G_2, \tilde{B} \rangle$ and $\langle G_1, \tilde{A} \rangle = 0 = \langle G_1, \tilde{B} \rangle$). In the latter two cases $T$ is convex, and hence $q(T)$ is spherically convex. In the parabolic case Lemma 2.3 shows that $\text{clo}(q(T))$ is spherically convex. It follows from Lemma 2.2 that $\text{clo}(q(\pi_W C))$ is spherically convex, as claimed. \Halmos

The proof given above immediately generalizes to infinite-dimensional spaces:

**Theorem 4.2.** Let $(V, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, and let $f, q : V \to \mathbb{R}$ be continuous quadratic functions defined on $V$ by
\begin{align*}
f &:= x \mapsto c_f + 2\langle v_f, x \rangle + \langle x, M_f x \rangle, \\
q &:= x \mapsto c_q + 2\langle v_q, x \rangle + \langle x, M_q x \rangle,
\end{align*}
where $M_f, M_q : V \to V$ are continuous self-adjoint operators, $v_f, v_q \in V$ and $c_f, c_q \in \mathbb{R}$ are constants. Let $q$ further have the property that there exists $\varpi \in V$ for which $q(\varpi) > 0$. Then $f$ is copositive with $q$ if and only if there exists $\xi \geq 0$ such that $f(x) - \xi q(x) \geq 0$ for all $x \in V$.

**Proof.** The proof is identical to that of Theorem 4.1 bar the following construction: let $H := V \oplus \mathbb{R}$, where $\oplus$ denotes the direct sum of Hilbert spaces, and let us write $\langle \cdot, \cdot \rangle_H$ for the inner product on $H$. Let $\mathcal{S}$ be the space of self-adjoint operators on $H$. Such operators are automatically continuous, and it is easy to see that $A, B \in \mathcal{S}$, where
\begin{align*}
A &:= H \to H, \\
(x, \tau) &\mapsto (M_f x + \tau v_f, \langle v_f, x \rangle + \tau c_f)
\end{align*}
and
\begin{align*}
B &:= H \to H, \\
(x, \tau) &\mapsto (M_q x + \tau v_q, \langle v_q, x \rangle + \tau c_q).
\end{align*}
Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis of $H$, and consider the operators
\[ E_{ij} : y \mapsto \frac{1}{1 + \delta_{ij}} \langle e_i, y \rangle_H e_j + \langle e_j, y \rangle_H e_i, \]
where $\delta_{ij}$ is the Kronecker delta. Defining furthermore
\[ \langle E_{ij}, E_{kl} \rangle_S := \begin{cases} 1 & \text{if } \{i, j\} = \{k, l\}, \\
0 & \text{otherwise}, \end{cases} \]
the \( E_{ij} \) generate a Hilbert space \((S, \langle \cdot, \cdot \rangle_S)\) for which \( \{ E_{ij} : \ i, j \in \mathbb{N} \} \) is an orthonormal basis. Furthermore, \( S \subseteq \mathcal{F} \), since \( E_{ij} \in \mathcal{F} \) for all \( i, j \). Every \( x \in V \) defines an operator \( R(x) \in \mathcal{F} \),

\[
R(x) : z \mapsto \langle (x, 1), z \rangle_H (x, 1),
\]

and if \((x, 1) = \sum_{i \in \mathbb{N}} \xi_i e_i\) then \( R(x) = \sum_{ij} \xi_i \xi_j E_{ij} \) and \( \sum_{ij} \xi_i^2 \xi_j^2 = (\sum_i \xi_i^2)(\sum_j \xi_j^2) < \infty \). This shows that \( C := \{ R(x) : x \in V \} \subset S \). Extending the map

\[
g : C \to \mathbb{R},
R(x) \mapsto \langle (x, 1), A(x, 1) \rangle_H
\]

by linearity and continuity, we obtain a bounded linear operator on the Hilbert space \((\text{clo}(\text{span}(C)), \langle \cdot, \cdot \rangle_S)\). Likewise, \( B \) defines a bounded linear operator \( h \) on the same space. Replacing \( \mathcal{F}_{n+1} \) by \( \text{clo}(\text{span}(C)) \) in the proof of Theorem 4.1, a repetition of the arguments presented there proves the claim of the present theorem.

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**REFERENCES**