Model Theory of Holomorphic Functions

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Abstract

This thesis is concerned with a conjecture of Zilber: that the complex field expanded with the exponential function should be *quasi-minimal*; that is, all its definable subsets should be countable or have countable complement. Our purpose is to study the geometry of this structure and other expansions by holomorphic functions of the complex field without having first to settle any number-theoretic problems, by treating all countable sets on an equal footing.

We present axioms, modelled on those for a Zariski geometry, defining a non-first-order class of "quasi-Zariski" structures endowed with a dimension theory and a topology in which all countable sets are of dimension zero. We derive a quantifier elimination theorem, implying that members of the class are quasi-minimal.

We look for analytic structures in this class. To an expansion of the complex field by entire holomorphic functions $\mathcal{R}$ we associate a sheaf $\mathcal{O}^{\mathcal{R}}$ of analytic germs which is closed under application of the implicit function theorem. We prove that $\mathcal{O}^{\mathcal{R}}$ is also closed under partial differentiation and that it admits Weierstrass preparation. The sheaf defines a subclass of the analytic sets which we call $\mathcal{R}$-analytic. We develop analytic geometry for this class proving a Nullstellensatz and other classical properties. We isolate a condition on the asymptotes of the varieties of certain functions in $\mathcal{R}$. If this condition is satisfied then the $\mathcal{R}$-analytic sets induce a quasi-Zariski structure under countable union. In the motivating case of the complex exponential we prove a low-dimensional case of the condition, towards the original conjecture.

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DPhil thesis

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Chapter 1

Introduction

The exponential function on the complex field is of natural mathematical interest, but was for a long time comparatively neglected by model theorists. Recently Zilber [27] has constructed a class of artificial "pseudo-exponential" analogues of the complex exponential in a manner which "ties together most of the model theory of the last fifty years" (Baldwin,[1]). Assuming that two significant open conjectures number-theoretic in character, Schanuel's conjecture and satisfaction of the strong exponential-algebraic closure axioms (a conjecture of Diophantine type), both hold, the complex exponential field can be identified with the unique pseudo-exponential field of continuum size.

Without settling these number-theoretic questions the strongest conjecture hoped for was posed by Zilber [28]:

Conjecture 1.1 *The complex exponential field* $\mathbb{C}_{\exp}$ *is homogeneous and quasi-minimal.*

A structure is *quasi-minimal* if every subset of the domain which is definable with parameters is either countable or of countable complement.

This is not a first-order property, in the sense that the elementary extension of a quasi-minimal structure is not necessarily quasi-minimal. It is somewhat better behaved in classes of homogeneous structures, especially the "excellent classes" of Shelah. It is only recently that such properties have enjoyed much attention in model theory. Even that the expansion of the complex field with a predicate for the integers, $\mathbb{C}_{\mathbb{Z}} = \langle C, +, \cdot, -, 0, 1, \mathbb{Z}\rangle$, is quasi-minimal was proved by Wilkie [25] only in 2002, for example.

In this thesis I take a geometrical approach to definability in the structure $\mathbb{C}_{\exp} = \langle \mathbb{C}, +, \cdot, -, 0, 1, \exp\rangle$ and other expansions of the complex field by entire holomorphic functions. To an expansion in which all the derivatives of the primitive functions are definable, we associate a sheaf of analytic germs closed under taking the coordinates of implicitly defined functions. This gives us a subclass of the complex analytic sets in which we can develop a geometry intermediate between the complex algebraic and analytic geometries. It is hoped that this construction may be of independent interest.

We isolate a tractable condition on the asymptotes of certain quantifier-free definable varieties which is sufficient to show any such expansion to be quasi-minimal, and verify a low-dimensional case of the condition for $\mathbb{C}_{\exp}$.

1.1 Background

For the history of the model-theoretic study of exponential functions see Macintyre's lectures [9]. Here I sketch briefly the context of the ideas in this thesis.

As early as 1937 Tarski conjectured that the real exponential field $\mathbb{R}_{\exp}$ should have decidable theory. Since the ring $\mathbb{Z}$ of integers is definable in $\mathbb{C}_{\exp}$, the theory of the complex exponential field is Gödelian and no analogous conjecture can be made. Becker, Henson and Rubel proved in 1970 [2] that the ring of all entire functions (that is, the functions holomorphic on all of $\mathbb{C}$ with addition and multiplication defined pointwise) is bi-interpretable with full second-order number theory. Apart from this result, model theorists neglected the study of expansions of the complex field by analytic functions until the 1990s.

On the other hand the p-adic and real analytic cases proved fruitful analogies of each other. Out of this programme was developed the theory of o-minimal geometry, culminating in Wilkie's proof [24] that $\mathbb{R}_{\exp}$ is o-minimal and model complete. The central ideas of the proof are:

(i) an expression of any existential formula from the language of $\mathbb{R}_{\exp}$ in a normal form as the projection of an exponential polynomial, possibly with (restricted analytic) Pfaffian functions as parameters; and

(ii) an understanding of the asymptotic behaviour of the zero sets of exponential polynomials through a valuation theory, in which the restricted analytic functions from a Pfaffian chain take value zero.

In conception the methods of this thesis are closely derived from these ideas, with the $\mathcal{R}$-holomorphic functions of chapter 3 taking the place of the Pfaffian chain, and a notion formally analogous to a valuation ("bounded difference in logarithm" applied to the coordinates of a real curve segment inside the variety of an exponential polynomial) motivating, now perhaps obscurely, the proof in section 6.2.

In 1960 Schanuel formulated the conjecture:

(SC) if $z_1, \ldots, z_n \in \mathbb{C}$, then

$$\text{tr.deg. } \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) - \dim \mathbb{Q}\langle z_1, \ldots, z_n \rangle > 0.$$

This summarizes all the known or conjectured transcendence theory of exp. Very few cases are known but a decision procedure for $\mathbb{R}_{\exp}$ would enable one to test, for example, whether e^e satisfies specific polynomials over $\mathbb{Q}(e)$ (the conjecture implies that in all cases it should not). Conversely Macintyre and Wilkie [10] showed that (SC) implies that $\text{Th}(\mathbb{R}_{\exp})$ is decidable, settling Tarski's conjecture positively under this hypothesis.

As in much of model theory, the complex numbers make their first appearance in the history of our problem as a model of the paradigmatically well-behaved first-order theory of algebraically closed fields of characteristic 0, an uncountably categorical theory. Baldwin and Lachlan analysed the models of uncountably categorical (*i.e.,* finite Morley rank) theories as being prime over a strongly minimal set. Zilber [26] identifies the essence of this analysis thus: "The key factor is measurability by a dimension and high homogeneity of the structure."

On philosophical grounds believing that logical phenomena of this nature should be classically prefigured, Zilber put forward his trichotomy conjecture, that the pregeometry (determining the dimension notion) in any strongly minimal set should be essentially algebraic and arise in some classical algebraic context. Hrushovski's construction of counterexamples was not fatal to the conjecture.

An initial response was Hrushovski and Zilber's development [5] of *Zariski geometries*, axiomatized as structures for which there is a dimension theory and a (compactifiable) topology emulating the Zariski topology on algebraic varieties. These have a notion of nonstandard analysis through specializations, and so the trichotomy theorem for Zariski structures can be derived as a non-standard version of Chow's theorem, that all globally analytic structure on a compact complex manifold is algebraic.

Further, in the spirit of the trichotomy conjecture, efforts were made to find the Hrushovski examples in nature. Hrushovski's method finds existentially closed structures inside classes (of structures expanding some theory by a new function symbol or predicate, say the characteristic zero algebraically closed fields with a unary function H) with the amalgamation property. The idea is that tuples in the class should have a predimension δ measuring the number of explicit dependencies between elements of the tuple in the new language, in a way appropriate to the desired formal properties of the expansion. It is natural to postulate that the predimension is non-negative; we can express, in terms of such a predimension, the condition for the amalgamation to be possible.

In the case that H is to be a homomorphism from the additive to the multiplicative groups of the field, such a construction cannot give a strongly minimal, or even a stable, amalgamation. Nevertheless this case shows clearly the resemblance between these constructions and *analytic* objects. The appropriate pre-dimension is

$$\delta(z_1, \ldots, z_n) = \text{tr.deg. } \mathbb{Q}(z_1, \ldots, z_n, H(z_1), \ldots, H(z_n)) - \dim \mathbb{Q}\langle z_1, \ldots, z_n\rangle$$

and the condition that this should be non-negative is exactly of the form of

Schanuel's conjecture (SC) above. With an appropriate notion of existential closure, described by Zilber in [29], the *pseudo-exponential fields* can be constructed. They are axiomatized with an $L_{\omega_1\omega}$ sentence and a (non-$L_{\omega_1\omega}$) condition, the countable closure property, which is satisfied by $\mathbb{C}_{\text{exp}}$. The class of such structures is a quasi-minimal excellent class, a development of Shelah's work on abstract elementary classes towards analogues of Morley's theorem for infinitary logic. In particular it is an uncountably categorical class, so subject to the number-theoretic conjectures mentioned above we can identify $\mathbb{C}_{\text{exp}}$ with the pseudo-exponential field of continuum cardinality.

Working from the opposite perspective, Wilkie [23] defines the Liouville function, an entire function satisfying the Schanuel-type condition for a maximally free unary function expanding the field $\mathbb{C}$. Koiran [7] recognised the theory of the Liouville function as the limit theory of a generic polynomial and thereby proved the corresponding existential closure condition. This is therefore indeed an analytic model of the desired kind.

Peatfield and Zilber [15] have framed the *analytic Zariski* axioms extending those for Zariski geometries, to describe abstractly the geometric properties of an analytic structure. At least one of the Hrushovski structures (a free ternary relation on an algebraically closed field) falls into this class. But it is not yet known whether any natural analytic structures do.

The principle difference between Peatfield and Zilber's approach and that of this thesis is that the analytic Zariski axioms focus on the natural analytic map, *proper* projection, on open subsets of a compactified structure. The quasi-Zariski axioms presented in chapter 2 are framed for unrestricted projection which does not respect analyticity.

1.2 This thesis

In chapter 2 are presented axioms for a class of structures called *quasi-Zariski*, with a dimension notion on the positively definable sets. These are modelled on the axioms for a Zariski geometry but treat all countable sets on an equal footing, and allow a countable union to have the same dimension as the maximum dimension of the members of the union. We prove that with these axioms the positively definable sets (called *closed*) do indeed form a topology, and that in an appropriate language the theory of a quasi-Zariski structure eliminates quantifiers. In particular, any quasi-Zariski structure is quasi-minimal. In an analytic model of these axioms the closed sets of the topology will be of Borel class F_σ over the definable analytic sets.

At the start of chapter 3 we fix a class $\mathcal{R}$ of entire holomorphic mappings extending the polynomials over $\mathbb{C}$ and closed under composition with linear maps and partial differentiation. We define the ring $\mathcal{O}_U^{\mathcal{R}}$ of *$\mathcal{R}$-holomorphic functions* on an open set U, to contain those holomorphic functions f for which there is a finite partition of U, on each member of which f is *implicitly represented* over $\mathcal{R}$. A function is implicitly represented if it is a coordinate function of a mapping whose graph is contained in the set $\mathrm{Ift}(F)$ of points at which some $F \in \mathcal{R}$ satisfies the hypotheses of the implicit function theorem.

We prove some elementary properties of this presheaf, including closure under any further taking of implicit functions. More significantly, we prove that Weierstrass preparation is possible within $\mathcal{O}_D^{\mathcal{R}}$ for functions f for which D is a *preparation domain* and all of whose partial derivatives lie in $\mathcal{O}_D^{\mathcal{R}}$.

Chapter 4 is concerned with discharging this last condition and proving that any germ in $\mathcal{O}^{\mathcal{R}}$ has all its partial derivatives inside $\mathcal{O}^{\mathcal{R}}$. We show this by way of an analytic continuation theorem.

That is, we cover the boundary of $\mathrm{Ift}(F)$, where the hypotheses of the implicit function theorem fail, with the graphs of finitely many implicitly represented functions. Necessarily this involves a detailed understanding of the possible behaviours of F where its Jacobian determinant $J(F)$ takes value zero. We prove that the zero set of $J(F)$ can be covered with finitely many preparation domains and exploit Weierstrass preparation (under the conditions of the proof in chapter 3). Points on the boundary of $\mathrm{Ift}(F)$ are distinguished from other zeros of F by a technical notion of *accessibility* in $\mathrm{Ift}(F)$, which adds to the complexity of the argument.

The details of this proof give a good sense of the strength of our construction of $\mathcal{O}^{\mathcal{R}}$ but we do not need them after this chapter.

In chapter 5 we develop the geometry of $\mathcal{R}$-analytic sets, defined locally by functions from $\mathcal{O}^{\mathcal{R}}$, a subclass of the analytic sets. As far as we carry the programme, it follows exactly the classical development of analytic geometry. In particular, we prove a Nullstellensatz for $\mathcal{O}^{\mathcal{R}}$ and prove that any analytic component of an $\mathcal{R}$-analytic set is $\mathcal{R}$-analytic.

We prove that the class $\mathcal{C}(\mathcal{R})$ of countable unions of $\mathcal{R}$-analytic subsets of affine space satisfies all the QZ axioms except for the axiom (CP) of constructible projection and, in the absence of (CP), except that the additive formula (AF) may hold only in a "local" version. If the class satisfies (CP) then it gives us a quasi-Zariski geometry.

Chapter 6 presents an alternative formulation of axiom (CP) in this context, in terms of the asymptotes of the varieties of certain functions from $\mathcal{R}$. In the case that $\mathcal{R}$ contains the terms of the language of $\mathbb{C}_{\exp}$, this is a condition on the zeros of exponential polynomials in m variables. We prove the condition true when $m = 1$.

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Chapter 2

Quasi-Zariski Geometries

The axioms for a Zariski geometry for a set M given by Ehud Hrushovski and Boris Zilber in [5] describe what it is for a collection $\mathcal{C}$ of subsets of powers of M (*closed* sets) to behave like the algebraic sets over an algebraically closed field. In particular the projection of a closed set in a Zariski geometry is *constructible*, *i.e.,* a boolean combination of closed sets, as follows in algebraic geometry from the Nullstellensatz. Another important principle is that closed sets can be decomposed into a *finite* number of irreducible components.

For example, if M is a compact complex manifold and the $\mathcal{C}$ contains all the analytic sets over M, then the requisite finiteness follows immediately from compactness, and the projection condition is given by Remmert's theorem.

We wish to find an analogue of Zariski geometry which will describe "tame" analytic geometry over all of $\mathbb{C}$. This necessarily involves relaxing the finiteness condition. In particular, the proper analytic subsets of $\mathbb{C}$ are locally finite (in the usual topology) and have a countable number of connected components in $\mathbb{C}$.

Typical projections of countable analytic subsets of $\mathbb{C}^2$, by contrast, are not locally finite; as for example $\mathbb{Q}$ or $\mathbb{Q}[\sqrt{-1}]$. That these sets are dense in $\mathbb{R}$ and $\mathbb{C}$ respectively, sets to which we would want to assign dimension greater than zero, shows that we cannot hope to have a Zariski-type closure for these sets which agrees with the closure operation for the usual topology. By treating all countable sets on an equal footing, however, we are able to prove a quantifier elimination theorem analogous to that for Zariski geometries. Our closed sets (in analytic models of our axioms) will thus be of Borel class F_σ over the analytic sets. An immediate consequence of the quantifier elimination is that all models of the axioms are quasi-minimal.

The analytic models have cardinality $2^{\aleph_0}$ and their natural topology has a basis of cardinality $\aleph_0$. The arguments of this chapter make no essential use of the numbers $(2^{\aleph_0}, \aleph_0)$; we might consider any corresponding pair (λ, κ) with κ regular and $\lambda > \kappa$.

We follow the presentation of the axioms in Zilber's notes [30] (see example 2.11 below), giving separate axioms for the dimension function.

Definition. A *quasi-Zariski geometry* (alternatively a *QZ-structure*) is a triple $\langle M, \mathcal{C}, \dim \rangle$ where:

(i) M is a set;

(ii) $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_n$ is a subcollection of the subsets of M^n; and

(iii) $\dim : \mathcal{C} \to \mathbb{N} \cup \{-\infty\}$;

which satisfies the following axioms 1–10.

A subset $X \subseteq M^n$ is called *closed* if $X \in \mathcal{C}_n$. We shall prove below that in any structure satisfying axioms 1–7 the closed sets form a topology, so this language is appropriate. In this case there is a unique smallest closed set $\mathrm{cl}(Y)$ containing any $Y \subseteq M^n$.

A closed set X is called *irreducible* if it cannot be written as a proper countable union of closed sets: *i.e.,* whenever $X = \bigcup_{i \in \omega} X_i$ with each X_i closed, there is $i \in \omega$ with $X = X_i$. An *irreducible component* of X is a member of a minimal representation of X as a countable union of irreducibles.

We write $\pi_{n,m} : M^{n+m} \to M^n$ for the natural projections onto the first n coordinates. If $X \subseteq M^{n+1}$ then usually by $\pi(X)$ I shall mean $\pi_{n,1}(X) \subseteq M^n$.

2.1 Axioms

I will use (QZ) to denote the conjunction of the following axioms 1–10, and (QZ^-) to denote (QZ) \ (IM).

1 (L) *Language:*

(i) singletons $\{a\}$ for $a \in M$ are closed;

(ii) M is closed;

(iii) finite intersections and unions of closed sets are closed;

(iv) cartesian products of closed sets are closed;

(v) The diagonals $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\}$, where $i \leq j \leq n$, are closed.

2 (IC) *Irreducible components:* for any closed set X there exist countably many irreducible closed sets $(X_i)_{i\in\omega}$ with $X = \bigcup_{i\in\omega} X_i$.

3 (CU) *Countable union:* if $X_i \in \mathcal{C}_n$ for each $i \in \omega$ and $X = \bigcup_{i\in\omega} X_i$ then $X \in \mathcal{C}_n$ also.

4 (DP) *Dimension of a point:* for all $a \in M^n$, $\dim\{a\} = 0$.

5 (IM) *Irreducible model* the domain M is irreducible and $\dim(M) = 1$.

6 (DU) *Dimension of unions:* $\dim(\bigcup_{i\in\omega} X_i) = \max\{\dim(X_i) : i \in \omega\}$.

7 (DI) *Dimension of irreducible sets:* if X is irreducible and X' is a proper closed subset of X then $\dim(X') < \dim(X)$.

8 (CP) *Constructible projection:* if X is a closed set then there is a closed X' such that $\mathrm{cl}(\pi(X)) = \pi(X) \cup X'$ and $\dim(X') < \dim \mathrm{cl}(\pi(X))$.

9 (FC) *Fibre condition:* for each closed $X \in \mathcal{C}_{n+m}$ and $k \in \mathbb{N}$, the set

$$\mathcal{P}^{n,m}(X, k) = \{a \in M^n : \dim(X \cap \pi_{n,m}^{-1}(a)) > k\}$$

is the projection of some closed subset X' of X each of whose fibres is of dimension greater than k, *i.e.,*

$$\mathcal{P}^{n,m}(X, k) = \mathcal{P}^{n,m}(X', k) = \pi(X').$$

10 (AF) *Additive formula:* if X is irreducible, then $\dim(X) = \dim \mathrm{cl}(\pi(X)) + \min\{\dim(\pi^{-1}(a)) \cap X : a \in \pi(X)\}$.

Axiom (IC) together with the dimension axiom (DI) takes the place of the Noetherianity condition on Zariski geometries: any descending chain *of irreducibles* stabilises. Clearly in the absence of a global finiteness condition some such modification is needed. The statement (AF) is adopted without change from the (not necessarily compact) Zariski case. By contrast the exact formulation of (FC), needed for the proof of quantifier elimination, is quite sensitively dependent on the forms of (AF) and (CP) since we wish in particular to verify it in analytic cases where (CP) may fail.

The form of (CP) is intended to be suggestive of how we might prove it to hold in an analytic structure: assuming we are given an irreducible set X, it would be enough to show that the boundary of $\pi(X)$ is *contained in* a set of low dimension. We take up this theme in section 6.1.

Lemma 2.1 (L) *The projection mappings* $\pi : M^n \to M^{n-1}$ *are continuous.*

Proof. Suppose X in M^{n-1} is closed. Then $\pi^{-1}(X) = X \times M$ is closed too, by axiom (L). □

Lemma 2.2 (L,IC,CU,DI) *The irreducible components of* X *are defined up to an enumeration uniquely.*

Proof. Minimal representations exist, since if $X = \bigcup_{i\in\omega} X_i \subseteq M^n$ with each X_i irreducible and all the X_i distinct, we may consider the set

$$I = \{i \in \omega : \forall j \in \omega\, (X_i \subseteq X_j \to X_i = X_j)\}.$$

I is nonempty, as $\dim M^n$ is finite and so by axiom (DI) there are no ascending chains of irreducibles. Moreover, if $x \in X_j$ for some $j \in \omega$ then there is $i \in I$ such that $x \in X_j \subseteq X_i$. So $X = \bigcup_{i\in I} X_i$. But this is indeed a minimal representation, since for each $i \in I$, $X_i \not\subseteq \bigcup_{j\in I\setminus\{i\}} X_j$ (as all the X_i are irreducible).

Now suppose $X = \bigcup_{i\in\alpha} X_i = \bigcup_{j\in\beta} X'_j$, two minimal representations of X as unions of irreducibles (with $\alpha, \beta \leq \omega$).

For each $i \in \alpha$ we can write $X_i = \bigcup_{j\in\beta}(X_i \cap X'_j)$, so as X_i is irreducible there is some $j = f(i)$ such that $X_i \subseteq X'_{f(i)}$. Similarly each X'_j is contained in some $X_{g(j)}$.

Then for each i, $X_i \subseteq X'_{f(i)} \subseteq X_{g\circ f(i)}$. But since the X_i were chosen minimally, this means that $X_i = X_{g\circ f(i)}$, and hence $X_i = X'_{f(i)}$.

So each X_i features among the X'_j, as required. □

Theorem 2.3 (L,IC,CU,DP,DI,DU) *An arbitrary intersection of closed sets is closed.*

I am grateful to the examiners for communicating to me a much improved proof of this theorem, which follows on page 15.

Proof. Let X_α be closed for each ordinal α and suppose that for some $\beta \in \mathrm{On}$, whenever $0 < \gamma < \beta$ then $Y_\gamma = \bigcap_{\alpha<\gamma} X_\alpha$ is closed.

We wish to show that $\bigcap_{\alpha<\beta} X_\alpha$ is closed too. If β is a successor ordinal then this is immediate by axiom (L).

So suppose β is a limit ordinal and consider $\langle \mathfrak{I}_\beta, \subseteq \rangle$, where

$$\mathfrak{I}_\beta = \{A : A \text{ is an irreducible component of } \bigcap_{\alpha<\gamma} X_\alpha \text{ for some } \gamma < \beta\}.$$

Then any chain in $\mathfrak{I}_\beta$ is of length no greater than $\dim X_0 + 1$, by axiom (DI); and so $\mathfrak{I}_\beta$ is closed under intersections of chains. I claim: if $A \in \mathfrak{I}_\beta$, then A has only countably many maximal proper subsets in $\mathfrak{I}_\beta$.

Certainly, if we write

$$S_\gamma = \{\alpha \in \mathrm{On} : \text{ for some } A \text{ a component of } Y_\gamma,\ \alpha \text{ is minimal s.t. } A \not\subseteq Y_\alpha\}$$

then S_γ is a countable set, and so is its closure in the class of ordinals $\bar{S_\gamma}$. If B is a maximal proper subset in $\mathfrak{I}_\beta$ of some component of Y_γ, then B is a component of Y_{γ_1} for some $\gamma_1 \in \bar{S_\gamma}$. Suppose otherwise, for a contradiction, and let γ_1 be minimal such that B is a component of Y_{γ_1} , but $\gamma_1 \notin \bar{S_\gamma}$. Then there is $\gamma_2 < \gamma_1$

with $\gamma_2 \in \bar{S_\gamma}$, such that if A is any component of Y_γ and $B \subseteq A$, then A is not a component of Y_{γ_2}.

Now B is not a component of Y_{γ_2}, by hypothesis, but $B \subseteq Y_{\gamma_2}$ and B is irreducible, so there is some component C of Y_{γ_2} with $B \subsetneq C$. Similarly, C is not a component of Y_γ, by choice of γ_2, but $C \subseteq Y_\gamma$, so that for some A a component of Y_γ, $C \subsetneq A$. This contradicts maximality of B as a proper subset of a component of Y_γ.

So all maximal proper subsets in $\mathfrak{I}_\beta$ of any $A \in \mathfrak{I}_\beta$ are components of some Y_{γ_1} with $\gamma_1 \in \bar{S_\gamma}$, if A is a component of Y_γ. Thus there are, as claimed, at most countably many such, by axiom (IC) and the inductive hypothesis.

Finally, by axiom (IC) for X_0 there are only countably many maximal elements in $\mathfrak{I}_\beta$. From this, the claim, and the fact that all chains are finite we deduce that $\mathfrak{I}_\beta$ is a countable set.

Now if we define $\mathfrak{F}_\beta$ to be the closure of $\mathfrak{I}_\beta$ under taking irreducible components of finite intersections, so that $\mathfrak{F}_\beta = \bigcup_{i\in\omega} \mathfrak{F}_{\beta,i}$ with $\mathfrak{F}_{\beta,0} = \mathfrak{I}_\beta$ and $\mathfrak{F}_{\beta,i+1} = \{A : A \text{ is an irreducible component of } B_1 \cap B_2 \text{ for some } B_1, B_2 \in \mathfrak{F}_{\beta,i}\}$, then for each i, $\mathfrak{F}_{\beta,i} \subseteq \mathfrak{F}_{\beta,i+1}$ and $\mathfrak{F}_{\beta,i}$ is countable; so $\mathfrak{F}_\beta$ is countable too.

Moreover, since all the sets in $\mathfrak{F}_\beta$ are irreducible, any chain in $\mathfrak{F}_\beta$ has length no greater than $\dim X_0 + 1$. In particular, if $a \in A \in \mathfrak{F}_\beta$, we can find a minimal $A' \in \mathfrak{F}_\beta$ with the property that $a \in A' \subseteq A$.

Now suppose that $a \in \bigcap_{\alpha<\beta} X_\alpha$, and that A is an element of $\mathfrak{F}_\beta$ minimal such that $a \in A$. Then $A \subseteq \bigcap_{\alpha<\beta} X_\alpha$. For suppose $a' \in A$ but $a' \notin \bigcap_{\alpha<\beta} X_\alpha$. Then, as β is a limit ordinal,

$$\bigcap_{\alpha<\beta} X_\alpha = \bigcap_{\beta'<\beta} \bigcap_{\alpha<\beta'} X_\alpha,$$

so there is some $\gamma < \beta$ with $a' \notin \bigcap_{\alpha<\gamma} X_\alpha$. If $B \in \mathfrak{I}_\beta$ is a component of $\bigcap_{\alpha<\gamma} X_\alpha$ containing a, then $a \in (A \cap B)$ and there is a component $A' \in \mathfrak{F}_\beta$ of $A \cap B$ with $a \in A'$. This contradicts the minimality of A in $\mathfrak{F}_\beta$.

Finally let $X = \bigcap_{\alpha<\beta} X_\alpha$. We have

$$X = \bigcup_{a\in X} \bigcup \{A \in \mathfrak{F}_\beta : \text{A is minimal in } \mathfrak{F}_\beta \text{ with } a \in A\}.$$

But this is a union of elements of $\mathfrak{F}_\beta$ and thus is, in particular, a countable union of closed sets. Hence X is closed (by (CU)), as required. $\square$

Alternative proof (Zilber). Let $\{X_\alpha : \alpha \in I\}$ be an arbitrary collection of closed sets, and let

$$d = \min_{\alpha_1,\ldots,\alpha_k \in I; k \in \mathbb{N}} (\dim(X_{\alpha_1} \cap \cdots \cap X_{\alpha_k}))$$

be the least dimension of any finite intersection among the X_α.

Without loss of generality we may take $d = \dim X_0$, say, and $X_\alpha \subseteq X_0$ for all $\alpha \in I$. We shall prove the theorem by induction on d.

If $d = 0$ then X_0, and hence $\bigcap_{\alpha\in I} X_\alpha$, are all countable and therefore closed.

Otherwise, we may write $X_0 = Z \cup \bigcup_{j\in\omega} W_j$ with each W_j irreducible of dimension d and $\dim Z < d$. Then by induction we know that $Z \cap \bigcap_{\alpha\in I} X_\alpha$ is closed. Each W_j is irreducible, so satisfies either:

(i) for some $\alpha \in I$, $\dim(W_j \cap X_\alpha) < d$; or

(ii) $W_j \subseteq X_\alpha$ for all $\alpha \in I$.

By induction in case (i), and obviously in case (ii), $W_j \cap \bigcap_{\alpha\in I} X_\alpha$ is closed. Hence

$$\bigcap_{\alpha\in I} X_\alpha = X_0 \cap \bigcap_{\alpha\in I} X_\alpha = (Z \cap \bigcap_{\alpha\in I} X_\alpha) \cup \bigcup_{j\in\omega} (W_j \cap \bigcap_{\alpha\in I} X_\alpha)$$

which is a countable union of closed sets, and therefore closed. $\square$.

Definition. A subset X of M^n is *constructible* if it can be written $X = \bigcup_{i=1}^m (S_i \setminus P_i)$ for some $m \in \mathbb{N}$ and $S_1, \ldots, S_m, P_1, \ldots, P_m$ all closed.

Any constructible set X has a representation $X = \bigcup_{i=1}^m (S_i \setminus P_i)$ in which for each i, $P_i \subseteq S_i$ and no component of P_i is a component of S_i. In this case $\dim P_i < \dim S_i$. Call such a representation a *good construction* of X.

We extend the domain of definition of the dimension function to constructible sets. If $X = \bigcup_{i=1}^m (S_i \setminus P_i)$ is a good construction then write $\dim X = \dim \mathrm{cl}(X) = \max\{\dim S_i : i \leq m\}$.

Note, moreover, that if X is constructible of dimension d then we can find a good construction $\bigcup_{i=1}^{m} S_i \setminus P_i$ of X in which $d = \dim S_1 > \dim S_2 > \ldots \dim S_m$. This is a consequence of the identity

$$\bigcup_{i=1}^{m}(S_i \setminus P_i) = ((\bigcup_{i=1}^{m} S_i) \setminus (\bigcup_{i=1}^{m} P_i)) \cup \bigcup_{1\leq i\neq j\leq m} ((P_j \cap S_i) \setminus P_i)$$

applied recursively, with induction on d.

Lemma 2.4 (QZ$^-$) *A countable union of constructible subsets of M^n is constructible.*

Proof. Suppose $X = \bigcup_{i\in\omega} X_i$, with each X_i constructible. We will proceed by induction on $\max\{\dim X_i : i \in \omega\}$.

The closed sets of dimension zero are exactly the countable sets, so if $\max\{\dim X_i : i \in \omega\} = 0$ then each X_i is closed and X is closed by axiom (CU); and hence X is constructible.

Otherwise, suppose that the lemma holds for countable unions of constructible sets all of dimension less than $\max\{\dim X_i : i \in \omega\}$. We may find good constructions of each X_i and label them consecutively, so that

$$X = \bigcup_{i\in\omega} X_i = \bigcup_{i\in\omega}(S_i \setminus P_i)$$

with $\max\{\dim P_i : i \in \omega\} < \max\{\dim S_i : i \in \omega\} = \max\{\dim X_i : i \in \omega\}$. Now

$$\begin{aligned}\bigcup_{i\in\omega}(S_i \setminus P_i) &= (\bigcup_{i\in\omega} S_i) \setminus \bigcup_{i\in\omega}\Big(P_i \setminus \bigcup_{j\neq i}(S_j \setminus P_j)\Big) \\ &= (\bigcup_{i\in\omega} S_i) \setminus \bigcup_{i\in\omega}\Big(P_i \setminus \bigcup_{j\neq i}((S_j \cap P_i) \setminus (P_j \cap P_i))\Big)\end{aligned}$$

and for each i it is clear that $\bigcup_{j\neq i}((S_j \cap P_i) \setminus (P_j \cap P_i))$ is a countable union to which we may apply the inductive hypothesis (as $\max\{\dim(P_i \cap S_j) : j \in \omega\} \leq \max\{\dim P_i : i \in \omega\} < \max\{\dim X_i : i \in \omega\}$).

Hence, by the inductive hypothesis again, $\bigcup_{i\in\omega}(P_i \setminus \bigcup_{j\neq i}(S_j \setminus P_j))$ is constructible; and we conclude that since $\bigcup_{i\in\omega} S_i$ is closed, X is constructible also.

□

Lemma 2.5 (QZ$^-$) *If X is closed then $\pi(X)$ is constructible.*

Proof. The proof is by induction on $\dim \mathrm{cl}(\pi(X))$. Suppose that whenever Y is closed and $\dim \mathrm{cl}(\pi(Y)) < \dim \mathrm{cl}(\pi(X))$, then $\pi(Y)$ is constructible. If $\dim \mathrm{cl}(\pi(X)) = 0$ then $\pi(X) = \mathrm{cl}(\pi(X))$, a countable set.

Otherwise, by axiom (CP) we know that there is a closed X' such that $\mathrm{cl}(\pi(X)) = \pi(X) \cup X'$, with $\dim X' < \dim \mathrm{cl}(\pi(X))$. We have

$$X' \cap \pi(X) = \pi(\pi^{-1}(X') \cap X)$$

and the set $(\pi^{-1}(X') \cap X)$ is closed in consequence of lemma 2.1. But its projection is a subset of X', and is therefore of lower dimension than $\mathrm{cl}(\pi(X))$.

So the inductive hypothesis, applied to $(\pi^{-1}(X') \cap X)$, tells us that $(X' \cap \pi(X))$ is constructible.

Now writing

$$\pi(X) = (\mathrm{cl}(\pi(X)) \setminus X') \cup (X' \cap \pi(X))$$

shows that $\pi(X)$ is constructible too. □

Corollary 2.6 (QZ$^-$) *The sets $\mathcal{P}^{n,m}(X,k)$ of axiom (FC) are constructible too.* □

Theorem 2.7 (QZ$^-$) *The projection of a constructible set is constructible.*

Proof. With a view to using lemma 2.4, suppose that P is a proper subset of an irreducible set S, and (inductively) that if X' is constructible and $\dim(X') < \dim(S)$ then $\pi(X')$ is constructible.

By lemma 2.5 we know $\pi(S)$ is constructible and we may write

$$\pi(S) = (A \setminus B) \cup Y$$

for some closed A, B and constructible Y, with $\dim(Y) < \dim(A)$ and $\dim(B) < \dim(A)$. Write $\tilde{S} = \pi^{-1}(\mathrm{cl}(Y)) \cap S$. Since S is irreducible, $\dim(\tilde{S}) < (\dim S)$.

Let $d = \min\{\dim(\pi^{-1}(\{a\}) \cap S) : a \in A \setminus B\}$. Then, by axiom (FC) and corollary 2.6, there is some closed $P' \subseteq P$ such that

$$\begin{aligned}\mathcal{P}(P, d-1) &= \{a : \dim(\pi^{-1}(\{a\}) \cap P) \geq d\} \\ &= \mathcal{P}(P', d-1) \\ &= \pi(P')\end{aligned}$$

and $\mathcal{P}(P, d-1)$ is constructible. Write $\mathrm{cl}(\mathcal{P}(P, d-1)) = F$. Evidently, $(\pi(S) \setminus F) \subseteq \pi(S \setminus P) \subseteq \pi(S)$.

Suppose $\dim(F) = \dim A$. Then by the additive formula (AF) applied to P',

$$\dim(P') \geq \dim(F) + d = \dim S$$

and since S is irreducible this means $P' = P = S$, so $\pi(S \setminus P) = 0$.

Otherwise, $\dim F < \dim A$ and, since S is irreducible, $\dim(S \cap \pi^{-1}(F)) < \dim S$. Now

$$\begin{aligned}\pi(S \setminus P) = (A \setminus (B \cup F)) \quad &\cup \quad \pi((S \cap \pi^{-1}(F)) \setminus (P \cap \pi^{-1}(F))) \\ &\cup \quad \pi(\tilde{S} \setminus (P \cap \tilde{S})).\end{aligned}$$

The inductive hypothesis applied to the second and third terms in this union shows that $\pi(S \setminus P)$ is constructible.

Now given any constructible set $X = \bigcup_{i=1}^{m} S_i \setminus P_i$ we can decompose each S_i into irreducibles; so lemma 2.4 completes the proof. $\square$

Theorem 2.8 *Let $\langle M, \mathcal{C}, \dim\rangle$ be a quasi-Zariski geometry. Suppose that L is a first-order or $L_{\omega_1\omega}$ language and that $\mathfrak{M}$ is an L-structure with domain M such that:*

(i) the extension in $\mathfrak{M}$ of every function symbol or predicate of L lies in $\mathcal{C}$; and

(ii) all $C \in \mathcal{C}$ are quantifier-free definable in L (without parameters).

Then $Th(\mathfrak{M})$ eliminates quantifiers.

Proof. This is a restatement of theorem 2.7. If L is an $L_{\omega_1\omega}$ language then it is enough in (ii) to check that every irreducible set is definable. □

So far we have in fact made no use of the restriction imposed by axiom (IM).

Corollary 2.9 *If M is a quasi-Zariski geometry in which M is irreducible and* $\dim(M) = 1$*, then M is a quasi-minimal structure.* □

A further consequence of axiom (IM) is the fact that the dimension function is determined uniquely by the rest of the structure $\langle M, \mathcal{C}\rangle$. If one wanted actually to do model theory in the (non-first-order) class of QZ structures this would be important, presumably.

Theorem 2.10 (QZ) *The dimension of the empty set is $-\infty$. For any nonempty closed set X,* $\dim X$ *is equal to the length of a maximal chain of nonempty irreducible proper subsets of X.*

Proof. As a proper subset of $\{a\}$ for any $a \in M^n$, $\dim \emptyset < 0$ so $\dim \emptyset = -\infty$.

Let $\emptyset = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_k \subsetneq X$ be a maximal chain of irreducible proper subsets of X for X a closed subset of M^m. Clearly axiom (DI) implies that $k \leq \dim X$.

For the converse, it is sufficient to prove by induction on m the following statement: if $X \subseteq M^m$ is irreducible of dimension $n+1$, say, and if P is a proper closed subset of X, then there is a closed $Y \subseteq X$ such that $\dim Y = n$ and $Y \not\subseteq P$. (Note that if X is of zero dimension then its only proper subset is the empty set, since an irreducible zero-dimensional set must be a singleton, by axioms (DP) and (DI).)

So suppose $X \subseteq M^{m+1}$ and the statement holds for all irreducible subsets of M^m. Let $\pi : M^{m+1} \to M^m$ denote the natural projection, as usual. Consider first the case when $\dim \mathrm{cl}(\pi(X)) < \dim X$. Then by the additive formula the minimal dimension of a fibre of this projection is 1; and hence $\pi^{-1}(a) \cap X = \{a\} \times M$ for each $a \in \pi(X)$. Thus the set $Y = X \cap (M^m \times \{b\})$ is a closed subset of X for each $b \in M$ and, by (AF) applied to a component of Y of maximal dimension, $\dim Y = \dim \mathrm{cl}(\pi(X))$. For at most countably many choices of b can such a Y be a subset of P (since $\dim P \leq \dim Y$). With any other choice of b, then, Y is as required.

Or in the case that $\dim \mathrm{cl}(\pi(X)) = \dim X$, write the good construction of $\mathrm{cl}(X) = \bigcup S_i \setminus P_i$. We may assume that S_1 is irreducible and of dimension $\dim X$. Then $P' := (\bigcup P_i \cup \mathrm{cl}(\mathcal{P}(X, 1)))$ is closed and of dimension strictly less than that of S_1; so by the inductive hypothesis there is a closed subset Y' of S_1 not contained in P' of dimension $\dim X - 1$. Then $Y = \pi^{-1}(Y') \cap X$ so Y is also of dimension $\dim X - 1$, as required. □

2.2 Examples of quasi-Zariski geometries

Definition. We say that a first order structure $\mathfrak{M}$ *induces* the quasi-Zariski geometry $\langle M, \mathcal{C}, \dim \rangle$ if M is the domain of $\mathfrak{M}$, the graphs of all relations and functions of $\mathfrak{M}$ lie in $\mathcal{C}$, and every irreducible set of $\mathcal{C}$ is a component of a closed set definable in $\mathfrak{M}$.

Example 2.11 *Any essentially uncountable Zariski geometry induces a quasi-Zariski geometry.*

A (complete) Zariski geometry is a first-order structure M with a distinguished collection $\mathcal{Z} = \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n$ of its definable sets (the closed sets) and a dimension function $\dim : \mathcal{Z} \to \mathbb{N} \cup \{-\infty\}$, satisfying the following axioms 1–9:

1 (zL)

(i) The graphs of relation symbols and functions (including equality) are closed;

(ii) singletons $\{a\} \in M$ are closed;

(iii) M is closed;

(iv) finite intersections and unions of closed sets are closed;

(v) cartesian products of closed sets are closed.

2 (zP) The projection mappings $\pi : M^n \to M^{(n-1)}$ are continuous and closed.

3 (zDCC) Descending chain condition for closed subsets.

4 (zDU) Dimension of unions: $\dim(X_1 \cup X_2) = \max\{\dim X_1, \dim X_2\}$.

5 (zDI) Dimension of irreducible sets: if X is irreducible and X' is a proper closed subset of X then $\dim(X') < \dim(X)$.

In this context an irreducible set is a closed set which cannot be written as a *finite* union of proper subsets. That any closed subset can be written as a finite union of irreducibles follows from (zDCC) and (zL).

6 (zDP) Dimension of a point is 0.

7 (zFC) Fibre condition: the set

$$\mathcal{P}^{n,m}(X,k) = \{a \in M^n : \dim(X \cap \pi^{-1}(a)) > k\}$$

(where $\pi : M^{n+m} \to M^n$) is closed for any $X \in \mathcal{Z}_{n+m}$.

8 (zAF) Additive formula: if X is irreducible, then $\dim(X) = \dim(\pi(X)) + \min\{\dim(\pi^{-1}(a) \cap X) : a \in \pi(X)\}$.

A Zariski geometry is said to be *essentially uncountable* if in addition it satisfies the following condition (EU).

9 (EU) If a closed $X \in \mathcal{Z}_n$ is a union of countably many closed subsets, then there are finitely many among the subsets, the union of which is X.

To an essentially uncountable Zariski geometry $\langle M, \mathcal{Z}, \dim\rangle$, we can associate a structure $\langle M, \mathcal{C}, \dim\rangle$ where

$$\mathcal{C}_n = \{\bigcup_{i\in\omega} Z(i) : Z : \omega \to \mathcal{Z}_n\}$$

and $\dim(\bigcup_{i\in\omega} Z_i) = \max\{\dim(Z_i) : i \in \omega\}$. It is necessary to verify that this is a well-defined map.

Lemma 2.12 *An irreducible set in the sense of $\mathcal{C}$ is exactly an irreducible set in the sense of $\mathcal{Z}$.*

Proof. Suppose $X \in \mathcal{C}$ is irreducible in the sense of $\mathcal{Z}$, but that $X = \bigcup_{i \in \omega} X_i$ with each $X_i \in \mathcal{C}$.

Then $X_i = \bigcup_{j \in \omega} Z_{ij}$ for some $Z_{ij} \in \mathcal{Z}$, by construction of $\mathcal{C}$; so $X = \bigcup_{(i,j) \in \omega^2} Z_{ij}$. Now by (EU) for $\mathcal{Z}$, there are finitely many of the Z_{ij} such that X is their union. And irreducibility of X in the sense of $\mathcal{Z}$ means that for some choice of i and j, $X = Z_{ij}$. But then as $Z_{ij} \subseteq X_i \subseteq X$, $X = X_i$; so X is irreducible in the sense of $\mathcal{C}$ as required.

Conversely, any set irreducible in the sense of $\mathcal{C}$ must evidently be a member of $\mathcal{Z}$; and irreducibility over $\mathcal{Z}$ follows *a fortiori*. □

It is straightforward to check that $\mathcal{C}$ satisfies axiom (L), in particular that the intersection of two elements of $\mathcal{C}$ is again in $\mathcal{C}$. (IC) is a consequence of proposition 2.12 and the existence of irreducibles in $\mathcal{Z}$. (CU) is immediate from the construction.

Given these axioms (L) and (IC) we may prove lemma 2.2 for $\mathcal{C}$. So our purported definition of dim for $\mathcal{C}$ does indeed give a well-defined function. For suppose $X \in \mathcal{C}$. Then we may write $X = \bigcup_{i \in \omega} X_i$ with each X_i irreducible (and so in $\mathcal{Z}$). Let $d = \max\{\dim(X_i) : i \in \omega\}$. If we are given any other representation of X as a countable union from $\mathcal{Z}$, say $X = \bigcup_{j \in \omega} Z_j$, then each Z_j is a finite union of the X_i and all the X_i are contained in some Z_j. So, by (zDU),

$$\max\{\dim(X_i) : i \in \omega\} = \max\{\dim(Z_j) : j \in \omega\}$$

and the dimension of X is given unambiguously by d.

Now (DU), (DI) and (DP) follow immediately. (CP) is true because a stronger result holds: the projection π is a closed map in $\mathcal{Z}$ (we are in a compact structure). Projection and union commute; so π is closed in $\mathcal{C}$.

For (FC), suppose that $k \in \mathbb{N}$ and $X \in \mathcal{C}$ are given. Writing once more $X = \bigcup_{j\in\omega} Z_j$, we have (by (DU) for $\mathcal{C}$)

$$\begin{aligned}\mathcal{P}^{n,m}(X,k) &= \{a \in M^n : \dim(X \cap \pi^{-1}(a)) > k\} \\ &= \{a \in M^n : \exists j \in \omega \ \dim(Z_j \cap \pi^{-1}(a)) > k\} \\ &= \bigcup_{j\in\omega} \mathcal{P}^{n,m}(Z_j, k).\end{aligned}$$

So, as $\mathcal{P}^{n,m}(Z_j, k)$ is closed for each j by axiom (zFC), $\mathcal{P}^{n,m}(X,k)$ is closed too. Since π is continuous, $\pi^{-1}(\mathcal{P}^{n,m}(X,k)) \cap X$ is then a closed subset of X whose projection is $\mathcal{P}^{n,m}(X,k)$, as required.

Finally, (AF) is no more than (zAF) by proposition 2.12, and the proof is complete.

Corollary 2.13 *A compact complex manifold M in the language with predicates for all countable unions of analytic subsets of finite powers of M is a quasi-Zariski geometry.*

(The projective line over) an uncountable algebraically closed field in the $L_{\omega_1\omega}$ language of rings is a quasi-Zariski geometry.

We have thus proved by an alternative and much less informative route the result of Wilkie, that $\mathbb{C}_{\mathbb{Z}}$ is quasi-minimal, mentioned in the introduction.

Note that in the above example 2.11 we do not essentially use the completeness condition, and could relax axioms (zP) and (zFC) to specify that the projections of closed sets and of their high-dimensional fibres are merely constructible. In the context of Zariski geometries this makes no difference because, as a consequence of the trichotomy theorem, models of these relaxed axioms all have a completion.

The condition (EU) occurs naturally in this context. It is equivalent for Zariski geometries to ω_1-compactness. In Zilber's notes [30] it appears explicitly in this form in the proof that the elementary extension of a Z-structure is a Z-structure. Other presentations of the material (for example in [5]) define Zariski geometries so that they implicitly satisfy related conditions on all models of their theory.

Chapter 3

$\mathcal{R}$-holomorphic functions

Let $\mathcal{R}$ be a class of (entire) holomorphic functions on the finite powers of $\mathbb{C}$ satisfying:

(R1) if $f_1, f_2 \in \mathcal{R}$ and for some m $f_1 : \mathbb{C}^m \to \mathbb{C}^1, f_2 : \mathbb{C}^m \to \mathbb{C}^1$, then $f_1 + f_2$,$f_1 \cdot f_2$, and $-f_1 \in \mathcal{R}$ too.

(R2) for each m the ring of polynomials on m variables $\mathbb{C}[z_1, \ldots, z_m] \subseteq \mathcal{R}$;

(R3) $\mathcal{R}$ is closed under partial derivatives;

(R4) if $f_1 : \mathbb{C}^m \to \mathbb{C}^{k_1}, f_2 : \mathbb{C}^m \to \mathbb{C}^{k_2} \in \mathcal{R}$, then so is $(f_1, f_2) : \mathbb{C}^m \to \mathbb{C}^{k_1+k_2}$;

(R5) if $\lambda : \mathbb{C}^n \to \mathbb{C}^m$ is an affine map and $f : \mathbb{C}^m \to \mathbb{C}^k \in \mathcal{R}$, then $f \circ \lambda \in \mathcal{R}$;

(R6) if $\mu : \mathbb{C}^k \to \mathbb{C}^n$ is an affine map and $f : \mathbb{C}^m \to \mathbb{C}^k \in \mathcal{R}$, then $\mu \circ f \in \mathcal{R}$.

For example, the tuples of terms of the language L_{exp} with parameters from $\mathbb{C}$ form one such class $\mathcal{R}$. We wish to perform analytic geometry over $\mathcal{R}$, in the sense of obtaining a local description of a variety V defined by functions from $\mathcal{R}$, and of its singular set V^*. To this end we define a sheaf of rings $\mathcal{O}_U^{\mathcal{R}}$ extending $\mathcal{R}$ in which Weierstrass preparation can be performed; then a large part of chapters I–IV of Łojasiewicz [8] can be carried through with functions drawn only from these local rings.

The property of Weierstrass coefficients of analytic functions that is useful to us was observed by van den Dries in [18]. He obtains in that paper their *strong definability* from the set of partial derivatives of the original functions in an analytic expansion of the real field. (A set $X \subseteq A^n$ is strongly definable in an L-structure $\mathfrak{A}$ if it is existentially defined over $\mathfrak{A}$ in such a manner that there is exactly one witness to the formula for each $\bar{x} \in X$.) The apparatus of strong definability makes essential use of the definable ordering on $\mathbb{R}$, which is no good for our purposes: namely looking for analytic examples of QZ structures in a language which necessarily has no definable ordering on any uncountable subset of the domain $\mathbb{C}$. Instead we must be ready to make a non-definable selection of zeros of a function F at which the Jacobian of F is non-singular. If we were to equip the structure with predicates for suitable neighbourhoods of such zeros, the implicit function theorem tells us that these functions would be existentially definable. So Weierstrass coefficients can be thought of as being "locally existentially defined" over $\mathcal{R}$. Where van den Dries is able to define functions' Weierstrass coefficients in the original language, we will have to add a sheaf of new, locally defined functions.

This construction may be read as an exercise in the programme proposed by Peterzil and Starchenko in [16], of extending Whyburn's work [21] on topological, integration-free complex analysis into the theory of several variables.

In this and subsequent chapters, by "closed" or "open" sets I will mean the usual topology of ε-neighbourhoods on $\mathbb{C}^m$. For notational convenience, to keep the number of subscripts under control, I write in this chapter variables referring to a point in several-dimensional space both as x and as x_i; for the jth coordinate of x I write $\pi_j(x)$. A function f is holomorphic at the point $a \in \mathbb{C}^m$ if there is an open subset U of $\mathbb{C}^m$ containing a such that $f \restriction U$ is defined and holomorphic. I occasionally make tacit use of the uniqueness of analytic extension to identify holomorphic functions equal on the intersection of their domains. The germ of a holomorphic function f or analytic set X at a point a I write f_a or X_a respectively.

3.1 Definition and elementary properties

Recall the implicit function theorem from complex analytic geometry:

Proposition 3.1 (Implicit function theorem, Łojasiewicz, [8] C.1.13) *Let X, Y, Z be finite-dimensional complex vector spaces with $\dim Y = \dim Z$. Let $F(x, y)$ be a holomorphic mapping on a neighbourhood of a point $(a, b) \in X \times Y$ with values in Z, such that $F(a, b) = 0$ and the differential $\partial F/\partial y(a, b)$ is an isomorphism. Then we have equality of the germs of sets*

$$\{F(x, y) = 0\}_{(a,b)} = \{y = \phi(x)\}_{(a,b)}$$

for some holomorphic mapping ϕ defined on a neighbourhood of the point a with values in Y. Furthermore,

$$F(x, \phi(x)) = 0 \textit{ in a neighbourhood of } a,\ \phi(a) = b,$$

and the germ of a continuous function satisfying these conditions is uniquely determined.

We shall write the set of points at which F satisfies the hypotheses of the implicit function theorem as $\mathrm{Ift}(F)$, so that if F is defined on the open set $U \subseteq X \times Y$ then

$$\mathrm{Ift}(F) := \{(a, b) \in U : F(a, b) = 0 \text{ and } \det(\partial F/\partial y)(a, b) \neq 0\}.$$

Definition. Let $U \subseteq \mathbb{C}^m$ be a connected open set and $f : U \to \mathbb{C}$ be a holomorphic function on U. For any subset $X \subseteq U$ then an *implicit representation* of f on X is a triple $\langle k, F, \tilde{f}\rangle$ where $k \in \mathbb{N}$, $F(x, y) : U \times \mathbb{C}^k \to \mathbb{C}^k$ is the restriction of a member of $\mathcal{R}$, and $\tilde{f} : X \to \mathbb{C}^k$ is a function holomorphic at each point $a \in X$, such that for each $a \in X$,

$$F(a, \tilde{f}(a)) = 0 \text{ and } \det(\frac{\partial F}{\partial y}(a, \tilde{f}(a))) \neq 0 \ (\textit{i.e.}, (a, \tilde{f}(a)) \in \mathrm{Ift}(F)),$$

and $f(a) = \pi_1(\tilde{f}(a))$, where $\pi_1 : \mathbb{C}^k \to \mathbb{C}$ is the first coordinate function.

We define the *ring of $\mathcal{R}$-holomorphic functions on U*, $\mathcal{O}_U^{\mathcal{R}}$, as follows: $f \in \mathcal{O}_U^{\mathcal{R}}$ if f is holomorphic on U and there is a partition $U = X_1 \cup \cdots \cup X_t$ of U into disjoint sets, and on each X_s an implicit representation $\langle k_s, F_s(x, y_s), \tilde{f}_s \rangle$ of f with the property that $\tilde{f}_s$ can be extended *continuously* to $\mathrm{cl}(X_s) \cap U$. (Recall that we require the defining functions F_s each to be a member of the class $\mathcal{R}$).

If $a \in \mathbb{C}^m$ we define the ring of germs of $\mathcal{R}$-holomorphic functions at a, $\mathcal{O}_a^{\mathcal{R}}$, in the obvious way (as a quotient of the union over all U containing a of $\mathcal{O}_U^{\mathcal{R}}$, identifying functions equal on a neighbourhood of a). Then if $0 \in \mathbb{C}^m$ is the origin, we have $\mathcal{O}_a^{\mathcal{R}} \cong \mathcal{O}_0^{\mathcal{R}}$; we write this latter ring $\mathcal{O}_m^{\mathcal{R}}$.

So $\mathcal{O}_U^{\mathcal{R}}$ contains the coordinate functions of those $\tilde{f}$ defined implicitly on U by functions from $\mathcal{R}$, and we also allow a finite disjunction of these implicit definitions to cover all of U. Note that selection of a coordinate function is an example of projection, and that the condition that $\tilde{f}$ be continuous on the closure in U of the set on which it represents f ensures that this projection is locally a proper map. The projection and disjunction commute.

The following technical lemma shows the sense in which we may find a member of $\mathcal{O}_U^{\mathcal{R}}$ by "patching together" a disjunction of members of $\mathcal{O}_V^{\mathcal{R}}$ for $V \subseteq U$. The non-trivial condition to check is that of continuous extension to the boundary.

Lemma 3.2 *Let $f : U \to \mathbb{C}$ be holomorphic on U and suppose that we are given a partition $U = X_1 \cup \cdots \cup X_r$ and open sets $U_1, \ldots U_r$ with $X_i \subseteq U_i$ for each $i \leq r$. Suppose further that for each i there is $f_i \in \mathcal{O}_{U_i}^{\mathcal{R}}$ such that $f_i \restriction X_i = f \restriction X_i$. Let each U_i be partitioned into sets $X_{i,s}$ so that f_i has implicit representation $\langle k_{i,s}, F_{i,s}, \tilde{f}_{i,s} \rangle$ on $X_{i,s}$. If each $\tilde{f}_{i,s}$ can be extended continuously to $\mathrm{cl}(X_{i,s} \cap X_i) \cap U$, then $f \in \mathcal{O}_U^{\mathcal{R}}$. In particular this is the case if each $k_{i,s} = 1$.*

Where the conditions on f_i for this lemma hold, we shall say (slightly loosely) that the implicit representations of f_i extend to $\mathrm{cl}(X_i) \cap U$.

Proof. If $\langle k_{i,s}, F_{i,s}, \tilde{f}_{i,s} \rangle$ is an implicit representation of f_i on $X_{i,s}$ then the same triple $\langle k_{i,s}, F_{i,s}, \tilde{f}_{i,s} \rangle$ is an implicit representation of f on $X_{i,s} \cap X_i$.

When $k_{i,s} = 1$ then $\tilde{f}_{i,s} = f$ on $X_{i,s} \cap X_i$. But f extends continuously (indeed holomorphically) to $\mathrm{cl}(X_{i,s} \cap X_i) \cap U$, as we require. □

The X_i featuring in our definition of $\mathcal{O}_U^{\mathcal{R}}$ do not need to have any particular structure, and will not in general be analytically constructible subsets of U. Locally, however, they can be taken to be definable in the o-minimal structure $\mathbb{R}_{an}$.

Lemma 3.3 *Let $U \subseteq \mathbb{C}^m$ be bounded and definable in $\mathbb{R}_{an}$ as a subset of $\mathbb{R}^{2m}$. Let $U = X_1 \cup \cdots \cup X_t$, and let $f \in \mathcal{O}_U^{\mathcal{R}}$ have implicit representation $\langle k_s, F_s, \tilde{f}_s \rangle$ on X_s for each $s = 1, \ldots, t$. Suppose each $\tilde{f}_s$ has continuous extension to $\mathrm{cl}(X_s)$. Then there is a partition $U = Y_1 \cup \cdots \cup Y_r$ such that each Y_i is definable in $\mathbb{R}_{an}$ and there is an implicit representation of f on Y_i.*

Proof. For each s, $\tilde{f}_s$ extends continuously to the compact set $\mathrm{cl}(X_s)$: let $\rho = \max\{|\tilde{f}_s(x)| : x \in \mathrm{cl}(X_s)\}$. Without loss of generality (after a rescaling) we may assume that $U \subseteq \{x : |x| < 1/2\}$ and that $\rho < 1/2$. Then there are function symbols in L_{an} for the real and imaginary parts of the coordinates of F_s, and we have

$$\{(a, \tilde{f}(a)) : a \in X_s\} \subseteq \big\{(a,b) \in U \times \{|y| \leq \rho\} : F_s(a,b) = 0 \quad \text{and } \det(\frac{\partial F_s}{\partial y})(a,b) \neq 0\big\},$$

an $\mathbb{R}_{an}$-definable set which we shall denote A.

In the o-minimal structure $\mathbb{R}_{an}$, A admits a cell decomposition, $A = C_1 \cup \cdots \cup C_{r_s}$. Each cell C is the graph of a continuous function from $\pi(C) \subseteq U$ to $\mathbb{C}^{k_s}$; and by the implicit function theorem this is the restriction of a holomorphic function $\tilde{g}$ on some open set $V \supseteq \pi(C)$. If $\{(a, \tilde{f}(a)) : a \in X^s\} \cap C \neq 0$, then $\tilde{g}$ is the unique holomorphic extension of $\tilde{f}$ to V.

In particular, $\langle k_s, F_s, \tilde{g} \rangle$ is an implicit representation of f on $\pi(C)$, and $\tilde{g}$ has continuous extension to $\mathrm{cl}(\pi(C)) \subseteq \mathrm{cl}(U)$. As s varies between 1 and t, by selecting all such $\pi(C)$ we will obtain a cover of U, $U = C_1 \cup \cdots \cup C_r$. We set $Y_i = C_i \setminus (\bigcup\{C_j : j < i\})$, and take the implicit representation of f on Y_i to be

$\langle k_s, F_s, \tilde{g}\rangle$ as found for C_i. Alternatively we may break such Y_i down further into individual cells. □

Lemma 3.4 (Elementary properties)

1. *If $U \subseteq \mathbb{C}^m$ and $F \in \mathcal{R}$ with $F : \mathbb{C}^m \to \mathbb{C}^k$, then the restriction to U of each coordinate $\pi_j(F) \in \mathcal{O}^{\mathcal{R}}_U$.*
2. *If $f \in \mathcal{O}^{\mathcal{R}}_U$ and $\lambda : \mathbb{C}^k \to \mathbb{C}^m$ is affine, then $f \circ \lambda \in \mathcal{O}^{\mathcal{R}}_{\lambda^{-1}(U)}$.*
3. *Suppose $f \in \mathcal{O}^{\mathcal{R}}_V$ with $V \subseteq C^n$ and $g_1, \ldots, g_n \in \mathcal{O}^{\mathcal{R}}_U$. If we write $g = (g_1, \ldots, g_n) : U \to C^n$ and $U' = g^{-1}(V) \cap U$, then $f \circ g \in \mathcal{O}^{\mathcal{R}}_{U'}$.*
4. *(Implicit function theorem) Suppose $g : U \to V \subseteq \mathbb{C}^n$ is holomorphic and there are $f_1(x, y), \ldots, f_n(x, y) \in \mathcal{O}^{\mathcal{R}}_{U\times V}$ such that writing $f = (f_1, \ldots, f_n) : U \times V \to C^n$, $f(x, g(x)) = 0$ on U and $\partial f/\partial y(a, g(a))$ is an isomorphism for each $a \in U$. Then $g \in (\mathcal{O}^{\mathcal{R}}_U)^n$.*

Proof. 1. By property (R6) of $\mathcal{R}$, if $F(x) : \mathbb{C}^m \to \mathbb{C}^k \in \mathcal{R}$ then $\pi_j(F) \in \mathcal{R}$ also. Then $G(x, y) = y - \pi_j(F)(x) \in \mathcal{R}$ and $\partial G/\partial y = 1$ on $\mathbb{C}^{m+1}$. So $\pi_j(F)(x) \in \mathcal{O}^{\mathcal{R}}_{\mathbb{C}^m} \subseteq \mathcal{O}^{\mathcal{R}}_U$.

2. Suppose $f \in \mathcal{O}^{\mathcal{R}}_U$ and $\lambda : \mathbb{C}^k \to \mathbb{C}^m$ is affine; and that $a \in \lambda^{-1}(X_s)$. Then letting $G_s(x, y) = F_s(\lambda(x), y)$, we have $G_s \in \mathcal{R}$ and

$$G_s(a, \tilde{f}_s \circ \lambda(a)) = F(\lambda(a), \tilde{f}_s(\lambda(a))) = 0.$$

Moreover $\partial G/\partial y(a, \tilde{f}_s \circ \lambda(a)) = \partial F_s/\partial y(\lambda(a), \tilde{f}_s(\lambda(a)))$ so is non-singular, while $\mathrm{cl}(\lambda^{-1}(X_s)) \cap \lambda^{-1}(U) \subseteq \lambda^{-1}(\mathrm{cl}(X_s) \cap U)$.

Thus $f \circ \lambda = \pi_1(\tilde{f}_s \circ \lambda)$ has the implicit representation $\langle k_s, G_s, \tilde{f}_s \circ \lambda\rangle$ on $\lambda^{-1}(X_s)$.

The sets $\lambda^{-1}(X_s)$ cover $\lambda^{-1}(U)$. So $f \circ \lambda \in \mathcal{O}^{\mathcal{R}}_{\lambda^{-1}(U)}$ as required.

3. We may take $X^* \subseteq U'$ small enough that each g_i has implicit representation $\langle k_i, G_i(x, y_i), \tilde{g}_i\rangle$ on X^*, and that f has implicit representation $\langle k_0, F(x, z), \tilde{f}\rangle$ on $g(X^*)$. In particular, this means that $\tilde{f} \circ g$ is continuous on $\mathrm{cl}(X^*) \cap U$.

Then we may define $H : U \times \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n} \times \mathbb{C}^{k_0} \to \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n} \times \mathbb{C}^{k_0}$ by

$$H(x, y_1, \ldots, y_n, z) = \big(G(x, y_1), \ldots G(x, y_n), F(\pi_1(y_1), \ldots, \pi_1(y_n), z)\big).$$

Now certainly $H \in \mathcal{R}$, $H(a, \tilde{g}_1(a), \ldots, \tilde{g}_n(a), \tilde{f} \circ g(a)) = 0$, and we have the block-lower-triangular matrix representation

$$\frac{\partial H}{\partial(y_1, \ldots, y_n, z)} = \begin{pmatrix} \frac{\partial G_1}{\partial y_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\partial G_2}{\partial y_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial G_n}{\partial y_n} & 0 \\ * & * & \cdots & * & \frac{\partial F}{\partial z} \end{pmatrix}$$

which is nonsingular at $(a, \tilde{g}_1(a), \ldots, \tilde{g}_n(a), \tilde{f} \circ g(a))$ for $a \in X^*$, as each of the diagonal blocks is.

Thus (after composing H with a permutation μ to bring $\pi_1(z)$ to the fore, using property (R5) of $\mathcal{R}$) we have an implicit representation of $f \circ g$ on X^*. But finitely many such X^* cover U'; so we are done.

4. We may again take some $X^* \subseteq U \times V$ such that each f_i has implicit representation $\langle k_i, F_i(x, y, z_i), \tilde{f}_i(x, y)\rangle$ on X^*. Define $G \in \mathcal{R}$, $G : U \times V \times \mathbb{C}^{k_1} \times \ldots \mathbb{C}^{k_n} \to \mathbb{C}^{k_1} \times \ldots \mathbb{C}^{k_n} \times \mathbb{C}^n$ by

$$G(x, y, z_1, \ldots, z_n) = \big(F_1(x, y, z_1), \ldots, F_n(x, y, z_n), \pi_1(z_1), \ldots, \pi_1(z_n)\big).$$

Now certainly for each $(a, b) \in \mathrm{cl}(X^*) \cap U$ with $b = g(a)$ it is the case that

$$G(a, g(a), \tilde{f}_1(a, g(a)), \ldots, \tilde{f}_n(a, g(a))) = 0;$$

and we must check that $\partial G/\partial(y, z_1, \ldots, z_n)$ evaluated at this point is an isomorphism when $(a, g(a)) \in X^*$.

Let $E_i \in M_{n,k_i}(\mathbb{C})$ denote the $n \times k_i$ matrix with entry 1 in the ith row of the first column. Then $\partial/\partial z_i(\pi_1(z_1), \ldots, \pi_1(z_n)) = E_i$ and we have:

$$\frac{\partial G}{\partial(y, z_1, \ldots, z_n)} = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & 0 & \cdots & 0 & \frac{\partial F_1}{\partial y} \\ 0 & \frac{\partial F_2}{\partial z_2} & & 0 & \frac{\partial F_2}{\partial y} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial F_n}{\partial z_n} & \frac{\partial F_n}{\partial y} \\ E_1 & E_2 & \cdots & E_n & 0 \end{pmatrix}.$$

Now by the chain rule applied to the implicit representation of f_i on X^* we have that the differential

$$\partial\tilde{f}_i/\partial y(a, g(a)) = -((\partial F_i/\partial z_i)^{-1} \circ \partial F_i/\partial y)(a, g(a), \tilde{f}_i(a, g(a))).$$

So premultiplying $\partial G/\partial(y, z_1, \ldots, z_n)$ by the non-singular matrix, Δ say, represented by its diagonal blocks

$$\Delta = \text{Diagonal}\big((\frac{\partial F_1}{\partial z_1})^{-1}, \ldots, (\frac{\partial F_n}{\partial z_n})^{-1}, \text{Id}_n, \big),$$

(where Id_n is the identity in $M_n(\mathbb{C})$) we have that

$$\Delta\frac{\partial G}{\partial(y, z_1, \ldots, z_n)} = \begin{pmatrix} \text{Id}_{k_1} & 0 & \cdots & 0 & -\frac{\partial \tilde{f}_1}{\partial y} \\ 0 & \text{Id}_{k_2} & & 0 & -\frac{\partial \tilde{f}_2}{\partial y} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \text{Id}_{k_n} & -\frac{\partial \tilde{f}_n}{\partial y} \\ E_1 & E_2 & \cdots & E_n & 0 \end{pmatrix}.$$

Finally we may annul the submatrices E_i of this matrix with row operations, subtracting the row corresponding to $\pi_1(F_i)$ from that corresponding to z_i as coordinates of G. But this transcribes a copy of $\partial f/\partial y$ in the last diagonal block of $\Delta\partial G/\partial(y, z_1, \ldots, z_n)$, and recalling the hypothesis that $\partial f/\partial y$ is an isomorphism at $(a, g(a))$ we are done.

So $\partial G/\partial(y, z_1, \ldots, z_n)$ is indeed an isomorphism at the required point for each $(a, g(a)) \in X^*$ and (using property (R5) again) we have an implicit representation of $\pi_j(g)$ on X^* for each $j \leq n$. And finitely many such X^* cover the graph of g. So $\pi_j(g) \in \mathcal{O}_U^{\mathcal{R}}$ as required. □

Corollary 3.5 *For any U, $\mathcal{O}_U^{\mathcal{R}}$ is a ring. The units of $\mathcal{O}_U^{\mathcal{R}}$ are exactly those $f \in U$ which do not take value zero on U.*

Proof. The functions $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ are both in $\mathcal{O}_{\mathbb{C}^2}^{\mathcal{R}}$ by lemma 3.4 (1) since $\mathbb{C}[x, y] \subseteq \mathcal{R}$, while $(x \mapsto -x) \in \mathcal{O}_{\mathbb{C}}^{\mathcal{R}}$. Then if $g_1, g_2 \in \mathcal{O}_U^{\mathcal{R}}$, so are $g_1 + g_2$, $g_1 \cdot g_2$, and $-g_1$ by 3.4 (2). So $\mathcal{O}_U^{\mathcal{R}}$ is a ring.

Moreover, suppose $f(x) \in \mathcal{O}_U^{\mathcal{R}}$ and $f(U) \subseteq \mathbb{C} \setminus \{0\}$; and let $\phi(x) = 1/f(x)$ be its multiplicative inverse. Then defining

$$h : (x, y) \mapsto f(x)y - 1$$

gives $h \in \mathcal{O}_{U\times\mathbb{C}}^{\mathcal{R}}$ such that $h(a, \phi(a)) = 0$ for each $a \in U$, and $\partial h/\partial y(a, \phi(a)) = f(a) \neq 0$ for each $a \in U$; so by the implicit function theorem, lemma 3.4 (4), $\phi \in \mathcal{O}_U^{\mathcal{R}}$.

More generally, if $g(x) \in \mathcal{O}_U^{\mathcal{R}}$ as well, then defining $h' : (x, y) \mapsto f(x)y - g(x)$ shows directly in the same manner that $g/f \in \mathcal{O}_U^{\mathcal{R}}$. □

As a subring of the ring of analytic functions on U, $\mathcal{O}_U^{\mathcal{R}}$ is an integral domain.

The following observation is useful in conjunction with lemma 3.2.

Corollary 3.6 *Let $U \subseteq X \subseteq \mathrm{cl}(U)$ and suppose that, in addition to the conditions of lemma 3.4 (4), the functions $f_1, \ldots, f_n$ have implicit representations which can be extended continuously to $X \times V$, and g can be extended continuously to X. Then the implicit representations of g found in that lemma extend continuously to X too.* □

Definition. Where the conditions of corollary 3.6 hold, and $h(x) = \pi_1(g)(x)$ for all $x \in W \subseteq V$, we shall say that $\langle n, f, g\rangle$ is *an implicit representation of h on W over $\mathcal{O}_{U\times V}^{\mathcal{R}}$*. This vocabulary will be useful in the proof of theorem 4.14, but we shall not need it before then.

3.2 Weierstrass preparation

The preceding lemma 3.4 demonstrates for $\mathcal{O}_U^{\mathcal{R}}$ the elementary properties needed for a proof of Weierstrass preparation analogous to van den Dries' argument in [18], to which we now turn our attention.

Recall that a function $f(x, y)$ holomorphic in a neighbourhood of zero in $\mathbb{C}^m \times \mathbb{C}$ is called *regular* in y if $f(0, b) \neq 0$ for some b in any neighbourhood of zero in $\mathbb{C}$; *i.e.,* if $(f(0, y))_0$ is not the zero germ. In this case there is some $d \in \mathbb{N}$ for which $f(0, y) = c(y)y^d$ in a neighbourhood of zero, where $c(y)$ does not take value 0 on this neighbourhood. Then f is called *regular of order d*.

Suppose that $0 \in U \subseteq \mathbb{C}^{m+1}$ and that $f(x, y)$ is a function holomorphic on U and regular in y of order d. Then there exist open neighbourhoods D of the origin on which we can identify "these" d zeros. Namely, we can find $0 \in D \subseteq U$ such that (if $\pi : \mathbb{C}^{m+1} \to \mathbb{C}^m$ and $D_1 = \pi(D)$):

1. D is a bounded set definable in the o-minimal structure $\mathbb{R}_{an}$ as a subset of $\mathbb{R}^{2m+2}$,
2. D is simply connected and each fibre $D(a) = (\pi^{-1}(a) \cap D)$ is simply connected,
3. $f(0, b) \neq 0$ for $0 \neq b \in D(0)$,
4. f is holomorphic at (a, b) for $a \in D_1$, $b \in \mathrm{cl}(D(a))$ and has continuous extension to $\mathrm{cl}(D)$,
5. $f(a, b) \neq 0$ for $a \in D_1$, $b \in (\mathrm{cl}(D(a)) \setminus D(a))$, and moreover
6. $f(a, y)$ has exactly d zeros $y_1(a), \ldots, y_d(a)$ (counted with their multiplicities) on $D(a)$.

Definition. We call a set D satisfying these conditions 1–6 a *preparation domain.*

Given any such f there is a preparation domain D which is a polydisk $D_1 \times D_2$, as found in the classical proof of the Weierstrass preparation theorem for analytic functions (see van den Dries [18] 3.3 or Łojasiewicz [8] C.2.4). The existence of such a polydisk (indeed, of arbitrarily small such polydisks) is assured by Rouché's theorem. We will, though, need the statement of the preparation theorem for this slightly more general D to allow us to decompose sets into finitely many such domains in the spirit of lemma 3.2. Requiring that D be definable in $\mathbb{R}_{an}$ will let us exploit the cell decomposition theorem for such sets.

If D is a preparation domain and $C \subseteq D_1$ is open and simply connected with $0 \in C$, then $D \cap \pi^{-1}(C)$ is also a preparation domain. If f is a distinguished polynomial then $D_1 \times \mathbb{C}$ is a preparation domain.

Given any set $D \subseteq U$ and $(a, b) \in D$, if $\{(x, y) \in \mathbb{C}^{n+1} : (a + x, b + y) \in D\}$ is a preparation domain for $f(x - a, y - b)$ then we will say D is a *preparation domain for f at (a, b)*. In this case D is also a preparation domain for f at every $(a', b') \in D$ at which f is regular of the same order as at (a, b).

Lemma 3.7 *Suppose that $f(x, y) \in \mathcal{O}_U^{\mathcal{R}}$ is regular in y of order d and let D be a preparation domain for f. Let $g(x, y) \in \mathcal{O}_D^{\mathcal{R}}$; and suppose that $\partial^k g/\partial y^k$ and $\partial^l f/\partial y^l \in \mathcal{O}_D^{\mathcal{R}}$ for each $k, l \leq d$. If there is h holomorphic on D with $g = hf$, then $h \in \mathcal{O}_D^{\mathcal{R}}$.*

Proof. On the open set $V_0 = D \cap \{(x, y) : f(x, y) \neq 0\}$ we have $h = g/f$ and, by corollary 3.5, $h \in \mathcal{O}_{V_0}^{\mathcal{R}}$. Moreover, the conditions of the observation 3.6 hold for this h on $D \subseteq \mathrm{cl}(V_0)$, so the implicit representation of h on V_0 extends continuously to D.

Elsewhere (that is, on the set of (a, b) such that $f(a, b) = 0$), we invoke L'Hôpital's rule: if

$$f(a, b) = \frac{\partial f}{\partial y} f(a, b) = \cdots = \frac{\partial^{k-1} f}{\partial y^{k-1}}(a, b) = 0, \frac{\partial^k f}{\partial y^k}(a, b) \neq 0, \tag{3.1}$$

then

$$h(a, b) = \frac{\partial^k g}{\partial y^k}(a, b) / \frac{\partial^k f}{\partial y^k}(a, b).$$

Given $k \leq d$, call the set of points $(a, b) \in D$ for which condition (3.1) holds V_k. For each k such that V_k is nonempty, there is an open subset U_k

of D, with $V_k \subseteq U_k$, on which $(\partial^k g/\partial y^k(a,b))/(\partial^k f/\partial y^k(a,b))$ is holomorphic and hence in $\mathcal{O}^{\mathcal{R}}_{U_k}$. Again we may appeal to corollary 3.6, extending this function $(\partial^k g/\partial y^k)/(\partial^k f/\partial y^k)$ and its implicit representations continuously to cover $(\mathrm{cl}(V_k) \cap D) \cup U_k$.

We know by the hypotheses on D that all such k are no greater than d; in particular that there are only finitely many such V_k. And we have checked the continuity conditions for our representation of h on the closure of each V_k in U. So we may appeal to lemma 3.2 and show that $h \in \mathcal{O}^{\mathcal{R}}_D$, as required. □

Corollary 3.8 *If, in addition, for some $j \in \mathbb{N}$ we have $\partial^k g/\partial y^k$ and $\partial^l f/\partial y^l \in \mathcal{O}^{\mathcal{R}}_D$ for each $k, l \leq d + j$, then $\partial^j h/\partial y^j \in \mathcal{O}^{\mathcal{R}}_D$ as well.*

Proof. Differentiating the identity $(g = hf)$ with respect to y repeated l times gives:

$$\frac{\partial^l g}{\partial y^l} = \sum_{i=0}^{l} \binom{l}{i} \frac{\partial^{l-i} h}{\partial y^{l-i}} \frac{\partial^i f}{\partial y^i}.$$

In particular, if $(a, b) \in V_k$, we may take $l = k + 1$ and deduce that

$$\frac{\partial^{k+1} g}{\partial y^{k+1}}(a,b) = (k+1)\frac{\partial h}{\partial y}(a,b)\frac{\partial^k f}{\partial y^k}(a,b) + h(a,b)\frac{\partial^{k+1} f}{\partial y^{k+1}}(a,b),$$

and hence (recalling that we have already proved $h \in \mathcal{O}^{\mathcal{R}}_D$)

$$\frac{\partial h}{\partial y}(a,b) = \left(\frac{\partial^{k+1} g}{\partial y^{k+1}}(a,b) - h(a,b)\frac{\partial^{k+1} f}{\partial y^{k+1}}(a,b)\right) / (k+1)\frac{\partial^k f}{\partial y^k}(a,b).$$

As before, this expression lets us find a function in $\mathcal{O}^{\mathcal{R}}_{U_k}$ equal to $\partial h/\partial y$ on V_k. Once again the finite collection of sets $V_0, \ldots, V_d$ cover D and the implicit representations extend as required. We conclude that $\partial h/\partial y \in \mathcal{O}^{\mathcal{R}}_D$. The same argument, taking $l = k + j$, will complete the proof by induction on j. □

Proposition 3.9 (Weierstrass division theorem, Łojasiewicz C.2.5) *Let $f(x, y) : D \to \mathbb{C}$ be regular of order d, and let D be a preparation domain for f. If $g(x, y)$ is any holomorphic function on D then we have*

$$g(x,y) = Q(x,y)f(x,y) + R_{d-1}(x)y^{d-1} + \cdots + R_0(x)$$

where Q is holomorphic on D and each R_i is holomorphic on D_1. Moreover Q and $R = R_{d-1}(x)y^{d-1} + \cdots + R_0(x)$ are uniquely determined by the functions f and g.

Theorem 3.10 *Suppose that in addition to the hypotheses of Proposition 3.9, $f(x,y) \in \mathcal{O}_D^{\mathcal{R}}$, $g(x,y) \in \mathcal{O}_D^{\mathcal{R}}$, and the partials $\partial^k g/\partial y^k$ and $\partial^l f/\partial y^l \in \mathcal{O}_D^{\mathcal{R}}$ for each $k, l \leq d$. Then $Q \in \mathcal{O}_D^{\mathcal{R}}$ and $R \in \mathcal{O}_{D_1}^{\mathcal{R}}[y]$.*

Proof. We have chosen D so that for each $a \in D_1$ there are exactly d zeros $y_1, \ldots, y_d$ of the function $f(a, y)$ in D_2, counted with multiplicity. The first part of the proof is to find sets on which we can make a holomorphic selection of these roots by appeal to the implicit function theorem.

Let S be the set of tuples of positive numbers summing to d, with re-orderings considered equivalent (alternatively, with each tuple in decreasing order), so

$$S = \{(\underbrace{1, \ldots, 1}_{d \text{ times}}), (2, \underbrace{1, \ldots, 1}_{d-1 \text{ times}}), \ldots, (d)\}.$$

For each $s \in S$ let V_s be the set of points $a \in D_1$ for which the coincidence of the roots of $f(a, y)$ matches s: if s is a k-tuple,

$$V_s = \big\{x \in D_1 : \exists y_1, \ldots, y_k \in D(x)\big(\bigwedge_{1 \leq i < j \leq k} (y_i \neq \bar{y}_j)$$
$$\wedge (0 = f(x, y_1) = f'(x, y_1) = \cdots = f^{(s_1)}(x, y_1) \wedge \ldots \wedge (0 = f(x, y_k) = \cdots = f^{(}s_k)(x, y_k))\big)\big\}.$$

These sets V_s are a partition of D_1. Each V_s is an analytically constructible subset of D_1 (being the proper projection of an analytic set). The closure in D_1 of each V_s is a union of members of the partition, all of which (apart from V_s itself) are of lower analytic dimension.

Now it follows that each V_s admits a cover by finitely many sets $X_1, \ldots, X_r$ open in V_s, simply connected, and with simply connected closure in D_1.

Indeed, these sets V_s are definable in the o-minimal structure $\mathbb{R}_{an}$ (as subsets of $\mathbb{R}^{2m}$). If C is a connected component of V_s then C is of constant dimension $2k \leq 2m$ in $\mathbb{R}_{an}$; so there is a cover $C = C_1 \cup \ldots C_l$ such that each C_j is open in C and definably homeomorphic to an open subset of $\mathbb{R}^{2k}$. If $\pi_1, \dots, \pi_{\binom{2m}{2k}}$ are the projections of $\mathbb{R}^{2m}$ onto its $2k$-dimensional subspaces generated by the canonical coordinates, then the C_j may be taken to be the components of the definable sets $\{x \in C : \dim(C \cap \pi_{j'}^{-1}(\pi_{j'}(x))) = 0\}$ as j' varies. Then the C_j are homeomorphic to their projections, which are *bounded* open subsets of $\mathbb{R}^{2k}$.

The following proposition is due to Wilkie [22].

Proposition 3.11 *Let $\mathcal{M} = \langle M, +, \cdot, <, 0, 1, \dots \rangle$ be an o-minimal expansion of a real closed field and let U be a definable, bounded, open subset of M^n. Then there exists a finite collection of open cells in M^n whose union is U.*

Appealing to this proposition we may cover the projections $\pi_{j'}(C_j)$ with open cells, whose homeomorphic inverse images in C under $\pi_{j'}$ are then simply connected with simply connected closure in D_1, and open in V_s as required. Label these sets $X_1, \dots, X_r$.

Suppose $a \in X_j$ and y_i is a zero of $f(a, y)$ with multiplicity k. Then the implicit function theorem applied to $\partial^k f / \partial y^k$ at the point (a, y_i) gives a germ of a holomorphic function $y_i(x)$ at a; and this can be extended holomorphically to a (simply connected) open set $U_j \supseteq X_j$ and continuously on the union of U_j with the closure of X_j in D_1. We may take U_j small enough that if y_i and $y_{i'}$ are distinct on X_j, they are distinct on all of U_j too.

Thus $y_i(x)$ so found is a function in $\mathcal{O}^{\mathcal{R}}_{U_j}$, by lemma 3.4 (4), to which the additional conditions of lemma 3.2 apply on the set $(\mathrm{cl}(X_j) \cap D_1) \cup U_j$. It is a holomorphic selection of a root (indeed, of k roots) of f on X_j as we required.

Now we find implicit representations of $R_0, \dots, R_{d-1}$ using the holomorphic selection of $y_1, \dots, y_d$. The representation depends on the partition s; I write out just two cases.

Suppose for example that $s = (1, \dots, 1)$ is the partition into d points, and that X is a member of the cover of V_s found above. So $y_1(x), \dots, y_d(x) \in \mathcal{O}^{\mathcal{R}}_U$ are holomorphic selections of the zeros of f on X; for each $x \in U$ they are all distinct.

Then we may use the identity

$$g = Qf + (R_0 + R_1 y + \cdots + R_{d-1}y^{d-1}) \tag{3.2}$$

at each (x, y_i) to establish the corresponding row of the relation

$$\begin{pmatrix} 1 & y_1 & \cdots & y_1^{d-1} \\ \vdots & & & \vdots \\ 1 & y_d & \cdots & y_d^{d-1} \end{pmatrix} \begin{pmatrix} R_0(x) \\ \vdots \\ R_{d-1}(x) \end{pmatrix} = \begin{pmatrix} g(x, y_1(x)) \\ \vdots \\ g(x, y_d(x)) \end{pmatrix}.$$

Now the Vandermonde matrix is non-singular on the (open) set $U \supseteq X$ and its determinant is therefore a unit in $\mathcal{O}_U^{\mathcal{R}}$ by corollary 3.5; while the coefficients of the adjugate matrix are in $\mathcal{O}_U^{\mathcal{R}}$ too. So we may invert the matrix and show that $R_0, \ldots, R_{d-1} \in \mathcal{O}_U^{\mathcal{R}}$.

What is more, since the implicit representation of each y_i extends continuously to $\mathrm{cl}(X) \cap D_1$, and the coefficients of the inverse matrix are rational functions of these y_i, observation 3.6 applied to the last part of 3.5 shows that the implicit representation of $R_0, \ldots, R_{d-1}$ extends continuously to $\mathrm{cl}(X) \cap D_1$ too.

Finitely many such X cover V_s, and this case is complete. As an interim result, an appeal to lemma 3.2 shows that $R_0, \ldots, R_{d-1} \in \mathcal{O}_{V_s}^{\mathcal{R}}$; and the implicit representations extend to $\mathrm{cl}(V_s) \cap D_1$.

Consider now the case $s = (2, 1, \ldots, 1)$ where for $x \in X \subseteq V_s$ we have $y_1(x) = y_2(x)$ and all the other $y_i(x)$ are distinct. Differentiating equation 3.2 we get

$$\frac{\partial g}{\partial y} = Q \cdot \frac{\partial f}{\partial y} + \frac{\partial Q}{\partial y} f + (R_1 + 2R_2 y + \cdots + (d-1)R_{d-1}y^{d-2}), \tag{3.3}$$

and may obtain the following relation:

$$\begin{pmatrix} 1 & y_1 & & \cdots & y_1^{d-1} \\ 0 & 1 & 2y_2 & \cdots & (d-1)y_2^{d-2} \\ 1 & y_3 & & \cdots & y_3^{d-1} \\ \vdots & & & & \vdots \\ 1 & y_d & & \cdots & y_d^{d-1} \end{pmatrix} \begin{pmatrix} R_0(x) \\ R_1(x) \\ R_2(x) \\ \vdots \\ R_{d-1}(x) \end{pmatrix} = \begin{pmatrix} g(x, y_1(x)) \\ (\partial g/\partial y)(x, y_2(x)) \\ g(x, y_3(x)) \\ \vdots \\ g(x, y_d(x)) \end{pmatrix}.$$

Now again this matrix is nonsingular on an open set U containing X and can be inverted within $\mathcal{O}_U^{\mathcal{R}}$, giving us holomorphic functions $R'_0, \ldots, R'_{d-1} \in \mathcal{O}_U^{\mathcal{R}}$ with $R'_i(x) = R_i(x)$ on X. Again the implicit representations extend continuously to $\mathrm{cl}(X) \cap D_1$, and again finitely many such X cover V_s.

The other possible partitions in S are treated similarly, taking further derivatives of equation 3.3 to construct the appropriate nonsingular matrix in each case.

Since D is covered by the V_s, and we have checked all the conditions of continuous extension throughout, we may appeal again to lemma 3.2. This completes the proof that $R_0, \ldots, R_{d-1} \in \mathcal{O}_{D_1}^{\mathcal{R}}$.

Now writing

$$g - (R_0 + R_1 y + \cdots + R_{d-1} y^{d-1}) = Qf$$

we may take $g := g - (R_0 + R_1 y + \cdots + R_{d-1} y^{d-1})$ and $h := Q$ in lemma 3.7 to complete the proof. □

For the following lemma and henceforward it is convenient to revert to the usual notation for the coordinates of $x \in \mathbb{C}^m$, writing $x = (x_1, \ldots, x_m)$. In particular $\partial/\partial x_i$ is the partial derivative with respect to x_i.

Lemma 3.12 *Suppose that in addition to the hypotheses of theorem 3.10,* all *the partial derivatives*

$$\left\{ \frac{\partial^{|\nu|}\partial^j}{\partial x^\nu \partial y^j} f, \frac{\partial^{|\nu|}\partial^j}{\partial x^\nu \partial y^j} g : \nu \in \mathbb{N}^m, j \in \mathbb{N} \right\}$$

lie in $\mathcal{O}_D^{\mathcal{R}}$. Then each of the partial derivatives

$$\frac{\partial^{|\nu|} R_0}{\partial x^\nu}, \ldots, \frac{\partial^{|\nu|} R_{d-1}}{\partial x^\nu} \in \mathcal{O}_{D_1}^{\mathcal{R}} \ \textit{and}\ \frac{\partial^{|\nu|}\partial^j}{\partial x^\nu \partial y^j} Q \in \mathcal{O}_D^{\mathcal{R}}.$$

Proof. To show that $\partial^j Q/\partial y^j \in \mathcal{O}_D^{\mathcal{R}}$ under these hypotheses it is enough to take $g := g - (R_0 + R_1 y + \cdots + R_{d-1} y^{d-1})$ and $h := Q$ in corollary 3.8.

For the rest, we will proceed by induction on ν.

As in the preceding few proofs, an example case is enough to show the method. Differentiating the identity

$$g = Qf + (R_0 + R_1 y + \cdots + R_{d-1} y^{d-1})$$

with respect to x_1 we get the identity

$$\frac{\partial g}{\partial x_1} = Q\frac{\partial f}{\partial x_1} + \frac{\partial Q}{\partial x_1} \cdot f + (\frac{\partial R_0}{\partial x_1} + \frac{\partial R_1}{\partial x_1} y + \cdots + \frac{\partial R_{d-1}}{\partial x_1} y^{d-1}) \tag{3.4}$$

holding on D, so

$$\left(\frac{\partial g}{\partial x_1} - Q\frac{\partial f}{\partial x_1}\right) = \frac{\partial Q}{\partial x_1} \cdot f + (\frac{\partial R_0}{\partial x_1} + \frac{\partial R_1}{\partial x_1} y + \cdots + \frac{\partial R_{d-1}}{\partial x_1} y^{d-1})$$

which is the statement of Weierstrass division for the functions f and $(\partial g/\partial x_1 - Q\partial f/\partial x_1)$.

All the hypotheses of theorem 3.10 apply, since inductively we know $Q \in \mathcal{O}_D^{\mathcal{R}}$, while f is still regular in y with the desired properties for D; and we deduce that

$$\frac{\partial R_0}{\partial x_1}, \ldots, \frac{\partial R_{d-1}}{\partial x_1} \in \mathcal{O}_{D_1}^{\mathcal{R}} \text{ and } \frac{\partial Q}{\partial x_1} \in \mathcal{O}_D^{\mathcal{R}}.$$

The same argument applied inductively completes the proof. $\square$

Definition. By $\widehat{\mathcal{O}_U^{\mathcal{R}}}$ we shall denote the set of those $f \in \mathcal{O}_U^{\mathcal{R}}$ whose implicit representations extend to $\mathrm{cl}(U)$ and all of whose partial derivatives $\partial^{|\nu|} f/\partial x^\nu$ are also in $\mathcal{O}_U^{\mathcal{R}}$ and also have implicit representations extending to $\mathrm{cl}(U)$.

So in this notation lemma 3.12 can be expressed: if $f, g \in \widehat{\mathcal{O}_D^{\mathcal{R}}}$, with D a preparation domain for the function f which is regular of order d, then $g = Qf + R_{d-1}y^{d-1} + \cdots + R_0$ with $Q \in \widehat{\mathcal{O}_D^{\mathcal{R}}}$ and each $R_i \in \widehat{\mathcal{O}_{D_1}^{\mathcal{R}}}$.

It is immediate that all the elementary properties of $\mathcal{O}_U^{\mathcal{R}}$ listed in lemma 3.4 hold when relativised to $\widehat{\mathcal{O}_U^{\mathcal{R}}}$.

Let $\widehat{\mathcal{O}_a^{\mathcal{R}}}$ denote the subring of $\mathcal{O}_a^{\mathcal{R}}$ containing the germs of functions in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$ for some U with $a \in U$. In the next chapter we shall show that in fact $\widehat{\mathcal{O}_a^{\mathcal{R}}} = \mathcal{O}_a^{\mathcal{R}}$. This will complete the proof that $\mathcal{O}_a^{\mathcal{R}}$ admits Weierstrass preparation.

Chapter 4

Partial derivatives and analytic continuation

It is evident from the basic lemmas that functions in $\mathcal{O}_U^{\mathcal{R}}$ allow implicit representation of their partial derivatives almost everywhere in U; for if $\langle k, F(x,y), \tilde{f}\rangle$ is an implicit representation of f on X, then by the chain rule we have

$$\frac{\partial \tilde{f}}{\partial x}(a) = -\Big(\frac{\partial F}{\partial y}(a, \tilde{f}(a))\Big)^{-1}\Big(\frac{\partial F}{\partial x}(a, \tilde{f}(a))\Big) \tag{4.1}$$

everywhere in the interior of X (taken relative to the ambient space U), and we have already seen that expressions of this form give us an implicit representation in $\mathcal{R}$.

In general, however, there may be $a \in U$ which do not lie in the interior of any X in the partition of U. Representation of the derivative is thus an example of the problem of analytic continuation: if $V \subseteq U \subseteq \mathrm{cl}(V)$ and $f \in \mathcal{O}_V^{\mathcal{R}}$, when is $f \in \mathcal{O}_U^{\mathcal{R}}$?

If $a \in \mathrm{cl}(X) \cap U$ and $\langle k, F, \tilde{f}\rangle$ represents f on X with the continuous extension $\tilde{f}(a) = b$, and $\det(\partial F/\partial y)(a,b) \neq 0$, then $\tilde{f}$ extends holomorphically to a neighbourhood of a. So provided that the implicit representations of f extend continuously to U (and f itself extends holomorphically), the problem reduces to finding implicit representations which cover the zero set of the Jacobian determinant $J(F) = \det(\partial F/\partial y)$.

We decompose this set $Z(J(F))$ into pieces A_d according to the orders of the zeros of the F_i (with respect to some vector for which the orders are defined) and exploit Weierstrass preparation in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$ to find implicit representations of f on each A_d.

Applying this result, not to the representations of equation 4.1 directly but to the functions

$$g_i(x,t) = \frac{f(x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_m) - f(x)}{t}$$

which have continuous extension to $t = 0$, will give us the representations that we need to show $\partial f/\partial x_i \in \mathcal{O}_a^{\mathcal{R}}$.

The most significant tool required is the existence of finitely many preparation domains covering the zero set of a regular holomorphic function.

Lemma 4.1 *Let $U \subseteq \mathbb{C}^{m+1}$ be a bounded $\mathbb{R}_{an}$-definable open set and let f be an $\mathbb{R}_{an}$-definable function holomorphic on U and on the closure of each fibre of U with respect to the canonical projection. If $d \in \mathbb{N} \setminus \{0\}$ let A_d denote the set*

$$\{(x,y) \in U : f \text{ is regular at } (x,y) \text{ of order } d\}.$$

Then there are finitely many sets $D_1,\ldots,D_r$ such that $A_d \subseteq D_1 \cup \cdots \cup D_r$ and every D_j is a preparation domain for f (at some point (a_j,b_j) of A_d).

Proof. Working in the o-minimal structure $\mathbb{R}_{an}$, we can take a C^1 cell decomposition of the definable set $A_d = C_1 \cup \ldots C_r \subseteq \mathbb{R}^{2m+2}$. Let $\pi : \mathbb{R}^{2m+2} \to \mathbb{R}^{2m}$ be the canonical projection. Then each fibre of A_d with respect to π consists only of isolated points; so each cell C_j is the graph of a definable function g_j of class C^1 whose domain $\pi(C_j)$ is a cell in $\mathbb{R}^{2m}$.

Moreover if $\pi(C_j)$ is not itself open in $\mathbb{C}^m$ then there is a definable open set $V_j \supseteq \pi(C_j)$, $V_j \subseteq \pi(U)$, onto which the function g_j can be extended definably, continuously, and with continuous derivative.

Now the mapping $(x,y) \mapsto f(x, y+g_j(x))$ is of class C^1, and for each $a \in V_j$ $f(a, y+g_j(a))$ is holomorphic as a function of y. It has a zero of order d at

$y = 0$ if $a \in \pi(C_j)$, in which case we may find $\varepsilon_0(a) > 0$ such that

$$\varepsilon_0(a) = \min\{|y| : (y \neq 0 \wedge f(a, y + g_j(a)) = 0) \vee (a, y + g_j(a)) \notin U\}.$$

Suppose that $a' \in V_j$ and that there is a sequence $\{a_i : i \in \omega\} \subseteq \pi(C_j)$, converging to a', such that $\varepsilon_0(a_i)$ converges to zero. For sufficiently large i, the disc $\{(a_i, y + g_j(a_i)) : |y| = \varepsilon_0(a_i)\}$ is contained in the open set U, so there must exist b'_i such that $|b'_i| = \varepsilon_0(a_i)$ and $f(a_i, b'_i + g_j(a_i)) = 0$. Then $(a', g_j(a'))$ is a limit of points $(a_i, g_j(a_i))$ in A_d but also of points $(a_i, b'_i + g_j(a_i))$ which are zeros of f not in C_j. So $f(a', y)$ has a zero of order strictly greater than d at $y = g_j(a')$, or is not regular in y. So, as $C_j \subseteq A_d$, $(a', g_j(a')) \notin C_j$.

We can therefore find an $\mathbb{R}_{an}$-definable *continuous* function $\varepsilon : \pi(C_j) \to \mathbb{R}$ such that $0 < \varepsilon(a) < \varepsilon_0(a)$ for each $a \in \pi(C_j)$. (For example, for some n there is a definable homeomorphism $\sigma : \pi(C_j) \to (-1, 1)^n = \{t \in \mathbb{R}^n : \|t\| < 1\}$, the open unit box in $\mathbb{R}^n$ with l_∞ norm. We may take

$$\varepsilon(a) = \frac{1}{2} \min\{\varepsilon_0(\sigma^{-1}(t)) : \|t\| \leq \sigma(a)\}.$$

Then the argument above shows that $\varepsilon(a) > 0$ for each $a \in \pi(C_j)$.)

Now for each $a \in \pi(C_j)$ let

$$\delta_0(a) := \min\big\{\|x - a\| : \exists y \in \mathbb{C}(|y| = \varepsilon(a) \wedge \big(x, y + g_j(x)) \notin U \cap \pi^{-1}(V_j) \\ \vee |f(x, y + g_j(x)) - f(a, y + g_j(a))| \geq |f(a, y + g_j(a))|)\big)\big\}.$$

Then $\delta_0(a) > 0$ for each $a \in \pi(C_j)$; and moreover if the sequence $\{a_i : i \in \omega\}$ converges to a while $\delta_0(a_i)$ converges to zero, then $a \notin \pi(C_j)$. For suppose for a contradiction that (a_i) is a sequence converging to $a \in \pi(C_j)$ and that for some b_i with $|b_i| = \varepsilon(a)$ (so the b_i have some limit point b^*, say),

$$|f(a_i, b_i + g_j(a_i)) - f(a, b_i + g_j(a))| \geq |f(a, b_i + g_j(a))|.$$

Then by the mean value theorem, since $f(x, y + g_j(y))$ is continuously differentiable on $\pi^{-1}(V_j)$, there is a_i^* in the interval $[a_i, a] \subseteq \pi(U) \cap V_j$ such that

$$\|\frac{\partial f(x, b_i + g_j(x))}{\partial x}(a_i^*, b_i)\| \geq \frac{|f(a_i, b_i + g_j(a_i)) - f(a, b_i + g_j(a))|}{\|a_i - a\|} \\ \geq \frac{\min\{|f(a, y + g_j(a))| : |y| = \varepsilon(a)\}}{\|a_i - a\|} \neq 0.$$

Thus $\|\partial f(x, y+g_j(x))/\partial x(a_i^*, b_i)\|$ grows without bound as (a_i^*) converges to a, contradicting the hypothesis that $f(x, y+g_j(x))$ is of class C^1 at the point (a, b^*).

Again we may make a definable continuous selection $\delta : \pi(C_j) \to \mathbb{R}$ such that $0 < \delta(a) < \delta_0(a)$ for each $a \in \pi(C_j)$, requiring in addition that the set $\{a' \in \mathbb{C}^m : \exists a \in \pi(C_j)\, \|a' - a\| < \delta(a)\}$ be simply connected.

For such an a' we may apply Rouché's theorem to the two holomorphic functions of a single variable y, $f_1 := f(a', y + g_j(a')) - f(a, y + g_j(a))$ and $f_2 := f(a, y+g_j(a))$ on the circle $|y| = \varepsilon$. Since $|f_1| < |f_2|$ on this circle (by our choice of δ), we conclude that $f_1 + f_2$ has exactly as many zeros inside the circle as f_2 does, namely d. In other words, whenever $\|a' - a\| < \delta(a)$ the function $f(a', y + g_j(a'))$ has exactly d zeros (counted with their multiplicities) on the disc $|y| < \varepsilon(a)$ and does not take value zero on its boundary.

Now consider the set

$$D' = \big\{(x, y) \in \mathbb{C}^{m+1} : \exists a \in \pi(C_j)\, \big(\|x - a\| < \delta(a) \,\wedge\, |y| < \varepsilon(a)\big)\big\}.$$

Each fibre of D' is a union of open discs all centred on $y = 0$, so is itself a disc. Moreover by our selection of the function δ we have ensured that D' is simply connected.

It should now be evident that the set

$$D_j = \{(x, y + g_j(x)) : (x, y) \in D'\}$$

satisfies all the conditions to be a preparation domain for f at any $(a, g_j(a)) \in C_j$. □

It follows that if we are given n functions $f_1, \ldots, f_n$, holomorphic as above, and specified orders $d_1, \ldots, d_n$ then we can find finitely many sets $D_1, \ldots, D_r$ such that

1. every D_j is a preparation domain for *each* f_i; and

2. the set

$$A_{(d_1,\ldots,d_n)} := \{(x, y) \in U : f_i \text{ is regular at } (x, y) \text{ of order } d_i \text{ for each } i \leq n\}$$

is contained in the union $D_1 \cup \cdots \cup D_r$.

We shall also use the following straightforward application of Rouché's theorem for functions of several variables. Recall that $\mathrm{Ift}(f) = \{(x, y) : f(x, y) = 0 \wedge \det(\partial f/\partial y) \neq 0\}$.

Lemma 4.2 *For some open $U \subseteq \mathbb{C}^{m+k}$ let $f : U \to \mathbb{C}^k$ be holomorphic, and let $(a, b) \in \mathrm{Ift}(f)$. Let $\Omega \subseteq \mathbb{C}^n$ be a compact set containing the origin and let $\psi : \Omega \times \mathbb{C}^{m+k} \to \mathbb{C}^{m+k}$ be holomorphic and such that $\psi(0, x, y) : \mathbb{C}^{m+k} \to \mathbb{C}^{m+k}$ is the identity map. Define the family of maps $g_z : \psi^{-1}(\{z\} \times U) \to \mathbb{C}^k$, for $z \in \Omega$, by*

$$g_z : (x, y) \mapsto (f_1(\psi(z, x, y)), f_2(x, y), \ldots, f_k(x, y)).$$

Then there is an open neighbourhood $\Omega_0(a)$ of the origin in $\mathbb{C}^n$ such that for every $z \in \Omega_0(a)$, $a \in \pi_{m,k}(Z(g_z))$.

Proof. Fix a metric on $\mathbb{C}^k$ the codomain of f, and on $\mathbb{C}^n$ the parameter space for the family (g_z). Since f satisfies the hypotheses of the implicit function theorem at (a, b), there are neighbourhoods Γ of b and Δ of the origin in $\mathbb{C}^k$, such that $f(a, y)$ maps Γ bijectively onto Δ and the boundary $\partial\Gamma$ onto $\partial\Delta$. Let $\delta = \min\{\|c\| : c \in \partial\Delta\}$. Then by the uniform continuity of $f \circ \psi$ on the compact set $\Omega \times \{a\} \times \partial\Gamma$, we can find ε such that for all z in $\Omega_0(a) := \{\|z\| < \varepsilon\}$, and all $y \in \partial\Gamma$,

$$\|f_1(a, y) - f_1(\psi(z, a, y))\| = \|f_1(\psi(0, a, y)) - f_1(\psi(z, a, y))\| < \delta.$$

So for any fixed $\zeta \in \Omega_0(a)$, $\|f(a, y)\| > \|g(\zeta, a, y) - f(a, y)\|$ for all $y \in \partial\Gamma$, and we can apply Rouché's theorem to these functions of k variables to deduce that f and g_ζ have the same number of zeros in Γ, counted with multiplicity, namely 1. So $a \in \pi_{m,k}(Z(g_\zeta))$, as required. □

4.1 Regularity and good directions.

Definition. It is convenient to extend the definition of regularity of a holomorphic function on $U \subseteq \mathbb{C}^m$ to its affine transformations. Let $a = (a_1, \ldots, a_m) \in U$

be a point of the domain of $f : U \to \mathbb{C}$, and let $e \in (\mathbb{C}^m \setminus \{0\})$ be a non-zero vector. Then we shall say that f is *regular at a with respect to e* if there is a linear change of basis $\phi : \mathbb{C}^m \to \mathbb{C}^m$ such that $\phi : (0, \ldots, 0, 1) \mapsto e$ and the function $f'(x_1, \ldots, x_m) = f \circ \phi^{-1}(x_1 - a_1, \ldots, x_m - a_m)$ is regular in x_m.

We will exploit two results from the theory of o-minimal expansions of the real field.

Proposition 4.3 (Good Directions Lemma, van den Dries [19]) *Let $A \subseteq R^{n+1}$ be definable in an o-minimal expansion R of an ordered field, with* $\dim(A) \leq n$*; fix a metric $\| \cdot \|$. Call a unit vector $v \in R^{n+1}$ a* good direction *if for each $p \in R^{n+1}$, the set $\{t \in R : p + t \cdot v \in A\}$ is finite. Then the set of good directions is dense in $\{v \in R^{n+1} : \|v\| = 1\}$.*

A holomorphic function $f : U \to \mathbb{C}^k$ is regular at $a \in U \subseteq \mathbb{C}^m$ with respect to e if and only if for some neighbourhood V of a, e is a good direction for $Z(f) \cap V$ (as an $\mathbb{R}_{an}$-definable subset of $\mathbb{R}^{2m+2k}$). For if $(a + \mathbb{R} \cdot e) \cap Z(f \restriction V)$ is infinite, then $\{z \in \mathbb{C} : a + z \cdot e \in Z(f)\}_a$ is a 1-dimensional germ; while if f is regular at a with respect to e, f is regular with respect to e everywhere sufficiently close to a.

As a corollary, we have the following proposition which is used by Nishino in his proof of a theorem of Grauert (towards the existence of irreducible components of analytic sets); I refer to it in the proof of proposition 5.9 below. Nishino offers a direct proof.

Proposition 4.4 (Nishino, [13], Lemma 2.8) *Let f be a nonconstant holomorphic function on a domain $D \subseteq \mathbb{C}^m$, and let Δ be an open polydisk such that* $\mathrm{cl}(\Delta) \subseteq D$*. Let σ be the set of all points $e \in \Gamma$, the closed unit polydisk in $\mathbb{C}^m$, with the property: there exists $a \in \mathrm{cl}(\Delta) \cap Z(f)$ and a neighbourhood $V \subseteq D$ of a, such that $Z(f)$ contains the portion of the complex line L passing through a in direction e that lies in V. Then σ is a closed, nowhere dense subset of Γ.*

For closedness, observe that σ is a projection of the infinite intersection of the compact sets $\{(e, a) \in \Gamma \times \mathrm{cl}(\Delta) : \partial^n f / \partial e^n = 0\}$, for $n \in \mathbb{N}$. $\square$

Corollary 4.5 *Let $\{U_i : i \in \omega\}$ be a countable collection of open subsets of $\mathbb{C}^m$ and let $\{f_i : i \in \omega\}$ be a countable collection of non-zero functions, with f_i defined and holomorphic on U_i. Let $n \in \mathbb{N}$ and let V be any nonempty open subset of $\mathbb{C}^{n \cdot m}$. Then there exist $v_1, \ldots, v_n$ in $\mathbb{C}^m$ such that $(v_1, \ldots, v_n) \in V$ and each f_i is regular at* every *point a of U_i with respect to each of $v_1, \ldots, v_n$. In particular, suppose $k < m$ and $\{e_1, \ldots, e_k\}$ is an independent set of vectors in $\mathbb{C}^m$. Then we can complete this set to a basis $\{e_1, \ldots, e_m\}$ such that the f_i are regular everywhere on their respective domains with respect to each of $e_{k+1}, \ldots, e_m$.*

Proof. We can cover U_i with countably many polydisks $\{\Delta_{ij} : j \in \omega\}$ such that each $\mathrm{cl}(\Delta_{ij}) \subseteq U_i$. Then by proposition 4.4 the set σ_{ij}, of vectors $e \in \mathbb{C}^m$ such that at some $a \in \mathrm{cl}(\Delta_{ij})$ f_i is *not* regular with respect to e, is closed and nowhere dense. So the complement $\mathbb{C}^m \setminus \sigma_{ij}$ is open and dense in $\mathbb{C}^m$. Hence by Baire's theorem, the set

$$E = \bigcap_{i \in \omega} \bigcap_{j \in \omega} (\mathbb{C}^m \setminus \sigma_{ij})$$

is dense in $\mathbb{C}^m$. Therefore $E^n = E \times E \times \cdots \times E$ is dense in $\mathbb{C}^{n \cdot m}$. It thus has nonempty intersection with the open set V, as required.

In particular, the set

$$\{(v_{k+1}, \ldots, v_m) \in \mathbb{C}^{(m-k)m} : \mathrm{rank}(e_1, \ldots, e_k, v_{k+1}, \ldots, v_m) = m\}$$

is (Zariski) open and non-empty, and so has non-empty intersection with E^{m-k}. Any choice of $(e_{k+1}, \ldots, e_m)$ in this intersection completes the basis. $\square$

A second property of o-minimal structures is the following theorem of Miller. The structure $\mathbb{R}_{an}$ is polynomially bounded. If the $\mathbb{R}_{an}$-definable domain D is bounded, then all the functions f of $\mathcal{O}_D^{\mathcal{R}}$ that have continuous extension to $\mathrm{cl}(D)$ (and hence in particular all of $\widehat{\mathcal{O}_D^{\mathcal{R}}}$) are $\mathbb{R}_{an}$-definable.

Proposition 4.6 (Miller, [12]) *Assume that the expansion of the real field $\mathfrak{R}$ is polynomially bounded and o-minimal. Let $f : A \to \mathbb{R}$ be definable in $\mathfrak{R}$, with $A \subseteq \mathbb{R}^{m+n}$ ($m \geq 0$ and $n \geq 1$). Then there exists $N \in \mathbb{N}$ such that for all $(a, b) \in A$, if b is in the interior of A_a and $f(a, y)$ is N-flat at b (i.e., all partial derivatives of $f(a, y)$ of order less than or equal to N vanish at b), then $f(a, y)$ vanishes identically in a neighbourhood of y.*

In particular, if $f \in \mathcal{O}_D^{\mathcal{R}}$ for some bounded domain $D \subseteq \mathbb{C}^{m+1}$, f has continuous extension to $\mathrm{cl}(D)$, and $b \in D$, then (considered as an $\mathbb{R}_{an}$-definable mapping on $\mathbb{R}^{2m+2}$) f is N-flat at b if f is regular at b of order at least N or if f is not regular there.

Consequently the set

$$\{d \in \mathbb{N} : \exists a \in D\,(f_i \text{ is regular of order } d_i \text{ at } a)\}$$

is finite. We may contrast this with the theorem of Weierstrass, [17] 8.11, which asserts the existence of holomorphic functions with zeros of all orders.

Lemma 4.7 *Let U be a bounded open subset of $\mathbb{C}^m$, let $f \in \widehat{\mathcal{O}_U^{\mathcal{R}}}$ and let $\Omega \subseteq \mathbb{C}^m$ be a collection of vectors such that at every point a of U, f is regular with respect to some $e(a) \in \Omega$. Then there is a finite subset of Ω with the same property.*

Proof. For any finite subset S of Ω, let C_S denote the set

$$\{a \in U : f \text{ is not regular at } a \text{ with respect to any vector in } S\}.$$

Then C_S is the variety of finitely many functions in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$ and is therefore an analytic subset of U definable in $\mathbb{R}_{an}$. In particular, C_S has finitely many analytic components. If points $a_1, \ldots, a_n$ are chosen, one from each component, then for each a_i there is a vector $e_i \in \Omega$ such that f is regular at a_i with respect to e_i. Thus if $S' = S \cup \{e_1, \ldots, e_n\}$, the dimension of $C_{S'}$ is strictly less than that of C_S. An induction on this dimension completes the proof. $\square$

4.2 Operations on distinguished polynomials.

Recall that a *distinguished polynomial* in y is a polynomial $y^d + p_{d-1}(x)y^{d-1} + \cdots + p_0(x)$ whose leading coefficient is 1 and whose other coefficients $p_i(x)$ are holomorphic in some neighbourhood U of the origin of $\mathbb{C}^n$, each taking value 0 at the origin.

In this section we shall first discuss the Euclidean algorithm as applied to distinguished polynomials. If each coordinate of f is distinguished, then the algorithm preserves $\mathrm{Ift}(f)$. Secondly, we consider functions averaging the zeros of an implicit polynomial, with a view to replacing the polynomial with linear functions. Using the more complicated machinery of the next section this can also be done so as to preserve $\mathrm{Ift}(f)$.

Lemma 4.8 *Let f and g be distinguished polynomials with coefficients in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$, with $\deg(f) \geq \deg(g)$. Then the unique $Q, R : U \times \mathbb{C} \to \mathbb{C}$ such that $f = gQ + R$ are such that Q is a distinguished polynomial of degree $\deg(f) - \deg(g)$; and, for any $(a, b) \in U \times \mathbb{C}$ at which f is regular of order $\deg(f)$ and g is regular of order $\deg(g)$, Q is regular of order $\deg(f) - \deg(g)$ at (a, b), while R is not regular there. Both Q and R are in $\widehat{\mathcal{O}_{U\times\mathbb{C}}^{\mathcal{R}}}$.*

Proof. The uniqueness of Q and R follows from the Weierstrass division theorem 3.9. The rest is long division in $\mathcal{O}_U^{\mathcal{R}}[y]$.

Let $\mathcal{I}$ denote the ideal of $\mathcal{O}_U^{\mathcal{R}}$ containing those functions taking value 0 at the origin. Then the distinguished polynomial g can be written

$$g(x, y) = y^d + g_{d-1}(x)y^{d-1} + \cdots + g_0(x)$$

where $d = \deg(g)$ and $g_0, \ldots, g_{d-1} \in \mathcal{I}$.

It is the case that if h is any function in $\mathcal{I}[y]$ then there are $q, r \in \mathcal{I}[y]$ with $\deg(r) < \deg(g)$ and $h = qg + r$, as is easily proved by induction on $\deg(h)$.

Now if the degree of the distinguished polynomial f is c, we have $f - y^{c-d}g \in \mathcal{I}[y]$. Thus $f - y^{c-d}g = qg + r$ for some $q, r \in \mathcal{I}[y]$. So (since $\deg(q) < (c - d)$) we have that $Q := y^{c-d} + q$, $R := r$ are of the required form, in that Q is regular of order $c - d$ while R, being an element of $\mathcal{I}[y]$, is not regular at 0.

Note moreover that if f is regular at order $\deg(f)$ at $(a,b) \in U \times \mathbb{C}$ and g is regular of order $\deg(g)$ there, then $f(x-a, y-b)$ and $g(x-a, y-b)$ are distinguished polynomials of degree c and d respectively, while

$$f(x-a, y-b) = Q(x-a, y-b)g(x-a, y-b) + R(x-a, y-b)$$

and the uniqueness of such a representation shows that $Q(x-a, y-b)$ must be regular of order $c-d$ and $R(x-a, y-b) \in \mathcal{I}[y]$. □

Lemma 4.9 *Let $U \subseteq \mathbb{C}^{m+k}$ be a preparation domain for the functions $f_1, \ldots, f_k \in \widehat{\mathcal{O}_U^{\mathcal{R}}}$, which are each regular at the origin with respect to a vector $e \in \mathbb{C}^{m+k}$, of orders $d_1, \ldots, d_k$ respectively. Let A_d denote the set*

$$\{a \in U : \forall i \leq k \, (f_i \text{ is regular at } a \text{ of order } d_i)\}.$$

If $d_1 > d_2$ then there is a function $f_1' \in \mathcal{O}_U^{\mathcal{R}}$ such that:

1. *U is a preparation domain for f_1', which is regular of order d_2 at the origin and at all $a \in A_d$;*
2. *if we write $f' = (f_1', f_2, \ldots, f_k)$, then $\mathrm{Ift}(f') = \mathrm{Ift}(f)$.*

Note that in stating and proving this lemma we need make no distinction between the cases of regularity in the variables y_i and x_i. This contrasts with the approach needed in lemma 4.11 below.

Proof. If f_1 or f_2 is not a distinguished polynomial then we may use the Weierstrass preparation theorem (theorem 3.9 applied to $f := f_i \circ \phi, g := y_k^{d_i}$, where ϕ is a linear bijection on $\mathbb{C}^{m+k}$, e_{y_k} is in the canonical basis and $\phi : e_{y_k} \mapsto e$) on the preparation domain U to find distinguished polynomials $f_i^* = h_i f_i$ associated to f_i by the units h_i of $\mathcal{O}_U^{\mathcal{R}}$ $(i = 1, 2)$. Then the hypotheses of the lemma will apply equally to $(f_1^*, f_2^*, f_3, \ldots, f_k)$. So it is without loss that we assume that they are both distinguished polynomials, of degree d_1, d_2 respectively.

Appealing to lemma 4.8 we find polynomials Q, R such that

$$f_1 = Q \cdot f_2 + R \tag{4.2}$$

and observe that if we take $f_1' := f_2 + R$ this certainly satisfies condition 1 above.

Moreover we have $\mathrm{Ift}(f) = \mathrm{Ift}(f')$. For clearly if $f_1(a) = f_2(a) = 0$, then $R(a) = f_1'(a) = 0$ too. And at such an a,

$$\frac{\partial f_1}{\partial y} = f_2 \frac{\partial_Q}{\partial y} + Q \frac{\partial f_2}{\partial y},$$

so

$$\begin{aligned} \operatorname{rank} \frac{\partial f}{\partial y} &= \operatorname{rank}\Big(\frac{\partial R}{\partial y}, \frac{\partial f_2}{\partial y}, \dots \frac{\partial f_k}{\partial y}\Big) \\ &= \operatorname{rank}\Big(\frac{\partial f_2}{\partial y} + \frac{\partial R}{\partial y}, \frac{\partial f_2}{\partial y}, \dots \frac{\partial f_k}{\partial y}\Big) = \operatorname{rank} \frac{\partial f'}{\partial y}. \end{aligned}$$

Thus $\mathrm{Ift}(f) \subseteq \mathrm{Ift}(f')$, the set of such points at which $\operatorname{rank}(\partial f'/\partial y) = k$. The converse also holds, since if $f_1'(a) = f_2(a) = 0$, then $f_1(a) = 0$. □

Definition. We define functions averaging the zeros of a distinguished polynomial.

Let $f(x,y) = u(x,y)(y^d + p_{d-1}(x)y^{d-1} + \cdots + p_0(x)) \in \widehat{\mathcal{O}_U^{\mathcal{R}}}$ be the Weierstrass preparation of a function f on $U \subseteq \mathbb{C}^{n+1}$, where U is an open set containing the origin. The functions $p_0, \dots, p_{d-1}$ we call the *symmetric functions of* f. Let A_d denote the set $\{a \in U : f \text{ is regular at } a \text{ of order } d\}$, as before, and observe that if $(\vec{a}, a_{n+1}) \in A_d$ and some $p_r(\vec{a}) = 0$, then $p_i(\vec{a}) = 0$ for each i and $a_{n+1} = 0$. Indeed, in this case the function $(\partial^r f/\partial y^r)(\vec{a}, y)$ has a zero at $y = 0$, but all its zeros must coincide with the d coincident zeros of $f(\vec{a}, y)$.

We can apply lemma 4.1 to the set $U \setminus Z(\prod_{r=0}^{d-1} p_r)$, and cover

$$A_d \setminus Z(\prod_{r=0}^{d-1} p_r) = A_d \setminus \{y = 0\}$$

with finitely many preparation domains $D_1, \dots, D_s \subseteq U \setminus Z(\prod_{r=0}^{d-1} p_r)$.

Then on any such preparation domain, and in particular on each $D_j, 1 \leq j \leq s$, we can associate to f the collection of d *averaging functions* $G_0(f), \dots, G_{d-1}(f) \in \widehat{\mathcal{O}_{D_j}^{\mathcal{R}}}$, given by

$$G_r(f) : (x,y) \mapsto y + \binom{d}{r}^{-\frac{1}{d-r+1}} p_r^{\frac{1}{d-r+1}}(x) \quad (r = 0, \dots, d-1) \tag{4.3}$$

where the function $p_r^{1/(d-r+1)}$ is a holomorphic selection of the root on the simply connected set $\pi(D_j)$. So $G_0(f)$ takes value zero only at (a choice of) the geometric mean of the zeros of f, $G_{d-1}(f)$ only at the arithmetic mean. (Note also that in fact $G_{d-1}(f) \in \widehat{\mathcal{O}_D^{\mathcal{R}}}$.) At every point b of $A_d \cap D_j$, each $G_r(f)(b) = 0$. Conversely, if $b \in D_j$ and $G_0(f)(b) = \cdots = G_{d-1}(f)(b) = 0$ then $b \in A_d$.

In a similar manner for each $r = 0, \ldots, d-1$ the zero set of the function $(x, y) \mapsto y + p_r(x)$ covers $A_d \cap \{y = 0\}$, and any simply connected neighbourhood D_0 of $\pi(U) \times \{0\}$ is a preparation domain for this function. Thus we have covered A_d with finitely many sets, each of which is the intersection of the zero sets of a collection of d functions; each of the functions, moreover, being linear in y.

Suppose now that $1 \leq j \leq s$ and $a \in A_d \cap D_j$ (or, with $j = 0$, that $a \in A_d \cap \{y = 0\}$) and in addition that f is not regular at a with respect to a vector $e' \in \mathbb{C}^{n+1}$. Necessarily e' is linearly independent of the basis vector e_{n+1} corresponding to the variable y, so the affine set $\Lambda = a + \mathbb{C}e' + \mathbb{C}e_{n+1}$ is a (complex 2-dimensional) plane. If none of the averaging functions $G_0(f), \ldots, G_{d-1}(f)$ (or, respectively, the symmetric functions $p_0, \ldots, p_{d-1}$) are regular at a with respect to e', then the line $(a + \mathbb{C}e') \cap D_j$ (respectively, $(a + \mathbb{C}e') \cap D_0$) is contained within A_d. So in particular this line contains all the zeros of $f \restriction \Lambda$. We exploit the contrapositive of this observation—unless all the zeros are on this line, some averaging function is regular—in the following lemma.

Definition. Suppose $U \subseteq \mathbb{C}^{m+k}$ and $f : U \to \mathbb{C}^k$ are given. Given a vector $e \in \mathbb{C}^m$, we say that a point $b \in Z(f)$ is *accessible along e (in* $\mathrm{Ift}(f)$*)* if for every open neighbourhood V of b, there is $\alpha \in \mathbb{C}$ and $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$, $(b + \alpha t e) \in \pi_{m,k}(\mathrm{Ift}(f) \cap V)$. Let $\mathrm{Acc}(b, f) \subseteq \mathbb{C}^m$ denote the set of all such e. If $W \subseteq Z(f)$ then write $\mathrm{Acc}(W, f) = \bigcap_{b \in W} \mathrm{Acc}(b, f)$.

Lemma 4.10 *Let $U \subseteq \mathbb{C}^{m+k}$ and let the coordinate functions $f_i, 1 \leq i \leq k$ of $f = (f_1, \ldots, f_k)$ be in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$. Suppose that f_1 is regular in x_m at some $a \in U$ of order d, and that $D \subseteq U$ is a preparation domain for f_1 at a with respect to x_m; but that $a \in W$, where W is any subset of D satisfying:*

1. *W is a subset of A_d,*

$$A_d := \{b \in D : f_1 \textit{ is regular in } x_m \textit{ at } b \textit{ of order } d\}$$

2. *for every point $b \in W$, f_1 is* not *regular at b with respect to any linear combination of the variables $y_1, \ldots, y_k$;*

3. *every point b of W is accessible along e_m, the vector corresponding to x_m in the canonical basis, in* $\mathrm{Ift}(f)$.

Then we can cover W with finitely many sets $X_i \subseteq A_d$, $0 < i \leq s$, where, for some open $U_i \supseteq X_i$, each X_i is contained in the zero set of a function in $\widehat{\mathcal{O}^{\mathcal{R}}_{U_i}}$ which is either an averaging function or a symmetric function of f_1, and which is regular at every point of U_i in some linear combination of $(y_1, \ldots, y_k)$.

Proof. Let the cover of A_d by preparation domains $D_0, D_1, \ldots, D_s$ be found by application of lemma 4.1 as in the above definition of the averaging functions; suppose $b \in W \cap D_j$, for some $0 \leq j \leq s$.

Consider the $k+1$-dimensional fibre $S = (\pi_{m-1,k+1})^{-1}(\pi_{m-1,k+1}(b))$ containing $b = (b_1, \ldots, b_{k+m})$, and its k-dimensional subspace $S_1 := (x_m = b_m)$. Certainly property (2) implies that f_1 takes value zero everywhere on $S_1 \cap D$; but it cannot do so everywhere with multiplicity d. For otherwise this accounts for all the zeros of f_1 on $S \cap D$, counted with their multiplicities, so $Z(f_1) \cap (S \cap D) \subseteq S_1$. Hence $\pi_{m,k}(Z(f_1) \cap D) \cap S = \{b\}$. But there is by hypothesis some real interval contained in $\pi_{m,k}(\mathrm{Ift}(f) \cap D) \cap S$; a contradiction.

Thus $A_d \cap S_1$ is a proper analytic subset of $S_1 \cap D$. So we may pick a line $b + \mathbb{C}e'$ passing through b and contained in S_1, such that only finitely many points of A_d lie on the line. So if $b + ze' \in D_j \setminus A_d$, there is *another* zero of f_1 on $b + ze' + \mathbb{C}e_m$ (where e_m is the canonical basis vector corresponding to x_m) and contained in the preparation domain D_j. Hence $\Lambda = b + \mathbb{C}e' + \mathbb{C}e_m$ is a plane as the observation above, and we conclude that one of the averaging functions $G_r(f_1) \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_j}}$ (or, if $j = 0$, one of the symmetric functions $p_r \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_0}}$) is regular at b with respect to e'. So the sets

$$\{z \in D_j : G_r(f_1) \text{ is regular in some linear combination of } y_1, \ldots, y_k \text{ at } z\}$$

(or the analogous sets where a symmetric function is regular, if $j = 0$) cover $A_d \cap D_j$ and are candidates, numbered sequentially as j, r vary, for the U_i of the lemma. □

4.3 A discussion of the resultant.

Recall that if $P(X) = p_c X^c + \cdots + p_0$ and $Q(X) = q_d X^d + \cdots + q_0$ are polynomials over some integral domain $\mathcal{I}$, of degrees c and d respectively, then their *resultant* is defined as the determinant of a matrix in $M_{c+d}(\mathcal{I})$,

$$\mathrm{Res}(P,Q) = \det \begin{pmatrix} p_c & p_{c-1} & \cdots & p_0 & 0 & & \cdots & 0 \\ 0 & p_c & p_{c-1} & \cdots & p_0 & & \cdots & 0 \\ \vdots & & & \ddots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & p_c & p_{c-1} & \cdots & p_0 \\ q_d & q_{d-1} & \cdots & & q_0 & 0 & \cdots & 0 \\ 0 & q_d & q_{d-1} & \cdots & & q_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & q_d & q_{d-1} & \cdots & & q_0 \end{pmatrix}.$$

This has the property that if $\mathcal{K}$ is an extension of $\mathcal{I}$ over which P and Q factorise completely, so $P(X) = p_c(X - \alpha_1)\ldots(X - \alpha_c)$ and $Q(X) = q_d(X - \beta_1)\ldots(X - \beta_d)$, then

$$\mathrm{Res}(P,Q) = {p_c}^c {q_d}^d \prod_{\substack{1 \le i \le c \\ 1 \le j \le d}} (\alpha_i - \beta_j). \tag{4.4}$$

In particular $\mathrm{Res}(P,Q) = 0$ if and only if P and Q have a root in common.

If $P(x,y)$ and $Q(x,y)$ are distinguished polynomials in $\widehat{\mathcal{O}_U^{\mathcal{R}}}[y] \subseteq \widehat{\mathcal{O}_{U\times\mathbb{C}}^{\mathcal{R}}}$, therefore, then $\mathrm{Res}(P,Q) \in \mathcal{O}_U^{\mathcal{R}}$ and $\mathrm{Res}(P,Q)(a) = 0$ at exactly those a for which $P(a,y)$ and $Q(a,y)$ have a common zero in $\mathbb{C}$. In particular, if $P(x,y)$ is distinguished in $\widehat{\mathcal{O}_U^{\mathcal{R}}}[y]$ of degree d, then $\partial P/\partial y \in \widehat{\mathcal{O}_U^{\mathcal{R}}}[y]$ is distinguished also (when multiplied by $\frac{1}{d}$), and hence the discriminant of f, defined to be

$$D(f) := -1^{d(d-1)/2}\,\mathrm{Res}(f, \frac{\partial f}{\partial y}),$$

is an element of $\widehat{\mathcal{O}_U^{\mathcal{R}}}$.

Suppose a is such that $P(a,y)$ and $Q(a,y)$ have exactly one common zero (counted with multiplicity), meaning that $P(a,b)=Q(a,b)=0$ for some b but $\partial P/\partial y(a,b)\neq 0$ and $\partial Q/\partial y(a,b)\neq 0$. Then by the implicit function theorem we may find holomorphic $\alpha_1(x),\beta_1(x)$, defined in a neighbourhood of a, such that $\alpha_1(a)=\beta_1(a)=b$. This justifies our differentiating equation 4.4 formally with respect to x to get

$$\begin{aligned}\frac{\partial\,\mathrm{Res}(P,Q)}{\partial x}(a) &= \Big(\prod_{\substack{1\le i\le c,\ 1\le j\le d\\ (i,j)\neq(1,1)}}(\alpha_i-\beta_j)\Big)(a)\left(\frac{\partial\alpha_1}{\partial x}-\frac{\partial\beta_1}{\partial x}\right)(a).\\ &= \Big(\prod_{\substack{1\le i\le c,\ 1\le j\le d\\ (i,j)\neq(1,1)}}(\alpha_i-\beta_j)\Big)(a)\left((\frac{\partial Q}{\partial y})^{-1}\frac{\partial Q}{\partial x}-(\frac{\partial P}{\partial y})^{-1}\frac{\partial P}{\partial x}\right)(a,b)\end{aligned} \tag{4.5}$$

The left hand side of this equation is the evaluation at a of a holomorphic function of x taking values in the tangent space of U. Each coordinate lies in $\widehat{\mathcal{O}_U^{\mathcal{R}}}$. The right hand side shows that if $\partial Q/\partial x$ and $\partial P/\partial x$ are linearly independent when evaluated at the common root (a,b), then the value lies in the span of these two vectors together but is not a multiple of either of them.

Observe now that if we are given a monic polynomial $p(y)\in\mathbb{C}[y]$ of degree d, if $p(\alpha)=0$ and if $B\subseteq\mathbb{C}$ is a finite set not containing α, then there are z_1,z_2 arbitrarily close to 0 such that $\frac{\alpha+z_1}{1+z_2}=\alpha$ and

$$p'(z_1,z_2,y):=(1+z_2)^d\cdot p(\frac{y+z_1}{1+z_2})$$

has a root at $y=\alpha$ but not at any point of B. Moreover $p'(z_1,z_2,y)$ is a monic polynomial in y of degree d and $p'(0,0,y)\equiv p(y)$.

In the situation where P and Q are distinguished polynomials, and there is a common zero of $P(a,y)$ and $Q(a,y)$ (as above), therefore, by choosing suitable values of these new variables $(z_1,z_2)=(\zeta_1,\zeta_2)$, we can ensure that $P'(z_1,z_2,x,y),Q(x,y)$ have exactly the one common solution (counted *without* multiplicity; it might be of any order) in y at the fixed (ζ_1,ζ_2,a).

Moreover if D is a preparation domain for $P(x,y)$ at some (a,b) then there is $D' \subseteq (\mathbb{C}^2 \times D)$ which is a preparation domain for $P'(z,x,y)$ at $(0,a,b)$; and $D' \cap (z=0) = \{(0,0)\} \times D$.

We have now got all the machinery needed for the following two similar lemmas. The same construction is involved in each, but the details differ as a consequence of the fact that the hypotheses of the implicit function theorem are not invariant under a change of basis. In lemma 4.11, which is the central step in the diagonalizing construction of theorem 4.14 below, the preparation variable is y_1, one of the implicit variables. By contrast in lemma 4.12 the preparation is with respect to a variable in the domain of the implicit function; interpolating this step into the induction restores regularity where we need it.

Lemma 4.11 *Let $D \subseteq \mathbb{C}^{m+k}$ be a preparation domain for each of the functions $f_1, \ldots, f_k \in \widehat{\mathcal{O}^{\mathcal{R}}_D}$, each of which is a distinguished polynomial with respect to y_1, each of degree d. Suppose, writing $f = (f_1, \ldots, f_k)$, that the closure of* $\mathrm{Ift}(f)$ *contains a subset W of A_d, where*

$$A_d = \{(x,y) \in D : \forall i \leq k\, (f_i \text{ is regular in } y_1 \text{ at } (x,y) \text{ of order } d)\}.$$

Then there is $V = U \times D$ (for some neighbourhood U of the origin in $\mathbb{C}^2$) and $g = (g_1, \ldots, g_k)$ satisfying:

1. *each coordinate $g_1, \ldots, g_k \in \widehat{\mathcal{O}^{\mathcal{R}}_V}$*
2. *g_1 is a distinguished polynomial in y_1 of degree* 1 *(that is, g_1 is linear in y_1);*
3. *$g_2, \ldots, g_k$ are not dependent on y_1;*
4. *$\{(0,0)\} \times A_d \subseteq Z(g)$; and*
5. *$\{(0,0)\} \times W$ is contained within the closure of the set*

$$\mathrm{Ift}(g) = \{(z,x,y) \in V : g(z,x,y) = 0 \text{ and } \det(\partial g/\partial y)(z,x,y) \neq 0\}.$$

Proof. Since all the f_i are distinguished polynomials of the same degree, we may take any matrix $\lambda \in M_k(\mathbb{C})$ of full rank and of row-weight 1 (that is, for each $i \le k$, $\lambda_{i1}+\cdots+\lambda_{ik}=1$), and the linear combinations $\lambda_1(f),\ldots,\lambda_k(f)$ will be distinguished polynomials too, also satisfying the hypotheses of the lemma. In particular, $A_d = \{(x,y) \in D : \forall i \le k\,(\lambda_i(f) \text{ is regular at } (x,y) \text{ of order } d)\}$ and $\mathrm{Ift}(\lambda \circ f) = \mathrm{Ift}(f)$.

It is without loss of generality, therefore, that we assume that W is contained in the closure of $X(f)$, the set of points $(x,y) \in D$ satisfying:

(i) $f(x,y) = 0$;

(ii) $\det(\partial f/\partial y)(x,y) \neq 0$ (so by conditions (i) and (ii), $X(f) \subseteq \mathrm{Ift}(f)$);

(iii) $\partial f_i/\partial y_1(x,y) \neq 0$ for each i, $1 \le i \le k$; and

(iv) $\det(\dfrac{\partial(f_2,\ldots,f_k)}{\partial y_2,\ldots,y_k})(x,y) \neq 0$.

Indeed for each analytic component C of $A_d \cap \mathrm{cl}(\mathrm{Ift}(f))$ we may find a permutation λ_0 of $(f_1,\ldots,f_k)$ so that C is contained within the closure of the subset of $\mathrm{Ift}(\lambda_0 \circ f)$ defined by condition (iv). Requiring λ to be such that C is in the closure of the subset of $\mathrm{Ift}(\lambda \circ f)$ defined by condition (iv) is thus a nonempty as well as evidently Zariski open condition on λ (in the irreducible manifold of nonsingular matrices of unit row-weight); we reason similarly for condition (iii). For almost all nonsingular λ of unit row weight, therefore, $A_d \subseteq \mathrm{cl}(X(\lambda \circ f))$.

Under this assumption we let

$$h_1(z,x,y_1,y_2,\ldots,y_k) = (1+z_2)^d \cdot f_1(x, \frac{y_1+z_1}{1+z_2}, y_2, \ldots, y_k) \tag{4.6}$$

and for $i = 2,\ldots,k$, let

$$g_i(z,x,y_2,\ldots,y_k) = \mathrm{Res}(f_i, h_1).$$

This choice of $g_2,\ldots,g_k$ evidently satisfies conclusions 3 and 4 of the lemma.

Now pick any $(a,b) \in X(f)$ and write $\vec{b} = (b_2,\ldots,b_k)$. There are $\zeta = (\zeta_1,\zeta_2)$ arbitrarily close to $(0,0)$ (in particular, with $|\zeta_2| < 1$) such that $b_1 = (b_1 + \zeta_1)/(1+\zeta_2)$, whence $h_1(\zeta,a,b_1,\vec{b}) = 0$, but such that $h_1(\zeta,a,y,\vec{b})$ does not take value zero on $B = \bigcup_{i=2}^k \{y : f_i(a,y,\vec{b}) = 0\} \setminus \{b_1\}$.

So we may evaluate, for each $2 \le i, j \le k$,

$$\frac{\partial \operatorname{Res}(f_i, h_1)}{\partial y_j}(a, b_1, \vec{b}) = \tau_i \cdot \left((\frac{\partial f_i}{\partial y_1})^{-1} \frac{\partial f_i}{\partial y_j} - (\frac{\partial h_1}{\partial y_1})^{-1} \frac{\partial h_1}{\partial y_j} \right) (\zeta, a, b_1, \vec{b}) \quad (4.7)$$

where the non-zero factor τ_i is as derived from equation 4.5 on page 55. Note also that $\partial h_1/\partial y_1(a, b_1, \vec{b}, \zeta) = (1+\zeta_2)^{d-1} \cdot \partial f_1/\partial y_1(a, b_1, \vec{b})$ is non-zero, so condition (iii) on $X(f)$ means that this expression makes sense. Thus

$$\frac{\partial(h_1, g_2, \ldots, g_k)}{\partial y_1, y_2, \ldots, y_k}(\zeta, a, b) = M_1 M_2, \text{ where}$$

$$M_1 = \begin{pmatrix} (1+\zeta_2)^{d-1} & 0 & \cdots & 0 \\ -\tau_2(\frac{\partial f_1}{\partial y_1})^{-1} & \tau_2(\frac{\partial f_2}{\partial y_1})^{-1} & & 0 \\ \vdots & & \ddots & \\ -\tau_k(\frac{\partial f_1}{\partial y_1})^{-1} & 0 & & \tau_k(\frac{\partial f_k}{\partial y_1})^{-1} \end{pmatrix} (\zeta, a, b)$$

$$\text{and } M_2 = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & (1+\zeta_2)\frac{\partial f_1}{\partial y_2} & \cdots & (1+\zeta_2)\frac{\partial f_1}{\partial y_k} \\ 0 & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_k} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\partial f_k}{\partial y_2} & \cdots & \frac{\partial f_k}{\partial y_k} \end{pmatrix} (\zeta, a, b),$$

and this is a product of nonsingular matrices (see conditions (iii) and (iv) on $X(f)$). So $(\zeta, a, b) \in \operatorname{Ift}(h_1, g_2, \ldots, g_k)$; but we could choose (a, b) arbitrarily close to any point of W. Thus $\{(0,0)\} \times W \subseteq \operatorname{cl}(\operatorname{Ift}(h_1, g_2, \ldots, g_k))$.

Now $h_1(z, x, y)$ is a distinguished polynomial in y_1 of degree d (with coefficients in $\widehat{\mathcal{O}^{\mathcal{R}}_{V_1}}$), say $h_1 = y_1^d + p_{d-1} y_1^{d-1} + \cdots + p_0$.

Let

$$g_1 = \frac{1}{(d-1)!} \frac{\partial^{d-1} h_1}{\partial y_1^{d-1}} = y_1 + \frac{1}{d} p_{d-1}.$$

This averaging function is a distinguished polynomial in y_1 of degree 1, and at any point of $\{(0,0)\} \times A_d$, g_1 takes value 0. Evidently $\partial g_1/\partial y_1 = 1$ everywhere, so $\partial g_1/\partial y$ is independent of $(\partial g_2/\partial y, \ldots, \partial g_k/\partial y)$.

Suppose moreover that $(\zeta, a, b_1, \vec{b}) \in \operatorname{Ift}(h_1, g_2, \ldots, g_k)$. Then if we set $b_1' = -p_{d-1}(\zeta, a, \vec{b})/d$ and recall that $g_2, \ldots, g_k$ do not depend on y_1, it is clear that

$$(\zeta, a, b_1', \vec{b}) \in \operatorname{Ift}(g).$$

As (ζ, a, b) approaches $\{(0,0)\} \times A_d$, moreover, b_1' approaches b_1 (since p_{d-1} is continuous). So in fact $\mathrm{cl}(\mathrm{Ift}(h_1, g_2, \ldots, g_k)) \cap (\{(0,0)\} \times A_d) = \mathrm{cl}(\mathrm{Ift}(g)) \cap (\{(0,0)\} \times A_d) \supseteq (\{(0,0)\} \times W)$, and conclusion 5 of the lemma holds. □

Lemma 4.12 *Let $D \subseteq \mathbb{C}^{m+k}$ be a preparation domain for the function $f_1 = x_m^d + \sum_{r=0}^{d-1} p_r(f_1) x_m^r \in \widehat{\mathcal{O}_D^{\mathcal{R}}}$, a distinguished polynomial with respect to x_m of degree d, and let $f_2, \ldots, f_k \in \widehat{\mathcal{O}_D^{\mathcal{R}}}$ be regular in x_m too, of orders $d_2, \ldots, d_k$ respectively. Let A_d be the set $\{a \in D : f_1$ is regular in x_m at b of order $d\}$. Suppose that either:*

(case 1) D is the domain of the averaging functions $G_0(f_1), \ldots, G_{d-1}(f_1)$, $W \subseteq A_d \cap \mathrm{cl}(\mathrm{Ift}(f))$, and for some $0 \le r \le d-1$, $f' = G_r(f_1)$; or

(case 2) $W \subseteq A_d \cap \mathrm{cl}(\mathrm{Ift}(f)) \cap \{x_m = 0\}$, and for some $0 \le r \le d-1$, $f' = x_m + p_r(f_1)$.

Then there is, for some $n \in \mathbb{N}$, a neighbourhood V of $\{\bar{0}\} \times D$ (where $\bar{0}$ is the origin in $\mathbb{C}^n$) and functions $g_1, \ldots, g_k \in \widehat{\mathcal{O}_V^{\mathcal{R}}}$ such that $g_1(\bar{0}, a) = f'(a)$ for all $a \in D$ and $\{\bar{0}\} \times W \subseteq \mathrm{cl}(\mathrm{Ift}(g))$.

Proof. For each i, since the function f_i is regular in x_m of order d_i, at any point a of D at least one of the functions $\partial^{s+1} f_i / \partial x_m^{s+1}$, $0 \le s \le d_i - 1$, takes non-zero value. Let $s_i(a)$ denote the least such s. If $a \in \mathrm{Ift}(f)$, we may evaluate the Jacobian

$$J(f_1 + \zeta_{1s_1} \frac{\partial^{s_1(a)} f_1}{\partial {x_m}^{s_1(a)}}, \ldots, f_k + \zeta_{ks_k} \frac{\partial^{s_k(a)} f_k}{\partial {x_m}^{s_k(a)}})(a)$$

and see that it is non-zero for all sufficiently small $\zeta = (\zeta_{1s_1}, \ldots, \zeta_{ks_k}) \in \mathbb{C}^k$, while if each coordinate of ζ is non-zero then $\partial(f_i + \zeta_{is_i}(\partial^{s_i} f_i)/\partial x_m^{s_i})/\partial x_m \neq 0$, for each i.

We can treat all $a \in \mathrm{Ift}(f)$, no matter what the values of $s_i(a)$, simultaneously by choosing functions with a sufficient number of parameters: z_1, z_2 and

$(z_{is})_{i\le k; s<d_i}$. Let $h_1, h_2, \dots, h_k$ be defined by:

$$h_1(z, \vec{x}, x_m, y) = (1+z_1)^{d_1} f_1(\vec{x}, \frac{x_m + z_1}{1+z_2}, y) + \sum_{s=1}^{d_1-1} z_{1s}\big(\frac{\partial^s}{\partial {x_m}^s} f_1(\vec{x}, \frac{x_m+z_1}{1+z_2}, y)\big), \tag{4.8}$$

$$h_i(z, \vec{x}, x_m, y) = f_i(\vec{x}, x_m, y) + \sum_{s=1}^{d_i-1} z_{is}\big(\frac{\partial^s}{\partial {x_m}^s} f_i(\vec{x}, x_m, y)\big) \ (i = 2, \dots, k).$$

Then, given any $a \in \mathrm{Ift}(f)$, we can make a suitable choice of $\zeta = (\zeta_1, \zeta_2, (\zeta_{is}))$ arbitrarily close to the origin, requiring:

(i) $\dfrac{a_m + \zeta_1}{1+\zeta_2} = a_m$, and no other common zero of h_1 and any other h_i differs from (a, ζ) only in the x_m coordinate;

(ii) for each i, $\partial h_i / \partial x_m(a, \zeta) \neq 0$;

(iii) $J(h_1, \dots, h_k)(a, \zeta) \neq 0$;

and we see from equation 4.5 as before that the Jacobian

$$J(h_1, \mathrm{Res}(h_2, h_1), \dots, \mathrm{Res}(h_k, h_1))(a, \zeta) \neq 0$$

and that $h_1(a, \zeta) = \mathrm{Res}(h_2, h_1)(a, \zeta) = \cdots = \mathrm{Res}(h_k, h_1)(a, \zeta) = 0$. For each $2 \le i \le k$, let $g_i = \mathrm{Res}(h_i, h_1)$; then we have verified that $(a, \zeta) \in \mathrm{Ift}(h_1, g_2, \dots, g_k)$.

Now, h_1 is a distinguished polynomial of degree d_1 with respect to x_m, with symmetric functions $p_0(h_1), \dots, p_{d-1}(h_1)$. For each r and all $(\vec{x}, y) \in \pi(D)$, $p_r(h_1)(\bar{0}, \vec{x}, y) = p_r(f_1)(\vec{x}, y)$. In case 1, where the averaging functions $G_r(f_1)$ are defined on D, so too are the averaging functions $G_r(h_1)$ defined on a neighbourhood V' of $\{\bar{0}\} \times D$. We have

$$\frac{\partial G_r(h_1)}{\partial y} = \tbinom{d}{r}^{-\frac{1}{d-r+1}} \big(p_r(h_1)\big)^{\frac{r-d}{d-r+1}} \frac{\partial p_r(h_1)}{\partial y},$$

and this $\mathbb{C}$-linear multiple of $\partial p_r(h_1)/\partial y$ is independent of the variable x_m. In either case, for any $b \in \mathrm{Ift}(h_1, g_2, \dots, g_k)$, the derivative

$$\frac{\partial h_1}{\partial y}(b) = \sum_{j=0}^{d} x_m^r \frac{\partial p_j(h_1)}{\partial y}(b)$$

does not lie in the $\mathbb{C}$-linear span of $(\partial g_2/\partial y(b), \ldots, \partial g_k/\partial y(b))$ in $\mathbb{C}^k$.

It follows that for any $r \leq d_1 - 1$, and almost all choices of the new parameters $\zeta'_0, \ldots, \zeta'_{d_1-1}$, the functions

$$\Big(G_r(h_1) + \sum_{\substack{0 \leq j \leq d_1 - 1 \\ j \neq r}} z'_j G_j(h_1)\Big) \text{ (in case 1) and } \Big(x_m + p_r(h_1) + \sum_{\substack{0 \leq j \leq d_1 - 1 \\ j \neq r}} z'_j p_j(h_1)\Big) \quad (4.9)$$

take value zero at $(\zeta'_0, \ldots, \zeta'_{d-1}, b')$ for some b' differing from b only in the x_m-coordinate; and at this point their derivatives with respect to y are not in the span of the derivatives of $g_2, \ldots, g_k$. If $a \in A_d$ and all the n parameters $\zeta = (\zeta_1, \zeta_2, (\zeta_{is}), (\zeta'_j))$ are chosen to be sufficiently small, then such points b' can be found arbitrarily close to a (for any r, and with the choice of function from (4.9) appropriate to the case). We can relabel the parameter space and point $\zeta = (\zeta_1, \zeta_2, (\zeta_{is}), (\zeta'_j))$ so that $\zeta = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n)$.

So setting g_1, according to the case of the lemma, to be that one of the functions of (4.9) for which $g_1(\bar{0}, x, y) = f'(x, y)$, we have that $\{\bar{0}\} \times W \subseteq \mathrm{cl}(\mathrm{Ift}(g_1, g_2, \ldots, g_k))$, as required. In case 2 the functions $g_1, \ldots, g_k$ lie in, for example, $\widehat{\mathcal{O}^{\mathcal{R}}_{\Gamma_n \times D}}$ where Γ_n is the open polydisk of radius 1/2 in $\mathbb{C}^n$. In case 1 they are in $\widehat{\mathcal{O}^{\mathcal{R}}_V}$ where, with $\pi : (z_1, z_2, z_3, \ldots, z_n, x, y) \mapsto (z_1, z_2, x, y)$, we define $V = \Gamma_n \times D \cap \pi^{-1}(V')$. □

Corollary 4.13 *If D, f and W satisfy the hypotheses either of lemma 4.11 (in which case, let $n = 2$) or of lemma 4.12, and if in addition the set*

$$\Omega = \{e \in \mathbb{C}^m : e \in \mathrm{Acc}(b, f) \textit{ for all } b \in W\} = \mathrm{Acc}(W, f)$$

has non-empty interior, then we can find V and g satisfying the conclusions of that lemma and also such that for every point b of W, the set $\Omega' = \{e \in \mathbb{C}^{n+m} : e \in \mathrm{Acc}((\bar{0}, b), g)$ for all $b \in W\}$ has non-empty interior too.

Proof. If $g(z_1, z_2, x, y) \in \widehat{\mathcal{O}^{\mathcal{R}}_V}$ is found as in the proof of lemma 4.11, then for some sufficiently large power $N \in \mathbb{N}$ the function

$$g'(z_1, z_2, x, y) = g(z_1^N, z_2^N, x, y) \in \widehat{\mathcal{O}^{\mathcal{R}}_{V'}},$$

where $V' = \{(z_1, z_2, x, y) \in \mathbb{C}^{2+m+k} : (z_1^N, z_2^N, x, y) \in V\}$, has this extra property. Evidently, whatever the choice of $N > 0$, such a g' satisfies the conclusions of lemma 4.11.

Recall the definition of accessibility on page 52. At every point b of W which is accessible along e in $\mathrm{Ift}(f)$ (that is, for which $e \in \mathrm{Acc}(b, f)$), there is $\varepsilon(e, b) > 0$ and $\alpha \in \mathbb{C}$ such that for all $t \in (0, \varepsilon(e, b))$, $\pi(b+\alpha te) \in \pi(\mathrm{Ift}(f))$. We can make a selection of an $\alpha = \alpha(e, b)$ (with unit modulus) and $\varepsilon(e, b)$ definably in $\mathbb{R}_{an}$.

Consider the function $h_1(z, x, y)$ as defined at equation (4.6) in lemma 4.11, and its dependence on the variables (z_1, z_2). The function $(h_1, f_2, \ldots, f_k)$, considered as a family of maps parametrized by (z_1, z_2), is in the configuration described by lemma 4.2. From that lemma (applied with the l_∞ metric) we conclude that at every point x of $\pi_{m,k}(\mathrm{Ift}(f))$ there is some $r = r(x) > 0$ such that if $\|(z_1, z_2)\| < r$ (*i.e.*, $|z_1| < r$ and $|z_2| < r$), then $\pi_{2+m,k}(z_1, z_2, x) \in \pi_{2+m,k}(Z(h_1, f_2, \ldots, f_k))$. Indeed, by counting the zeros of $(h_1, f_2, \ldots, f_k)$ we see that if X is any analytic component of $Z(h_1, f_2, \ldots, f_k)$ containing a point (z_1, z_2, x, y) with $\|(z_1, z_2)\| < r(x)$, then X contains points (z_1', z_2', x, y') for any other $\|(z_1', z_2')\| < r(x)$.

Further, provided that $(z_1, z_2, x, y) \in Z(h_1, f_2, \ldots, f_k)$ and $(x, y) \in \mathrm{Ift}(f)$, we have seen in the proof of lemma 4.11 that, writing $g_i = \mathrm{Res}(h_1, f_i)$ for $2 \le i \le k$, the point $(z_1, z_2, x, y) \in \mathrm{Ift}(h_1, g_2, \ldots, g_k)$ wherever $(z_1, z_2) \in \{y_1 = (y_1 + z_1)/(1 + z_2)\} \setminus B$, where B is a finite set dependent on (x, y). In particular, if X is an analytic component of $Z(h_1, f_2, \ldots, f_k)$ containing (z_1, z_2, x, y) then $X \cap Z(\det \partial(h_1, g_2, \ldots, g_k)/\partial(y_1, \ldots, y_k))$ is a proper analytic subset of X. Moreover since $\pi_{2+m,k} \restriction X$ is a proper map (since all fibres are finite), $\pi(X \cap Z(\det \partial(h_1, g_2, \ldots, g_k)/\partial(y_1, \ldots, y_k)))$ is, by the Remmert Proper Mapping Theorem, an analytic subset Ξ of $\pi(X)$.

Hence we have

$$\pi_{2+m,k}(\mathrm{Ift}(h_1, g_2, \ldots, g_k)) \supseteq \big\{(z_1, z_2, x) : x \in \pi_{m,k}(\mathrm{Ift}(f)),\ \|(z_1, z_2)\| < r(x)\big\} \setminus \Xi,$$

where Ξ is a proper analytic subset of the open subset of $\mathbb{C}^{2+m}$,

$$\{(z_1, z_2, x) : x \in \pi_{m,k}(\mathrm{Ift}(f)),\ \|(z_1, z_2)\| < r(x)\}.$$

And finally if $(z_1, z_2, x, y) \in \mathrm{Ift}(h_1, g_2, \ldots, g_k)$ then there is a point in $\mathrm{Ift}(g_1, \ldots, g_k)$ differing from (z_1, z_2, x, y) only in the y_1 coordinate, so we conclude that

$$\pi_{2+m,k}(\mathrm{Ift}(g)) \supseteq \pi_{2+m,k}(\mathrm{Ift}(h_1, g_2, \ldots, g_k)).$$

Now, after a definable choice in $\mathbb{R}_{an}$ of the function $r(x)$, the map

$$R : (e, b, t) \mapsto r(a) \text{ where } a = (\pi_{m,k}(b) + \alpha(e, b)te)$$

with domain $\{(e, b, t) : b \in W, e \in \mathrm{Acc}(b, f), 0 < t < \varepsilon(e, b)\}$ is definable in the *polynomially bounded* o-minimal structure $\mathbb{R}_{an}$. So from proposition 4.6 there is some N such that for every $b \in W$ and $e \in \Omega_b$, $R(e, b, t) > t^N$ for all sufficiently small $t > 0$: say, for $0 < t < \varepsilon'(e, b)$. It follows that provided $\|z_1, z_2\| < t$ and $0 < t < \varepsilon'(e, b)$, we have

$$(z_1, z_2, a) \in \pi_{2+m,k}(Z(h_1(z_1^N, z_2^N, x, y), f_2(x, y), \ldots, f_k(x, y))),$$

where $a = (\pi_{m,k}(b) + \alpha(e, b)te)$, and if we take $g'(z_1, z_2, x, y) := g(z_1^N, z_2^N, x, y)$ then

$$(z_1, z_2, a) \in \pi_{2+m,k}(\mathrm{Ift}(g')) \text{ or } (z_1, z_2, a) \in \Xi'$$

where $\Xi' = \{(z_1, z_2, x) : (z_1^N, z_2^N, x) \in \Xi\}$ with Ξ as found above; Ξ' is an analytic set of dimension less than $2 + m$.

So if we take $g'(z_1, z_2, x, y) := g(z_1^N, z_2^N, x, y)$ and any point b of W, then for almost all $(\eta_1, \eta_2) \in \Gamma_2$ (the open polydisk of radius 1/2 in $\mathbb{C}^2$), $(\bar{0}, b)$ is accessible along (η_1, η_2, e) in $\mathrm{Ift}(g')$. For indeed, we have chosen $|\eta_1|, |\eta_2|$ small enough that

$$\big(\pi_{2+m,k}(\bar{0}, b) + \alpha(e, b)t(\eta_1, \eta_2, e)\big) \in \pi_{2+m,k}(Z(g'))$$

for every $t < \min(\varepsilon(e, b), \varepsilon'(e, b))$. So if accessibility along (η_1, η_2, e) fails, then for some (arbitrarily small) t in this interval, none of the holomorphic functions defining Ξ' at $(\pi_{2+m,k}(\bar{0}, b) + \alpha t(\eta_1, \eta_2, e))$ is regular in (η_1, η_2, e). But then, by proposition 4.4, the set of (η_1, η_2, e) with this property is closed and nowhere dense in $\Gamma_2 \times \Omega_0$ (where Ω_0 is an open polydisk contained in Ω).

The complement of this closed nowhere dense set is contained in $\Omega' = \{e \in \mathbb{C}^{n+m} : e \in \mathrm{Acc}((\bar{0}, b), g)$ for all $b \in W\}$, which therefore has non-empty interior. We have thus proved the corollary to lemma 4.11.

In lemma 4.12, similarly, we can take the function

$$g'(z_1, z_2, z_3, \ldots, z_n, x, y) = g(z_1^N, z_2^N, z_3, \ldots, z_n, x, y) \in \mathcal{O}_{V'}^{\mathcal{R}}$$

for a suitable $N \in \mathbb{N}$ and $V' \subseteq \mathbb{C}^{n+m+k}$.

The same argument carries through, since in this case too if $(x, y) \in \mathrm{Ift}(f)$ then the point $\{(z_1, z_2, z_3, \ldots, z_n, x, y)$ is in $\mathrm{Ift}(g)$ provided that it lies in $Z(g) \setminus \Xi$ for some proper analytic subset Ξ of V: the only conditions on the extra parameters $(z_3, \ldots, z_n)$ featuring in the proof of lemma 4.12 are that they do not lie on certain analytic subvarieties of V. □

4.4 Analytic continuation theorem

Theorem 4.14 (Analytic continuation) *Let $U \subseteq \mathbb{C}^m$ be a bounded neighbourhood, let the open set $V \subseteq U \subseteq \mathrm{cl}(V)$ be definable in $\mathbb{R}_{an}$ and let $g \in \mathcal{O}_V^{\mathcal{R}}$ have holomorphic extension to U. Suppose that the implicit representations of g extend continuously to $\mathrm{cl}(V)$. Then $g \in \mathcal{O}_U^{\mathcal{R}}$.*

The principle of the proof is to find representations $\langle k, f, \tilde{g} \rangle$ of g with the matrix $\partial f / \partial y$ in upper triangular form, and having unit diagonal entries.Then where $\tilde{g}$ extends continuously to a, the function $f(x, y)$ must necessarily satisfy the hypotheses of the implicit function theorem at $(a, \tilde{g}(a))$, so there is a neighbourhood of a on which $\langle k, f, \tilde{g} \rangle$ extends holomorphically.

There are three complications to this picture: first, that we work in the first instance with implicit representations of g over $\widehat{\mathcal{O}_D^{\mathcal{R}}}$ for a collection of domains D; secondly, that in view of lemma 4.10, we need to consider regularity with respect to bases other than the canonical one, and also the condition of accessibility; and thirdly, that in applying lemmas 4.11 and 4.12 we increase the dimension of the base space from m.

Proof. Observe that the conditions of lemma 3.3 apply to V, so we may assume that $V = X_1 \cup \cdots \cup X_N$ with each X_s definable in $\mathbb{R}_{an}$, and that g is represented by some $\langle k_s, G_s, \tilde{g}_s \rangle$ on each X_s.

The set $U \setminus V$ is then definable in $\mathbb{R}_{an}$ and of dimension (as a subset of $\mathbb{R}^{2m}$) strictly less than $2m$, since it has no interior. So, by lemma 4.3, we can find a dense subset $\Omega \subseteq \mathbb{C}^m$ of good directions for $U \setminus V$. If $e \in \Omega$ then, at every point $b \in U \setminus V$, when $\varepsilon > 0$ is chosen small enough the interval $\{b + te : 0 < t < \varepsilon\}$ is contained within X_s for some $s \leq N$. That is, $e \in \mathrm{Acc}((b, \tilde{g}_s(b)), G_s)$. (Note that e may also belong to $\mathrm{Acc}((b, \tilde{g}_{s'}(b)), G_{s'})$ for some other s', if for example $\{b + \alpha te : 0 < t < \varepsilon\} \subseteq X_{s'}$ for some $\alpha \in \mathbb{C}$.)

So we may choose an $\mathbb{R}_{an}$-definable disjoint covering of $(U \setminus V) \times \Omega) = \Delta_0 \cup \cdots \cup \Delta_N$ such that if $(b, e) \in \Delta_s$ then $e \in \mathrm{Acc}((b, \tilde{g}_s(b)), G_s)$.

Let $\pi : \mathbb{C}^{m+m} \to m$ be the canonical projection onto the first m coordinates and define

$$\Delta'_s = \{(b, e) \in \Delta_s : e \text{ is in the interior of } \pi^{-1}(b) \cap \Delta_s\}.$$

Then $\bigcup_{s \leq N} \Delta'_s$ is dense in $(U \setminus V) \times \mathbb{C}^m$, and the $\mathbb{R}_{an}$-definable set

$$\Sigma = \big((U \setminus V) \times \mathbb{C}^m\big) \setminus \big(\bigcup_{s \leq N} \Delta'_s\big)$$

has the property that for all $b \in (U \setminus V), \dim(\pi^{-1}(b) \cap \Sigma) < 2m$ (where dim denotes the dimension in $\mathbb{R}_{an}$).

It follows from this property that there is a definable partition of $U \setminus V$ into cells of the locally compact o-minimal structure $\mathbb{R}_{an}$, say $U \setminus V = C_1 \cup \cdots \cup C_t$, and open subsets $\Omega_1, \ldots, \Omega_t$ of $\mathbb{C}^m$ such that $(C_i \times \Omega_i) \cap \Sigma = \emptyset$ for each i. Then for each $b \in C_i$, $\{b\} \times \Omega_i \subseteq \Delta_s$ for some $s \leq N$.

Let $C_{is} = \{b \in C_i : \{b\} \times \Omega_i \subseteq \Delta_s\}$, and let $W_{is} = \{(b, \tilde{g}_s(b)) : b \in C_{is}\}$. We have constructed W_{is} so that $\Omega_i \subseteq \mathrm{Acc}(W_{is}, G_s)$ for each i, s.

Now consider $W = \bigcup_{i \le t, s \le N} W_{is}$. The projection $\pi_{m,k}(W) = U \setminus V$ and each W_{is} is contained within the graph of the extension of $\tilde{g}_s$ to $\mathrm{cl}(X_s) \cap U$. So if we can cover W with the graphs of implicitly represented functions, we will have representations of g on $U \setminus V$ as required.

We shall establish the following claim 4.15 by induction on r. By Y we denote the k-dimensional subspace of $\mathbb{C}^{n+m+k}$ generated by the vectors corresponding to $y_1, \ldots, y_k$ in the canonical basis. Where n is fixed, $\bar{0}$ denotes the origin in $\mathbb{C}^n$.

The partition $W = \bigcup_{i \le t, s \le N} W_{is}$ satisfies this claim for $r = 0$, with $n = 0$ and, corresponding to each W_{is}, the function $G_s \in (\widehat{\mathcal{O}^{\mathcal{R}}_{U \times \mathbb{C}^k}})^k$, open set Ω_i, and the canonical basis.

Claim 4.15 *For each $r = 0, \ldots, k$ we can find $n, N_r \in \mathbb{N}$, a partition $\{\bar{0}\} \times W = W_1 \cup \cdots \cup W_{N_r}$; and for each I $(1 \le I \le N_r)$, a basis $e^I = (e^I_1, \ldots, e^I_k)$ for Y, an open subset Ω_I of $\mathbb{C}^{n+m}$ and a function*

$$f^I : D_I \to \mathbb{C}^k,$$

where $D_I \subseteq \mathbb{C}^{n+m+k}$ is an R_{an}-definable domain containing W_I, such that $W_I \subseteq \mathrm{cl}(\mathrm{Ift}(f^I))$, such that $\Omega_I \subseteq \mathrm{Acc}(W_I, f^I)$, and such that, denoting the coordinates of f^I by $f^I_1, \ldots, f^I_k$, for each $j = 1, \ldots, r$ and each j', $j < j' \le k$,

$$\frac{\partial f^I_j}{\partial e^I_j} = 1 \text{ identically on } D_I, \text{ and } \frac{\partial f^I_{j'}}{\partial e^I_j} = 0;$$

and each coordinate function $f^I_j \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_I}}$, $1 \le j \le k$.

We have shown the case $r = 0$ above. Let $r > 0$ and assume that the claim holds for $r - 1$; so $\{\bar{0}\} \times W = W_1 \cup \cdots \cup W_{N_{r-1}}$, and domains $D_I \subseteq \mathbb{C}^{n+m+k}$, functions f^I, sets Ω_I and bases e^I are defined to satisfy this case of the claim.

Fix one I and W_I in this partition, and write $f := f^I$, $D := D_I$ and $(e_1, \ldots, e_k) := e^I$. It follows from the hypotheses on f that

$$\det\left(\frac{\partial(f_r, \ldots, f_k)}{\partial(e_r, \ldots, e_k)}\right)(a) \neq 0$$

at every point a of $\mathrm{Ift}(f)$; and clearly also that $f_r = \cdots = f_k = 0$ there. That is, $W_I \subseteq \mathrm{cl}(\mathrm{Ift}_{(e_r,\ldots,e_k)}(f_r,\ldots,f_k))$, and moreover every point of W_I is accessible along every $e' \in \Omega_I$ in $\mathrm{Ift}_{(e_r,\ldots,e_k)}(f_r,\ldots,f_k)$.

We will subdivide W_I in several stages. At each stage we will need a new indexing variable s_1, s_2, and so forth, taking values between 1 and, respectively, t_1, $t_2 = t_2(s_1)$, $t_3 = t_3(s_1,s_2)$, and so on. When I have introduced variable s_n, $\vec{s}$ will denote $(s_1,\ldots,s_{n-1})$.

(I.) First, let A_∞ denote the subset of D containing those points $a \in Z(f_r,\ldots,f_k)$ for which none of $f_r,\ldots,f_k$ are regular at a with respect to any vector in the span of $(e_r,\ldots,e_k)$. The first division of W_I is into $W_I \setminus A_\infty$ and $W_I \cap A_\infty$. We treat these cases separately.

(II.) There are, by lemma 4.7, finitely many vectors $v_1,\ldots,v_{t_1} \in \langle e_r,\ldots,e_k\rangle$ such that at every point of $Z(f_r,\ldots,f_k)\setminus A_\infty$, some one of $f_r,\ldots,f_k$ is regular with respect to one of $v_1,\ldots,v_{t_1}$. We partition $W_I \setminus A_\infty$ into $W'_1,\ldots,W'_{t_1}$, so that one of $(f_r,\ldots,f_k)$ is regular with respect to v_{s_1} at every point of W'_{s_1}. Let $e^{s_1}_r = v_{s_1}$, and complete the basis $e^{s_1} = (e^{s_1}_r,\ldots,e^{s_1}_k)$ for $\langle e_r,\ldots,e_k\rangle$ arbitrarily. Then $\mathrm{Ift}_{e^{s_1}}(f_r,\ldots,f_k) = \mathrm{Ift}_e(f_r,\ldots,f_k)$. Let ϕ_{s_1} be the bijection on $\mathbb{C}^{n+m+k}$ mapping $(e_1,\ldots,e_{r-1},e^{s_1}_r,\ldots,e^{s_1}_k)$ to the canonical basis for Y (and preserving the rest of the canonical basis).

(III.) Index the pairs of form $\langle j,S\rangle$ with $r \le j \le k$ and S a subset of $\{j+1,\ldots,k\}$ by the function $s_2 \mapsto \langle j(s_2),S(s_2)\rangle$. There are $t_2 = \sum_{j=r}^{k} 2^{k-j}$ of them. Let

$$W'_{s_1s_2} = \{a \in W'_{s_1} : f_i \text{ is regular w.r.t } e^{s_1}_r \text{ iff } i = j(s_2) \text{ or } i \in S(s_2)\}$$

and define

$$f^{s_1s_2}_i = \begin{cases} f_i & \text{if } i < r,\ i = j(s_2) \text{ or } i \in S(s_2), \\ f_i + f_{j(s_2)} & \text{otherwise.} \end{cases}$$

Then $\mathrm{Ift}(f^{s_1s_2}) = \mathrm{Ift}(f)$ and each coordinate of $f^{s_1s_2}$ is regular with respect to $e^{s_1}_r$ at every point of $W'_{s_1s_2}$, and is an element of $\widehat{\mathcal{O}^{\mathcal{R}}_D}$.

(IV.) It follows from Proposition 4.6 that there are only finitely many tuples $d \in \mathbb{N}^{k-r+1}$ for which $A_d \cap W'_{s_1 s_2}$ is non-empty when

$$A_d = \{a \in D : f_i^{s_1 s_2} \text{ is regular w.r.t. } e_r^{s_1} \text{ of order } d_i,\ r \leq i \leq k\}.$$

Index these by $d = d(s_3)$, $1 \leq s_3 \leq t_3$, and write $W'_{\vec{s}s_3} = W'_{s_1 s_2} \cap A_{d(s_3)}$.

(V.) Now, by lemma 4.1, there are finitely many preparation domains $D_1, \ldots, D_{t'}$ for $(f_r^{s_1 s_2}, \ldots, f_k^{s_1 s_2})$ with respect to $e_r^{s_1}$, covering $A_{d(s_3)}$. Let $W'_{\vec{s}s_4} = W'_{\vec{s}s_3} \cap D_{s_4}$ for each $1 \leq s_4 \leq t_4$, discarding those domains D_i for which $W'_{\vec{s}s_3} \cap D_i$ is empty.

On each D_{s_4} we may prepare $(f_r^{s_1 s_2}, \ldots, f_k^{s_1 s_2})$ with respect to $e_r^{s_1}$ (that is, apply theorem 3.10 with $f := f_i^{s_1 s_2} \circ \phi_{s_1}$, $g := {y_r}^{d_r(s_3)}$) to get $(f_i^{s_1 s_2}) = h_i f'_i$, with h_i invertible on D_{s_4}; each f'_i is a distinguished polynomial map with respect to $e_r^{s_1}$ of degree $d_i(s_3)$. (So, in other words, $f'_i \circ \phi_{s_1}$ is a monic polynomial in y_r of degree $d_i(s_3)$, and its non-leading coefficients vanish on $A_{d(s_3)}$.)

Then $\mathrm{Ift}_{e^{s_1}}(f') = \mathrm{Ift}_{e^{s_1}}(f^{s_1 s_2}) \cap D_{s_4}$. We apply lemma 4.9 repeatedly to $f'_r, \ldots, f'_k$ to reduce them all to the same degree with respect to $e_r^{s_1}$, say $d' = \min(d_r(s_3), \ldots, d_k(s_3))$, getting $f'' = (f''_r, \ldots, f''_k) \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_{s_4}}}$ such that $\mathrm{Ift}_{e^{s_1}}(f'') = \mathrm{Ift}_{e^{s_1}}(f^{s_1 s_2}) \cap D_{s_4}$ also. Then $W'_{\vec{s}s_4} \subseteq \mathrm{cl}(\mathrm{Ift}_{e^{s_1}}(f''))$; and every point of $W'_{\vec{s}s_4}$ is accessible along e' in $\mathrm{Ift}_{e^{s_1}}(f'')$ for every $e' \in \Omega_I$.

So we have satisfied the hypotheses of lemma 4.11 with $D := D_{s_4}$, $f := (f_r, \ldots, f_k)$, $W := W'_{\vec{s}s_4}$. We conclude that there is $V \subseteq \mathbb{C}^{2+n+m+k}$ and functions $f_r^*, \ldots, f_k^* \in \widehat{\mathcal{O}^{\mathcal{R}}_V}$ such that

1. $\{(0,0)\} \times W'_{\vec{s}s_4} \subseteq \mathrm{cl}(\mathrm{Ift}_{e^{s_1}}(f^*))$;
2. $\dfrac{\partial f_r^*}{\partial e_r^{s_1}} = 1$ identically on V; and
3. if $j > r$ then $\dfrac{\partial f_j^*}{\partial e_r^{s_1}} = 0$ identically.

(VI.) Moreover, by corollary 4.13, we can choose f^* so that there is an open subset $\Omega_{\vec{s}s_4}$ of $\Gamma_2 \times \Omega_I$ such that $\Omega^* \subseteq \mathrm{Acc}(\{\bar{0}\} \times W'_{\vec{s}s_4})$.

Then, as $s_1, \ldots, s_4$ vary, the sets $W'_{\vec{s}s_4}$ form a partition of $\{(0,0)\} \times W_I \setminus A_\infty$ for which the respective bases $(e_1, \ldots, e_{r-1}, e_r^{s_1}, \ldots, e_k^{s_1})$, domains D_{s_4}, sets $\Omega_{\vec{s}s_4}$ and functions $(f_1, \ldots, f_{r-1}, f_r^*, \ldots, f_k^*)$, meet the claim for r.

(VII.) It remains for us to witness the claim on a partition of $W_I \cap A_\infty$. Recall that $\Omega_I \subseteq \mathrm{Acc}(W_I, f)$ is an open subset of $\mathbb{C}^{n+m}$; so at every point of W_I, f_r is regular with respect to some $e' \in \pi_{m,k}^{-1}(\Omega_I) = \Omega_I \times Y$ (by proposition 4.4, as f_r is certainly non-constant because $\mathrm{Ift}(f)$ is non-empty). Hence, by lemma 4.7 there are $e'_1, \dots, e'_{t_1} \in \Omega_I \times Y$ and a partition $W_I \cap A_\infty = W''_1 \cup \dots \cup W''_{t_1}$ so that at every point of each W''_{s_1}, f_r is regular with respect to e'_{s_1}.

(VIII.) Once again, by Proposition 4.6 there is $d \in N$ such that f_r is regular with respect to e'_{s_1} everywhere in W''_{s_1} of order no greater than d. So we appeal to lemma 4.1 to cover W''_{s_1} with finitely many preparation domains for f_r with respect to e'_{s_1}, say $D_{s_1 1}, \dots, D_{s_1 t_2}$, such that $W''_{s_1} \cap D_{s_1 s_2} \subseteq A_{d(s_2)}$ for some $d(s_2) \le d$ (and each $D_{s_1 s_2}$ is a preparation domain for f_r at some point of $W''_{s_1} \cap D_{s_1 s_2}$). Write $W''_{s_1 s_2} = W''_{s_1} \cap D_{s_1 s_2}$.

(IX.) Let the bijection $\phi : \mathbb{C}^{n+m+k} \to \mathbb{C}^{n+m+k}$ fix Y and map e'_{s_1} to e_{x_m}, the vector corresponding to x_m in the canonical basis. Then

1. $\phi(W''_{s_1 s_2}) \subseteq A_{d(s_2)} := \{b \in \phi(D_{s_1 s_2}) : f_r \circ \phi^{-1}$ is regular in x_m at b of order $d(s_2)\}$;
2. at every point $b \in \phi(W''_{s_1 s_2})$, $f_r \circ \phi^{-1}$ is not regular with respect to any vector in Y; and
3. $e_{x_m} \in \mathrm{Acc}(\phi(W''_{s_1 s_2}), f \circ \phi^{-1})$.

Thus we may appeal to lemma 4.10 and conclude that there is a cover of $D_{s_1 s_2}$ by domains $D_{\vec{s} s_3}$, $1 \le s \le t_3$, and a partition of $W''_{s_1 s_2} = W''_{\vec{s}1} \cup \dots \cup W''_{\vec{s} t_3}$ with $W''_{\vec{s} s_3} \subseteq D_{\vec{s} s_3}$, such that for each s_3, either:

(case 1) $D_{\vec{s} s_3}$ is the domain of the functions $G'_0, \dots, G'_{d(s_2)-1} \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_{\vec{s} s_3}}}$ which satisfy $G'_\rho \circ \phi^{-1} = G_\rho(f_r \circ \phi^{-1})$ where $G_\rho(f_r \circ \phi^{-1})$ is an averaging function for $(f_r \circ \phi^{-1})$ defined on $\phi(D_{\vec{s} s_3})$, and for some $0 \le \rho(s_3) < d(s_2)$, $G'_{\rho(s_3)}$ is regular at every point $b \in W''_{\vec{s} s_3}$ with respect to some $e(b) \in Y$; or

(case 2) $W''_{\vec{s} s_3} \subseteq \phi^{-1}(x_m = 0)$ and for some $0 \le \rho(s_3) < d(s_2)$, the symmetric function $p_{\rho(s_3)}(f_r \circ \phi^{-1})$ is regular at every point b of $W''_{\vec{s} s_3}$ with respect to some $e(b) \in Y$, in which case the function $G'_{\rho(s_3)} \in \widehat{\mathcal{O}^{\mathcal{R}}_{D_{\vec{s}s_3}}}$ defined so that $G'_\rho \circ \phi^{-1} = x_m + p_{\rho(s_3)}(f_r \circ \phi^{-1})$ is also regular with respect to $e(b)$ at b.

Note that all such vectors $e(b)$, in either case, are not in the subspace $\langle e^I_1, \dots, e^I_{r-1}\rangle$ because, by the inductive hypothesis, $\partial f_r/\partial e^I_j$ is identically zero for each $j < r$.

(X.) So, appealing to lemma 4.7 again, we may in fact assume that $e(b)$ is a constant vector $e^{s_3}_r$ lying in Y and outside $\langle e^I_1, \dots, e^I_{r-1}\rangle$. Complete the basis for Y to, say, $(e^I_1, \dots, e^I_{r-1}, e^{s_3}_r, \dots, e^{s_3}_k)$ arbitrarily; let $e^{s_3} = (e^{s_3}_r, \dots, e^{s_3}_k)$.

(XI.) The functions $f_r, \dots, f_k$ on domain $D_{\vec{s}s_3}$, subset $W''_{\vec{s}s_3}$ of $A_{d(s_2)} \cap \mathrm{cl}(\mathrm{Ift}_{e^{s_3}}(f_r, \dots, f_k))$ and function $f' := G'_{\rho(s_3)}$ as defined in the appropriate case of (IX) above, then satisfy the hypotheses of lemma 4.12; moreover, $\Omega_I \subseteq \mathrm{Acc}(W''_{\vec{s}s_3}, (f_r, \dots, f_k))$.

We deduce that there are: $n(s_3) \in \mathbb{N}$; an open subset $V_{\vec{s}s_3}$ of $\mathbb{C}^{n(s_3)+n+m+k}$ containing $\{\bar{0}\} \times D_{\vec{s}s_3}$; and functions $f^{s_3}_r, \dots, f^{s_3}_k \in \widehat{\mathcal{O}^{\mathcal{R}}_{V_{\vec{s}s_3}}}$ such that $\{\bar{0}\} \times W''_{\vec{s}s_3} \subseteq \mathrm{cl}(\mathrm{Ift}_{e^{s_3}}(f^{s_3}_r, \dots, f^{s_3}_k))$, and such that $f^{s_3}_r(\bar{0}, a) = G'_{\rho(s_3)}(a)$ for all $a \in D_{\vec{s}s_3}$. In particular, $f^{s_3}_r$ is regular with respect to $e^{s_3}_r$ at every point of $\{\bar{0}\} \times W''_{\vec{s}s_3}$.

Moreover, by corollary 4.13, we may choose $V_{\vec{s}s_3}$ and $f^{s_3}_r, \dots, f^{s_3}_k$ so that in addition there is Ω_{s_3} open in $\mathbb{C}^{n(s_3)+n+m}$ such that every point of $\{\bar{0}\} \times W''_{\vec{s}s_3}$ is accessible in $\mathrm{Ift}_{e^{s_3}}$ along all $e' \in \Omega_{s_3}$.

(XII.) We have thus, for every member $W''_{\vec{s}s_3}$ of a partition of $W_I \cap A_\infty$, found $(f^{s_3}_r, \dots, f^{s_3}_k)$, $V_{\vec{s}s_3}$, $\{\bar{0}\} \times W''_{\vec{s}s_3}$, e^{s_3} and Ω_{s_3} which are in the same configuration as $(f_r, \dots, f_k)$, D, W'_{s_1}, e^{s_1} and Ω_I which we found at stage (II). Note especially that at least one of $(f^{s_3}_r, \dots, f^{s_3}_k)$ is regular in $e^{s_3}_r$ at every point of $\{\bar{0}\} \times W''_{\vec{s}s_3}$.

So we may now repeat stages (III) to (VI) of this argument. As before, we conclude that there is a partition of $\{0,0\} \times \{\bar{0}\} \times W''_{\vec{s}s_3} = W''_{\vec{s}1} \cup \dots \cup W''_{\vec{s}t_4}$ for which the basis $(e_1, \dots, e_{r-1}, e^{s_3}_r, \dots, e^{s_3}_k)$, and certain domains $D_{\vec{s}s_4} \subseteq \mathbb{C}^{2+n(s_3)+n+m+k}$, sets $\Omega_{\vec{s}s_4}$, and functions $(f_1, \dots, f_{r-1}, f^{\vec{s}s_4}_r, \dots, f^{\vec{s}s_4}_k)$ meet the claim for r.

(XIII.) And thirteenthly, we conclude by choosing $n_r \geq \max\{n(s_3) :$

$1 \le s_i \le t_i(s_1, \ldots, s_{i-1}),\ i = 1, 2, 3\}$ and embedding all the members $W'_{\vec{s}s_4}$ of the partition of $W_I \setminus A_\infty$, and the members $W''_{\vec{s}s_4}$ of the partition of $W_I \cap A_\infty$, in $\mathbb{C}^{n_r+m+k}$; and lifting the corresponding domains, sets, bases (extended canonically) and functions to this same space.

Labelling these members of a partition of $(\overbrace{0, \ldots, 0}^{n_r \text{ times}}) \times W$ sequentially as I varies completes the proof that the claim holds for r. By induction we are done.

Now, the claim for $r = k$ states that each W_I is contained in the closure of $\mathrm{Ift}(f^I) = \mathrm{Ift}_{e^I}(f^I)$; but $\partial(f^I)/\partial e^I$ is an upper triangular matrix with unit diagonal coefficients, so its determinant is never zero. Therefore $\mathrm{Ift}(f^I) = Z(f^I) = \mathrm{cl}(\mathrm{Ift}(f^I))$. In other words, f^I satisfies the hypotheses of the implicit function theorem at each point of W_I.

Moreover W_I is contained in the graph of the continuous extension of one of the $\tilde{g}_s$ as originally chosen on page 65 (lifted to the space $\mathbb{C}^{n_k} \times X_s$), so by the implicit function theorem W_I is contained in the graph of some g^I which is defined and holomorphic on an $\mathbb{R}_{an}$-definable domain $U_I \subseteq \pi_{n+m,k}(D_I)$ (and has continuous extension to $\mathrm{cl}(U_I)$).

And thus if we let $\hat{g} : (z, x) \mapsto g(x)$ denote the lifting of g to $\mathbb{C}^n \times U$, then $\langle k, f^I, g^I \rangle$ is an implicit representation of $\hat{g}$ on $\pi_{n+m,k}(W_I)$. This implicit representation is *over* $\widehat{\mathcal{O}^{\mathcal{R}}_{D_I}}$.

In particular, we have by part (4) of lemma 3.4 that $g^I \in \mathcal{O}^{\mathcal{R}}_{U_I}$, and the implicit representation of each g^I (over $\mathcal{R}$) extends continuously to $\mathrm{cl}(U_I)$ (by corollary 3.6). The function

$$\lambda(x_1, \ldots, x_m) \mapsto (\underbrace{0, \ldots, 0}_{n_k \text{ times}}, x_1, \ldots, x_m)$$

is affine, so by part (2) of lemma 3.4, $g^I \circ \lambda \in \mathcal{O}^{\mathcal{R}}_{\pi^*(U_I)}$ (where π^* is the projection inverse to λ, disposing of the z coordinates). The implicit representations extend to $\mathrm{cl}(\pi^*(U_I))$.

But $g^I \circ \lambda = \tilde{g}_s$ on $\pi^*(\pi_{m,k}(W_I))$, and in particular their first coordinates agree, and equal the function g. Writing $X^I = \pi^*(\pi_{m,k}(W_I))$, then, the implicit representation of g^I on X^I is also an implicit representation of g on this set, which has continuous extension to $\mathrm{cl}(X^I) \cap U$ since $\tilde{g}_s$ has continuous extension there. But $\bigcup_{1 \le I \le N_r} \{(x, g^I(x) : x \in X_I\} = W$, so $\bigcup_{1 \le I \le N_r} X_I = U \setminus V$.

And g is holomorphic on U by hypothesis. So finally by lemma 3.2 we see that indeed $g \in \mathcal{O}_U^{\mathcal{R}}$. □

Theorem 4.16 *If $f \in \mathcal{O}_U^{\mathcal{R}}$ where U is bounded and $\mathbb{R}_{an}$-definable, and the implicit representations of f extend continuously to $\mathrm{cl}(U)$, then each partial derivative of f $\partial f/\partial x_i \in \mathcal{O}_U^{\mathcal{R}}$.*

Proof. Let Γ be the open unit disk in $\mathbb{C}$. The function

$$g_i(x,t) = \frac{f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_m) - f(x)}{t}$$

defined on $V := U \times (\Gamma \setminus 0)$ certainly lies in $\mathcal{O}_V^{\mathcal{R}}$ by lemma 3.4 and corollary 3.5. The implicit representations extend continuously to $(t = 0)$, where $g_i(a, 0) = \partial f/\partial x_i(a)$; and g_i with this continuation is holomorphic on $U \times \Gamma$. So theorem 4.14 tells us that $g_i \in \mathcal{O}_{U \times \Gamma}^{\mathcal{R}}$. If we set $\lambda : U \to U \times \Gamma; x \mapsto (x, 0)$, then it is clear that $\partial f/\partial x_i = g_i \circ \lambda$ on U. Hence $\partial f/\partial x_i \in \mathcal{O}_U^{\mathcal{R}}$, as required. □

Corollary 4.17 *The ring of germs $\mathcal{O}_a^{\mathcal{R}} = \widehat{\mathcal{O}_a^{\mathcal{R}}}$.*

Proof. If $f \in \mathcal{O}_a^{\mathcal{R}}$ then there is U such that $a \in U$ and $\tilde{f} \in \mathcal{O}_U^{\mathcal{R}}$, where $\tilde{f}$ is the representative of the germ f on U. But then there is V, $a \in V \subseteq \mathrm{cl}(V) \subseteq U$ where V is bounded and $\mathbb{R}_{an}$-definable. Thus by theorem 4.16, all the partial derivatives of $\tilde{f} \restriction V$ lie in $\mathcal{O}_V^{\mathcal{R}}$. But these are the representatives in V of the partial derivatives of f, which themselves therefore lie in $\mathcal{O}_a^{\mathcal{R}}$. □

Chapter 5

The geometry of $\mathcal{O}^{\mathcal{R}}$

Recall that if $a \in \mathbb{C}^n$ we have defined $\mathcal{O}_a^{\mathcal{R}}$ to be the ring of germs of $\mathcal{R}$-holomorphic functions at a, and that $\mathcal{O}_a^{\mathcal{R}} \cong \mathcal{O}_n^{\mathcal{R}}$ (the ring of such germs at the origin in $\mathbb{C}^n$). We have for each n that $\mathcal{O}_n^{\mathcal{R}} \subseteq \mathcal{O}_n$, the ring of all germs of holomorphic functions at the origin; and $\mathcal{O}_0^{\mathcal{R}} = \mathcal{O}_0 = \mathbb{C}$.

In this chapter we shall introduce the class of $\mathcal{R}$-analytic sets, namely those sets defined at each point a as the common zeros in a sufficiently small neighbourhood of a of some germs in $\mathcal{O}_a^{\mathcal{R}}$, a subclass of the analytic sets. We prove a Nullstellensatz and the existence of irreducible sets in this class. In fact, the $\mathcal{R}$-analytic components of an $\mathcal{R}$-analytic set are exactly its analytic components.

Definition. Let $a \in \mathbb{C}^n$. A subset X of $\mathbb{C}^n$ is $\mathcal{R}$-*analytic at* a if there is a neighbourhood U of a, $k \in \mathbb{N}$, and functions $f_1, \dots, f_k \in \mathcal{O}_U^{\mathcal{R}}$ such that

$$X \cap U = Z(f_1, \dots, f_k) := \{z \in U : f_1(z) = \cdots = f_k(z) = 0\}.$$

An $\mathcal{R}$-*analytic germ at* a is the germ at a of a subset of $\mathbb{C}^n$ which is $\mathcal{R}$-analytic at a. An $\mathcal{R}$-analytic germ A is *irreducible* if it cannot be written as the union of two $\mathcal{R}$-analytic germs each properly contained in A.

A subset X of an open set $U \subseteq \mathbb{C}^n$ is $\mathcal{R}$-*analytic in* U (or a $\mathcal{R}$-analytic subset of U) if it is $\mathcal{R}$-analytic at every point $a \in U$ (but not necessarily at each point of the boundary ∂U). Such an X is also called a *locally* $\mathcal{R}$-*analytic* set. X is locally $\mathcal{R}$-analytic if and only if its germ at each $a \in X$ is $\mathcal{R}$-analytic.

Finally, a subset X of $\mathbb{C}^n$ is an $\mathcal{R}$-*analytic set* if it is $\mathcal{R}$-analytic at every point $a \in \mathbb{C}^n$; or equivalently, if its germ at each $a \in \mathbb{C}^n$ is $\mathcal{R}$-analytic; or equivalently, if X is locally $\mathcal{R}$-analytic and closed. Such an X is *irreducible* if it cannot be written as the union of two $\mathcal{R}$-analytic sets each properly contained in X.

These definitions are restrictions of the analogous concepts for holomorphic functions. Any (locally) $\mathcal{R}$-analytic set (germ) is (locally) analytic (an analytic germ). How closely the two classes are connected is expressed by the main theorem of this chapter:

Theorem 5.1 *If A is an $\mathcal{R}$-analytic germ (at $a \in \mathbb{C}^n$) and is irreducible as an $\mathcal{R}$-analytic germ, then A is irreducible as an analytic germ.*

It follows at once from this theorem that an $\mathcal{R}$-analytic *set* is irreducible if and only if it is irreducible as an analytic set.

We may similarly define the $\mathcal{R}$-analytic subsets of $\mathcal{R}$-*manifolds*; that is, of complex manifolds M such that M has an atlas Φ every transition map of which is $\mathcal{R}$-holomorphic on its domain. Given such an M the $\mathcal{R}$-holomorphic functions on M are those $f : M \to \mathbb{C}$ such that $f \circ \phi^{-1}$ is $\mathcal{R}$-holomorphic for each $\phi \in \Phi$. The definitions above carry through.

This definition of $\mathcal{R}$-holomorphic functions on M is not invariant over all atlases holomorphically equivalent to Φ: consider $\Psi = \{g \circ \phi : \phi \in \Phi\}$ where the bijection g is holomorphic on (a subset of) $\mathbb{C}^n$ but not $\mathcal{R}$-holomorphic. It is however invariant over $\mathcal{R}$-holomorphically equivalent atlases. Our definition of $\mathcal{R}$-analytic germs and subsets of M is therefore invariant over such atlases too.

To work with such manifolds is perhaps a distraction from our main objective of providing a framework for studying the projection map in the complex exponential field. Moreover the methods of this chapter are essentially local, and there is little to be gained by keeping in mind this additional abstraction. So I prefer to work with $\mathcal{R}$-analytic germs in $\mathbb{C}^n$ specifically. Nevertheless we shall require the concept of a *sub-$\mathcal{R}$-manifold* of $\mathbb{C}^n$, namely a (locally $\mathcal{R}$-analytic) subset of $\mathbb{C}^n$ which is an $\mathcal{R}$-manifold and for which the manifold structure is induced by an $\mathcal{R}$-holomorphic atlas of $\mathbb{C}^n$. That is, $X \subseteq \mathbb{C}^n$ is a sub-$\mathcal{R}$-manifold if for some $k \leq n$ every $a \in X$ has an open neighbourhood $U \subseteq \mathbb{C}^n$ for which there is a bijection $\phi \in (\mathcal{O}_U^{\mathcal{R}})^n$, such that $\phi : U \to V \subseteq \mathbb{C}^n$ and $\phi(a) = 0$, and such that $\phi(X)$ is an open subset of a k-dimensional vector subspace of $\mathbb{C}^n$.

A clarifying non-example is the subset $(w^2 = z^3) \subseteq \mathbb{C}^2$, which is an $\mathcal{R}$-manifold (being the bijective image of $\mathbb{C}$ under $t \mapsto (t^3, t^2)$), but is not a sub-$\mathcal{R}$-manifold at $(0, 0)$ for any $\mathcal{R}$. With the origin deleted this set becomes a sub-$\mathcal{R}$-manifold.

We start our work towards the Nullstellensatz and theorem 5.1 by developing some properties of $\mathcal{O}_a^{\mathcal{R}}$ analogous to those of $\mathcal{O}_a$. To the extent that this is possible the proofs follow Łojasiewicz, [8] chapter I.

Definition. A germ $f \in \mathcal{O}_a$ is *regular* if it is the germ of a function regular in z_n at $a \in \mathbb{C}^n$ and *distinguished* if it is the germ of a distinguished polynomial in $(z_n - a_n)$.

Lemma 5.2 *The ring $\mathcal{O}_a^{\mathcal{R}}$ is Noetherian and a unique factorization domain.*

Proof. It is sufficient to consider the case of $\mathcal{O}_n^{\mathcal{R}}$, and we prove both halves of the lemma by induction on n. Let $Q_n = \mathcal{O}_{n-1}^{\mathcal{R}}[z_n] \subseteq \mathcal{O}_n^{\mathcal{R}}$.

Clearly $\mathcal{O}_0^{\mathcal{R}} \cong \mathbb{C}$ is Noetherian; and assume that $n \geq 1$ and $\mathcal{O}_{n-1}^{\mathcal{R}}$ is. Then so is Q_n by the Hilbert basis theorem, since $Q_n \cong \mathcal{O}_{n-1}^{\mathcal{R}}[X]$.

Let I be a non-zero ideal of $\mathcal{O}_n^{\mathcal{R}}$. Any linear change of basis of $\mathbb{C}^n$ induces an automorphism of $\mathcal{O}_n^{\mathcal{R}}$, and for some such automorphism σ we have that σI contains a regular germ g, say. Now $J := \sigma I \cap Q_n$ is an ideal of Q_n and by the inductive hypothesis it is finitely generated, by $\{g_1, \ldots, g_s\}$, say.

Now by the Weierstrass Division Theorem for $\mathcal{O}_n^{\mathcal{R}}$, if $f \in \sigma I$ then $f = qg + r$, for some $q \in \mathcal{O}_n^{\mathcal{R}}$ and $r \in Q_n$. It follows that $r \in J$, so $r = q_1 g_1 + \cdots + q_s g_s$ for some $q_1, \ldots, q_s \in Q_n \subseteq \mathcal{O}_n^{\mathcal{R}}$. Thus σI is finitely generated, by $\{g, g_1, \ldots, g_s\}$; and since σ is an automorphism so is the ideal I. Thus all non-zero ideals of

$\mathcal{O}_n^{\mathcal{R}}$ are finitely generated, so $\mathcal{O}_n^{\mathcal{R}}$ is Noetherian. By induction this holds for each n.

In the light of this half of the lemma we may exploit the following characterization of unique factorization. See, for example, Matsumura [11], section 19.A.

Proposition 5.3 *A Noetherian integral domain R has unique factorization if and only if every irreducible element of R is prime.*

Again, $\mathcal{O}_0^{\mathcal{R}} \cong \mathbb{C}$ and so has unique factorization. Assume that $n \geq 1$ and that $\mathcal{O}_{n-1}^{\mathcal{R}}$ is a unique factorization domain. Then so is Q_n, by Gauss' Theorem, (as $Q_n \cong \mathcal{O}_{n-1}^{\mathcal{R}}[X]$).

Let f be an irreducible germ in $\mathcal{O}_n^{\mathcal{R}}$ and let $g, h \in \mathcal{O}_n^{\mathcal{R}}$ be such that f divides gh; so there is $r \in \mathcal{O}_n^{\mathcal{R}}$ with $rf = gh$. We may assume (by applying if necessary an automorphism of $\mathcal{O}_n^{\mathcal{R}}$ induced by a change of basis) that r, f, g and h are all regular. Write by the Weierstrass Preparation Theorem r^*, f^*, g^* and h^* for their associated distinguished germs in Q_n.

Then we have $r^*f^* = g^*h^*$. Indeed the two sides of this equality have quotient a unit of $\mathcal{O}_n^{\mathcal{R}}$, but are both distinguished germs; so by the statement of uniqueness in the Preparation Theorem the unit must be 1. Thus f^* divides g^*h^* in Q_n.

But f^* is irreducible in Q_n; for otherwise, if $f^* = c_1c_2$ in Q_n with neither c_1 nor c_2 a unit in Q_n, then the leading coefficients of c_1 and c_2 (as polynomials in z_n) are mutually inverse. So both c_1 and c_2 are of positive degree with leading coefficient non-zero at 0; so they are not invertible in $\mathcal{O}_n^{\mathcal{R}}$ either. This contradicts our choice of f.

Thus, by the proposition applied to Q_n, f^* divides g^* or f^* divides h^* in Q_n. So, multiplying by appropriate units of $\mathcal{O}_n^{\mathcal{R}}$, we see that f divides g or f divides h. Hence f is prime, and the inductive step is proved. □

Definition. Since lemma 5.2 tells us that all ideals of $\mathcal{O}_a^{\mathcal{R}}$ are finitely generated, we define for each ideal I its *locus*, namely the germ

$$V(I) := \big(Z(\tilde{f}_1, \dots, \tilde{f}_k)\big)_a$$

where $\tilde{f}_1, \dots, \tilde{f}_k$ are representatives on some neighbourhood of a of a generating set $\{f_1, \dots, f_k\}$ of I. Correspondingly if A is a $\mathcal{R}$-analytic germ at a we define the *ideal of* A

$$\mathcal{I}^{\mathcal{R}}(A) := \{f \in \mathcal{O}_a^{\mathcal{R}} : A \subseteq V(f)\}$$

of $\mathcal{R}$-analytic germs vanishing on A (equivalently, having some representative vanishing on a representative of A). I reserve the notation $\mathcal{I}(V)$ for the ideal of A in $\mathcal{O}_a$, the ring of all analytic germs at a.

Lemma 5.4 *A non-empty $\mathcal{R}$-analytic germ A is irreducible if and only if its ideal $\mathcal{I}^{\mathcal{R}}(A)$ is prime.*

Proof. If A is irreducible and $fg \in \mathcal{I}^{\mathcal{R}}(A)$, then $A \subseteq V(f) \cup V(g)$ and hence $A \subseteq V(f)$, so $f \in \mathcal{I}^{\mathcal{R}}(A)$; or $A \subseteq V(g)$ and $g \in \mathcal{I}^{\mathcal{R}}(A)$. Conversely, if $\mathcal{I}^{\mathcal{R}}(A)$ is prime and $A = B \cup C$ for germs of $\mathcal{R}$-analytic sets B, C then $\mathcal{I}^{\mathcal{R}}(A) = \mathcal{I}^{\mathcal{R}}(B) \cap \mathcal{I}^{\mathcal{R}}(C)$, by definition; but then $\mathcal{I}^{\mathcal{R}}(A) = \mathcal{I}^{\mathcal{R}}(B)$ or $\mathcal{I}^{\mathcal{R}}(A) = \mathcal{I}^{\mathcal{R}}(C)$ by primality, so $A = B$ or $A = C$. □

To motivate the proof of theorem 5.1 one may consider the simple case solved in the following lemma, which states that factorization in the class of polynomials in one variable over the $\mathcal{R}$-analytic germs is the same as in the analytic case. All the hard work has been done already.

Lemma 5.5 *Let $n \in \mathbb{N}$ and $p \in \mathcal{O}_n^{\mathcal{R}}[X]$ be a monic polynomial. Suppose that $p = qr$ where $q, r \in \mathcal{O}_n[X]$. Then $q, r \in \mathcal{O}_n^{\mathcal{R}}[X]$.*

Proof. Write $p = X^d + p_{d-1}X^{d-1} + \cdots + p_0$, and let $U \subseteq \mathbb{C}^n$ be a $\mathbb{R}_{an}$-definable neighbourhood on the origin on which $p_0, \dots, p_{d-1}$ have representatives $\tilde{p}_0, \dots, \tilde{p}_{d-1}$, respectively, which are holomorphic on a neighbourhood of the closure of U. Let k be the minimum number of distinct roots of $\tilde{p}(x, y) :=$

$y^d+\tilde{p}_{d-1}(x)y^{d-1}+\cdots+\tilde{p}_0(x)$ in a fibre $\pi^{-1}(x)$ for $x \in U$ (where $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $\pi : (x, y) \mapsto (x)$); write $\tilde{q}, \tilde{r} : U \times \mathbb{C} \to \mathbb{C}$ analogously. We have that $\tilde{p} = \tilde{q}\tilde{r}$ in the ring of holomorphic functions on $U \times \mathbb{C}$.

Then $V = \{x \in U : |\{y : \tilde{p}(x, y) = 0\}| = k\}$ is a Zariski open subset of U (it is the complement of $Z(\mathrm{Res}(\tilde{p}, \partial\tilde{p}/\partial y))$).

As in the proof of theorem 3.10 we may cover V with finitely many open sets $V_1, \ldots, V_r$, on each of which we may make a holomorphic selection of the roots of $\tilde{p}$, obtaining on each V_j functions $y_1(x), \ldots, y_k(x)$ with each $y_i \in \mathcal{O}^{\mathcal{R}}_{V_j}$ with continuous extension to $\mathrm{cl}(V_j) \cap U$.

Now as $\tilde{q}$ divides $\tilde{p}$ as holomorphic functions, there are for each V_j some $t_1, \ldots, t_k$ such that on $V_j \times \mathbb{C}$

$$\tilde{q}(x, y) = (y - y_1)^{t_1}(y - y_2)^{t_2} \ldots (y - y_k)^{t_k}$$

and so $\tilde{q} \in \mathcal{O}^{\mathcal{R}}_{V_j\times\mathbb{C}}$; this holds for each j. Therefore $\tilde{q} \in \mathcal{O}^{\mathcal{R}}_{V\times\mathbb{C}}$, and the implicit representations of $\tilde{q}$ extend continuously to $U \times \mathbb{C}$. By theorem 4.14 (applied on $U \times \{|y| < R\}$ where R is greater than the maximum modulus of any root of $\tilde{p}$ on U), then, $\tilde{q} \in \mathcal{O}^{\mathcal{R}}_{U\times\mathbb{C}}$. So now theorem 3.10 tells us that $q \in \mathcal{O}^{\mathcal{R}}_n[X]$, as we required. □

5.1 Normality and regularity of ideals

We now turn our attention to the Nullstellensatz. Necessarily the proof is similar to the analytic case, and there is a lot of machinery to develop. I follow the method and notation of Łojasiewicz, and many of the statements and proofs carry over without more than a slight change of notation; for these propositions I give references to the location of their analytic analogues in Łojasiewicz ([8]).

Łojasiewicz follows Hervé, after Cartan. Another presentation of the material is available in Nishino, following Oka. Nishino's proof does not separate out the Rückert descriptive lemma, although it is of course present. Proposition 5.14, not needed for the Nullstellensatz itself, is due to Łojasiewicz.

Since $\mathcal{O}_n^{\mathcal{R}}$ has unique factorization, it is integrally closed (in its field of fractions). We use this fact and Weierstrass division, following Łojasiewicz, to prove a lemma on separation of variables.

Let I be an ideal of $\mathcal{O}_n^{\mathcal{R}}$. We write $\hat{g} := g + I \in \mathcal{O}_n^{\mathcal{R}}/I$, and $\hat{\mathcal{O}_l^{\mathcal{R}}} = \{\hat{g} : g \in \mathcal{O}_l^{\mathcal{R}}\}$ for each $1 \leq l \leq n$. The factor map extends to polynomials over $\mathcal{O}_n^{\mathcal{R}}$: if $P = \sum a_\rho X^\rho \in \mathcal{O}_n^{\mathcal{R}}[X_1, \ldots, X_t]$, let $\hat{P} = \sum \hat{a}_\rho X^\rho$. Then we have that $P(g_1, \ldots, g_t) + I = \hat{P}(\hat{g}_1, \ldots, \hat{g}_t)$ for any $g_1, \ldots, g_t \in \mathcal{O}_n^{\mathcal{R}}$. If $I \cap \mathcal{O}_l^{\mathcal{R}} = 0$, then $\hat{\mathcal{O}_l^{\mathcal{R}}} \cong \mathcal{O}_l^{\mathcal{R}}$, and this isomorphism extends to $\hat{\mathcal{O}_l^{\mathcal{R}}}[X_1, \ldots, X_t] \cong \mathcal{O}_l^{\mathcal{R}}[X_1, \ldots, X_t]$.

Lemma 5.6 (Łojasiewicz, [8] III.2.2) *Let I be an ideal of $\mathcal{O}_n^{\mathcal{R}}$ and let $0 \leq k \leq n$. Then the following conditions are equivalent:*

(1) I contains a regular germ from $\mathcal{O}_l^{\mathcal{R}}$, for each $l = k+1, \ldots, n$;

(2) I contains a distinguished germ from $\mathcal{O}_k^{\mathcal{R}}[z_l]$ for each $l = k+1, \ldots, n$;

(3) $\hat{\mathcal{O}_n^{\mathcal{R}}} = \hat{\mathcal{O}_k^{\mathcal{R}}}[\hat{z}_{k+1}, \ldots, \hat{z}_n]$ and $\hat{z}_{k+1}, \ldots, \hat{z}_n$ are integral over $\hat{\mathcal{O}_k^{\mathcal{R}}}$;

(4) $\hat{\mathcal{O}_n^{\mathcal{R}}}$ is finite over $\hat{\mathcal{O}_k^{\mathcal{R}}}$ (i.e., generated over $\hat{\mathcal{O}_k^{\mathcal{R}}}$ by finitely many integral elements).

*Where one of these conditions holds I is said to be k-*normal.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are trivial.

Assume condition (1) and let $k+1 \leq l \leq n$. By the Preparation Theorem I contains a distinguished (and hence monic) germ from $\mathcal{O}_{l-1}^{\mathcal{R}}[z_l]$, so $\hat{z}_l$ is integral over $\hat{\mathcal{O}_{l-1}^{\mathcal{R}}}$. But by the Division Theorem $\mathcal{O}_l^{\mathcal{R}} \subseteq I + \mathcal{O}_{l-1}^{\mathcal{R}}[z_l]$; so $\hat{\mathcal{O}_l^{\mathcal{R}}} = \hat{\mathcal{O}_{l-1}^{\mathcal{R}}}[\hat{z}_l]$. Condition (3) follows by induction.

Now assume (4). As a purely algebraic fact, since $\hat{\mathcal{O}_n^{\mathcal{R}}}$ is finite over $\hat{\mathcal{O}_k^{\mathcal{R}}}$, each element of $\hat{\mathcal{O}_n^{\mathcal{R}}}$ is integral over $\hat{\mathcal{O}_k^{\mathcal{R}}}$. In particular, each $\hat{z}_l$ is integral over $\hat{\mathcal{O}_k^{\mathcal{R}}}$. Therefore the ideal I contains a monic (and hence z_l-regular) germ from $\mathcal{O}_k^{\mathcal{R}}[z_l]$. So, by the Preparation Theorem, I also contains a distinguished germ from $\mathcal{O}_k^{\mathcal{R}}[z_l]$. Thus condition (2) holds. $\square$

Corollary 5.7 *If I is a k-normal ideal of $\mathcal{O}_n^{\mathcal{R}}$ then I is generated by a finite subset of $\mathcal{O}_k^{\mathcal{R}}[z_{k+1}, \dots, z_n]$.*

Proof. Let $B = \mathcal{O}_k^{\mathcal{R}}[z_{k+1}, \dots, z_n]$. It is enough to prove that the ideal I' generated by $B \cap I$ contains I. By (2) above, I' is k-normal; and by (3) applied to I', we have $\mathcal{O}_n^{\mathcal{R}} = B + I'$. Therefore $I' \subseteq I \subseteq B + I'$; so (by a general algebraic proposition) $I \subseteq (B \cap I) + I' = I'$. □

Definition. An ideal I of $\mathcal{O}_n^{\mathcal{R}}$ is said to be *k-regular* (for $0 \leq k \leq n$) if I is k-normal and $\mathcal{O}_k^{\mathcal{R}} \cap I = 0$. Where such a k exists, I is said to be *regular*; then k is uniquely determined, since by equivalent (1) from lemma 5.6 it is the least k such that I is k-normal. Consistently with this definition we say that $\mathcal{O}_n^{\mathcal{R}}$ is (-1)-regular.

An $\mathcal{R}$-analytic germ A at $0 \in \mathbb{C}^n$ is k-normal (k-regular, regular) if its ideal $\mathcal{I}(A)$ in $\mathcal{O}_n^{\mathcal{R}}$ is k-normal (k-regular, regular).

After a suitable change of basis every ideal or $\mathcal{R}$-analytic germ at 0 is regular. The proofs, below, follow exactly as in the analytic case.

Proposition 5.8 (Łojasiewicz, [8] III.2.6) *A linear change ϕ of the coordinates $z_1, \dots, z_k$, or of $z_{k+1}, \dots, z_n$, preserves k-normality or k-regularity of an ideal of the ring $\mathcal{O}_n^{\mathcal{R}}$.*

Proof. Let $\psi_1 : \mathbb{C}^k \to \mathbb{C}^k$ and $\psi_2 : \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$ be linear bijections, and let Id_1 and Id_2 be the respective identity maps. Let $\phi = \psi_1 \times \mathrm{Id}_2$ or $\phi = \psi_2 \times \mathrm{Id}_1$. Then $\mathcal{O}_k^{\mathcal{R}} \circ \phi^{-1} = \mathcal{O}_k^{\mathcal{R}}$. Condition (4) above for an ideal I to be k-normal is equivalent to the condition that for some $g_1, \dots, g_m \in \mathcal{O}_n^{\mathcal{R}}$,

$$\mathcal{O}_n^{\mathcal{R}} = \sum_{1 \leq i \leq m} \mathcal{O}_k^{\mathcal{R}} g_i + I. \tag{5.1}$$

Applying the automorphism $f \mapsto f \circ \phi^{-1}$ to both sides of equation 5.1, we conclude that the ideal $I \circ \phi^{-1}$ is k-normal. Moreover if $\mathcal{O}_k^{\mathcal{R}} \cap I = 0$ then $\mathcal{O}_k^{\mathcal{R}} \cap (I \circ \phi^{-1}) = 0$ too. □

Proposition 5.9 (Łojasiewicz, [8] III.2.7) *Let $k \leq n$ and let $I_1, \ldots, I_m$ be ideals of $\mathcal{O}_n^{\mathcal{R}}$ all of which are k-normal. Then there is a change of the coordinates $(z_1, \ldots, z_k)$, ϕ say, such that each of $I_1, \ldots, I_m$ is regular with respect to the coordinates $\phi \times \mathrm{Id}_k$ (where Id_k denotes the identity map on $\mathbb{C}^{n-k}$); i.e., , each $I_j \circ (\phi \times \mathrm{Id}_k)^{-1}$ is regular. Indeed such a ϕ may be chosen from a dense subset of $L_0(\mathbb{C}^k, \mathbb{C}^k)$, the space of automorphisms of the vector space $\mathbb{C}^k$.*

Proof. For $0 \leq r \leq n$ define the condition

(c_r) : The ideal I is r-normal or s-regular for some $s \geq r$.

Let Id_r denote the identity map on $\mathbb{C}^{n-r}$, and suppose, for an induction, that $r > 0$ and some ideals $I'_1, \ldots, I'_m$ satisfy condition (c_r). Each of these ideals I'_j which is r-normal but not r-regular contains a non-zero germ from $\mathcal{O}_r^{\mathcal{R}}$. Applying corollary 4.5 we may find ϕ_r from a dense subset of $L_0(\mathbb{C}^r, \mathbb{C}^r)$ such that each of these germs is regular: that is, if $0 \neq g_j \in I'_j \cap \mathcal{O}_r^{\mathcal{R}}$, then $g_j \circ (\phi_r)^{-1}$ is regular with respect to z_r. (Since $\mathcal{O}_n^{\mathcal{R}}$ is Noetherian we need only consider finitely many such g_j for each j). Then, by equivalent (1) for $(r-1)$-normality, each of these ideals satisfies (c_{r-1}) with respect to the coordinates $\phi_r \times \mathrm{Id}_r$. Those I'_j which are either r-regular or not r-normal clearly satisfy (c_{r-1}) with respect to these coordinates too, by the preceding proposition 5.8.

Now proposition 5.8 shows also that the ideals $I_1, \ldots, I_m$ satisfy (c_k) with respect to $\phi_k \times \mathrm{Id}_k$ for any $\phi_k \in L_0(\mathbb{C}^k, \mathbb{C}^k)$. Fixing such a ϕ_k, we may find successively $\phi_{k-1}, \ldots, \phi_1$ so that the ideals $I_1, \ldots, I_m$ each satisfy condition c_0 with respect to the coordinates $(\phi_1 \circ \cdots \circ \phi_k) \times \mathrm{Id}_k$. Continuity of the composition map then implies that the set of all ϕ such that each I_j satisfies (c_0) with respect to $\phi \times \mathrm{Id}_k$ is dense in $L_0(\mathbb{C}^k, \mathbb{C}^k)$.

Note finally that every 0-normal ideal is either 0-regular or equal to $\mathcal{O}_n^{\mathcal{R}}$, and hence (-1)-regular. □

For any ideal I, $\mathcal{R}$-analytic germ A and affine bijection $\phi : \mathbb{C}^n \to \mathbb{C}^n$, it is immediate that $V(I \circ \phi^{-1}) = \phi(V(I))$ and $\mathcal{I}^{\mathcal{R}}(\phi(A)) = \mathcal{I}^{\mathcal{R}}(A) \circ \phi^{-1}$. Hence proposition 5.9 has an immediate corollary, referring to the regularity of $\mathcal{R}$-analytic germs at 0 (or at a), rather than of ideals.

Observe that every ideal of $\mathcal{O}_n^{\mathcal{R}}$, and hence every $\mathcal{R}$-analytic germ, is n-normal; so, given an arbitrary collection of finitely many ideals, they are simultaneously regular with respect to *some* coordinates.

5.2 Dimension for $\mathcal{R}$-analytic sets and the Rückert description

Definition. The *dimension* of a locally $\mathcal{R}$-analytic set (germ) is defined to be the largest k such that a neighbourhood of the origin in $\mathbb{C}^k$ can be embedded biholomorphically in the set (any representative of the germ) by an embedding drawn from $\mathcal{O}^{\mathcal{R}}$. Thus the dimension of a locally $\mathcal{R}$-analytic set is no greater than its dimension as a locally analytic set. (Recall that all $\mathcal{R}$-analytic sets are locally $\mathcal{R}$-analytic.)

Moreover the dimension, so defined, of a sub-$\mathcal{R}$-manifold coincides with its dimension as an analytic manifold. Consequently the rank theorem (proposition 5.21) can be interpreted as talking about this dimension. We shall use it once in the proof of Rückert's lemma but I defer the statement until section 5.4 below.

A *topographic sub-$\mathcal{R}$-manifold* Z in $\mathbb{C}^{k+n}$ is the graph of a mapping in $(\mathcal{O}_U^{\mathcal{R}})^n$ on U, a non-empty open subset of $\mathbb{C}^k$, with values in $\mathbb{C}^n$. Then $\dim Z = k$, and at every point $a \in Z$ the germ Z_a is irreducible of dimension k. If X is a locally $\mathcal{R}$-analytic subset of $\mathbb{C}^n$ and X is the graph of a holomorphic mapping on $U \subseteq \mathbb{C}^k$ (that is, X is a topographic submanifold in the analytic sense) then the implicit function theorem for $\mathcal{O}_k^{\mathcal{R}}$, lemma 3.4, implies that X is a topographic sub-$\mathcal{R}$-manifold.

A subset $X \subseteq \mathbb{C}^{k+n}$ is a *locally topographic sub-$\mathcal{R}$-manifold* if at every point $a \in X$ there is a neighbourhood U of a such that $X \cap U$ is a topographic $\mathcal{R}$-manifold.

A *Weierstrass set* in $\mathbb{C}^{k+n}$ is defined to be of form

$$\begin{aligned} W &= W(P_{k+1}, \dots, P_{k+n}) \\ &= \{z \in \mathbb{C}^{k+n} : (z_1, \dots, z_k) \in \Omega, P_{k+1}(z_1, \dots, z_k, z_{k+1}) = \dots \\ &\qquad \dots = P_{k+n}(z_1, \dots, z_k, z_{k+n}) = 0\} \end{aligned}$$

where Ω is a connected open neighbourhood of the origin in $\mathbb{C}^k$ and $P_{k+1}, \dots, P_{k+n}$ are distinguished polynomials in $\mathcal{O}^{\mathcal{R}}_{\Omega}[z_{k+1}], \dots, \mathcal{O}^{\mathcal{R}}_{\Omega}[z_{k+n}]$ respectively.

If W is a Weierstrass set then it is an $\mathcal{R}$-analytic subset of $\Omega \times \mathbb{C}^n$. The natural projection $\pi : W \to \Omega$ is proper, and $\pi^{-1}(0) = 0$. Its dimension $\dim W \leq k$ (see corollary 5.23, below).

Denote the union of the zero sets of the discriminants of $P_{k+1}, \dots, P_{k+n}$ by Z. Then Z is an $\mathcal{R}$-analytic subset of Ω. If none of the discriminants is identically zero, then Z is nowhere dense. By the implicit function theorem, the set $W_{\Omega \setminus Z} := \pi^{-1}(\Omega \setminus Z) \cap W$ is a locally topographic sub-$\mathcal{R}$-manifold of $(\Omega \setminus Z) \times \mathbb{C}^n$. Hence, if $Z \neq \Omega$, $\dim W = k$.

Theorem 5.10 (Rückert's Descriptive Lemma, Łojasiewicz III.3.1) *Let $A = V(I)$ be the locus of a k-regular prime ideal I of the ring $\mathcal{O}^{\mathcal{R}}_n$. Then there exist*

(i) an open connected neighbourhood of the origin $\Omega \subseteq \mathbb{C}^k$;

(ii) a representative X of the germ A, $\mathcal{R}$-analytic in $\Omega \times \mathbb{C}^{n-k}$; and

(iii) an $\mathcal{R}$-analytic nowhere dense subset Z of Ω

such that

1. *the natural projection $\pi : X \to \Omega$ is proper;*
2. *$X \cap \pi^{-1}(0) = 0$;*
3. *the set $X' := X \cap \pi^{-1}(\Omega \setminus Z)$ is a nonempty locally topographic sub-$\mathcal{R}$-manifold of $\Omega \times \mathbb{C}^{n-k}$.*

Thus $\pi(X') = \Omega \setminus Z$ and the restriction of the projection map $\pi \restriction X'$ is a finite covering of $\Omega \setminus Z$ of multiplicity > 0.

Proof. The ideal I is prime, so that the factor ring $\hat{\mathcal{O}}^{\mathcal{R}}_n$ is an integral domain; and k-regular, so that $\hat{\mathcal{O}}^{\mathcal{R}}_k \cong \mathcal{O}^{\mathcal{R}}_k$ and hence is a unique factorization domain. Thus by a general algebraic proposition every element of $\hat{\mathcal{O}}^{\mathcal{R}}_n$ which is integral over $\hat{\mathcal{O}}^{\mathcal{R}}_k$ has a unique minimal polynomial in $\hat{\mathcal{O}}^{\mathcal{R}}_k[T]$. By equivalent (4) for normality of I, all elements of $\hat{\mathcal{O}}^{\mathcal{R}}_n$ are such.

In particular, we first consider the elements $\hat{z}_j$, $j = k+1, \ldots, n$, of $\hat{\mathcal{O}}^{\mathcal{R}}_n$. Write their respective minimal polynomials as $\hat{p}_j \in \hat{\mathcal{O}}^{\mathcal{R}}_k[T]$, for what is (by the isomorphism mentioned above) a unique $p_j \in \mathcal{O}^{\mathcal{R}}_k[T]$. Then we have:

Claim 5.11 *The germs $p_j(z_j)$, $j = k+1, \ldots, n$ belong to the ideal I, are distinguished (in z_j respectively), and their discriminants are not identically zero.*

For since $\widehat{p_j(z_j)} = \hat{p}_j(\hat{z}_j) = 0$, certainly $p_j(z_j) \in I$. The polynomial $p_j[T]$ is irreducible in $\mathcal{O}^{\mathcal{R}}_k[T]$, so has a non-zero discriminant. By the preparation theorem, $p_j(z_j) = r_j(z_j)q_j(z_j)$ for some unit r of $\mathcal{O}^{\mathcal{R}}_{k+1}$ and distinguished polynomial $q_j(T)$ of $\mathcal{O}^{\mathcal{R}}_k[T]$. Then $\hat{q}_j(\hat{z}_j) = 0$ so the minimal polynomial $\hat{p}_j$ divides $\hat{q}_j$ in $\hat{\mathcal{O}}^{\mathcal{R}}_k[T]$, and hence $p_j(z_j)$ divides $q_j(z_j)$ in $\mathcal{O}^{\mathcal{R}}_k[z_j]$. But p_j is monic, so their quotient is 1, and $p_j(z_j)$ is therefore distinguished too.

Now by the corollary 5.7 there is a generating set

$$f_1(z_{k+1}, \ldots, z_n), \ldots, f_r(z_{k+1}, \ldots, z_n)$$

for I, where each $f_i \in \mathcal{O}^{\mathcal{R}}_k[Y_{k+1}, \ldots, Y_n]$; by the claim we may assume that this set contains $p_j(z_j)$ for each $j = k+1, \ldots, n$.

For any sufficiently small neighbourhood of the origin $\Omega \subseteq \mathbb{C}^k$, we can find representatives $\tilde{f}_1, \ldots, \tilde{f}_r$ of the f_i, each of which is a polynomial in $(z_{k+1}, \ldots, z_n)$ with coefficients in $\mathcal{O}^{\mathcal{R}}_\Omega$. Each of the minimal polynomials p_j has a representative $\tilde{p_j}$ equal to one of the $\tilde{f}_i$. Then $X := \{\tilde{f}_1 = \ldots \tilde{f}_r = 0\}$ is a representative of the germ A, $\mathcal{R}$-analytic on $\Omega \times \mathbb{C}^{n-k}$, and we see that

$$X \subseteq W = W(\tilde{p}_{k+1}, \ldots, \tilde{p}_n)$$

where W is a Weierstrass set. Moreover, if we are given any neighbourhood of the origin $\Delta \subseteq \mathbb{C}^{n-k}$, then for all Ω sufficiently small we have $X \subseteq W \subseteq \Omega \times \Delta$. For example, if $\Delta = \Delta_{k+1} \times \ldots \Delta_n$ is a polydisk, we may take Ω so that $\Omega \times \Delta_j$ is a preparation domain for each $\tilde{p}_j(z_1, \ldots, z_k, z_j)$ with respect to z_j;

Let $Z^* \subseteq \Omega$ denote the union of the zero sets of the discriminants of the $\tilde{p}_j$. Since $X \subseteq W$ it follows from the properties of Weierstrass sets that:

1. the natural projection $\pi : X \to \Omega$ is proper;
2. $X \cap \pi^{-1}(0) = 0$ (certainly $0 \in V$ as $I \neq \mathcal{O}_n^{\mathcal{R}}$);
3. the set $W_{\Omega \setminus Z^*} = W \cap \pi^{-1}(\Omega \setminus Z^*)$ is a locally topographic sub-$\mathcal{R}$-manifold of $\Omega \times \mathbb{C}^{n-k}$.

So it is enough to show that for suitably chosen Ω there is an $\mathcal{R}$-analytic $Z \supseteq Z^*$ which is nowhere dense in Ω and such that $X' := X \cap \pi^{-1}(\Omega \setminus Z)$ is nonempty and satisfies: for every $z \in X'$, the germ X'_z contains the germ of a sub-$\mathcal{R}$-manifold of dimension k at z. For at such a z we must have $X'_z = W_z$, since W_z is irreducible of dimension k, so X'_z is a topographic manifold at z.

Central to the argument is a thorough exploitation of the following algebraic proposition, "the primitive element theorem for integral domains". For a proof see, for example, Łojasiewicz [8] A.8.3.

Proposition 5.12 (Primitive element theorem) *Let R be a unique factorization domain of characteristic 0, and let S be an integral domain finite over R. Then there is a primitive element ζ of S over R, i.e., an element satisfying*

$$\forall \xi \in S \, \exists \alpha \in R \, (\alpha \neq 0 \wedge \alpha\xi \in R[\zeta]),$$

and if ζ is any primitive element of S then we may make a constant choice of $\alpha = \delta$, where $\delta \in R$ is the discriminant of the minimal polynomial of ζ over R, so

$$\delta S \subseteq R[\zeta].$$

Furthermore, if $S = R[\eta_1, \ldots, \eta_r]$ *for some* $\eta_1, \ldots, \eta_r \in S$*, and if* $R_1, \ldots, R_r$ *are any* infinite *subsets of the ring* R*, then there is a primitive element of the form* $\sum_1^r c_i \eta_i$ *where* $c_i \in R_i$*.*

The extension $\hat{\mathcal{O}}_n^{\mathcal{R}}$ of $\hat{\mathcal{O}}_k^{\mathcal{R}}$ satisfies the conditions of the primitive element theorem, so there is a primitive element of $\hat{\mathcal{O}}_n^{\mathcal{R}}$, of form $\hat{w}$ for some $w \in \mathcal{O}_n^{\mathcal{R}}$. The minimal polynomial for $\hat{w}$ over $\hat{\mathcal{O}}_k^{\mathcal{R}}$ may be written as $\hat{G} \in \hat{\mathcal{O}}_k^{\mathcal{R}}[T]$, where $G \in \mathcal{O}_k^{\mathcal{R}}[T]$. This G is uniquely determined as a representative of $\hat{G}$ and is an irreducible polynomial. Denote the discriminant of G by $\delta \in \mathcal{O}_k^{\mathcal{R}}$. Then $\hat{\delta}$ is the discriminant of $\hat{G}$, as we may see for example by direct calculation of $\mathrm{Res}(\hat{G}, \partial\hat{G}/\partial T)$. Since $\hat{\delta}$ is the discriminant of a minimal polynomial, $\hat{\delta}$ is not zero in $\hat{\mathcal{O}}_k^{\mathcal{R}}$; so $\delta \notin I$ and in particular $\delta \neq 0$. (Łojasiewicz distinguishes throughout between a representative δ of the discriminant of $\hat{G}$ and the discriminant δ_0 of its representative G; since he requires $\delta \in \mathcal{O}_k^{\mathcal{R}}$ it follows, as we have just seen, that $\delta = \delta_0$.)

By the property asserted of the primitive element $\hat{w}$ in the theorem, applied to $\hat{z_j}$ for each $j = k+1, \ldots, n$, therefore, we have that $\hat{\delta}\hat{z_j} = \hat{Q}_j(\hat{w})$ for some $Q_j \in \mathcal{O}_k^{\mathcal{R}}[T]$. Thus $\delta z_j - Q_j(w) \in I$, and so for some $a_{ij} \in \mathcal{O}_n^{\mathcal{R}}$,

$$\delta z_j - Q_j(w) = \sum_{i=1}^{r} a_{ij} f_i(z_{k+1}, \ldots, z_n). \tag{5.2}$$

Moreover $\hat{G}(\hat{w}) = 0$ implies $G(w) \in I$, so, for some $b_i \in \mathcal{O}_n^{\mathcal{R}}$,

$$G(w) = \sum_{i=1}^{r} b_i f_i(z_{k+1}, \ldots, z_n). \tag{5.3}$$

On the other hand, we may fix m large enough (at least the maximum of the degrees of the polynomials $f_i(Y_{k+1}, \ldots, Y_n)$) that for each $i = 1, \ldots, r$,

$$\delta^m f_i(z_{k+1}, \ldots, z_n) = F_i(\delta z_{k+1}, \ldots, \delta z_n) \tag{5.4}$$

for some $F_i \in \mathcal{O}_k^{\mathcal{R}}[Y_{k+1}, \ldots, Y_n]$. Then we evaluate

$$\hat{F}_i(\hat{Q}_{k+1}, \ldots, \hat{Q}_n)(\hat{w}) = \hat{F}_i(\hat{\delta}\hat{z_1}, \ldots, \hat{\delta}\hat{z_n}) = (\delta^m f_i(z_{k+1}, \ldots, z_n))\hat{} = 0$$

and see that the minimal polynomial $\hat{G}$ of $\hat{w}$ must divide $\hat{F}_i(\hat{Q}_{k+1}, \ldots, \hat{Q}_n)$ in $\hat{\mathcal{O}}_k^{\mathcal{R}}[T]$. That is, $\hat{F}_i(\hat{Q}_{k+1}, \ldots, \hat{Q}_n) = \hat{G}\hat{H}_i$ for some $H_i \in \mathcal{O}_k^{\mathcal{R}}[T]$. Once again we use the fact that $\mathcal{O}_k^{\mathcal{R}}[T] \cong \hat{\mathcal{O}}_k^{\mathcal{R}}[T]$ to deduce that $F_i(Q_{k+1},\ldots,Q_n) = GH_i$ and hence

$$F_i\big(Q_{k+1}(t),\ldots,Q_n(t)\big) = G(t)H_i(t) \tag{5.5}$$

where we have substituted into the formal polynomials the germ t at zero of the coordinate function $(z_1,\ldots,z_k,t) \mapsto t$ introducing a new variable t.

Now we may choose the neighbourhood $\Omega\times\Delta$ of the origin in $\mathbb{C}^n$, on which the f_i have representatives $\tilde{f}_i$ and such that $X \subseteq \Omega\times\Delta$, sufficiently small that the germs w, a_{ij}, b_i have representatives $\tilde{w}, \tilde{a}_{ij}, \tilde{b}_i \in \mathcal{O}^{\mathcal{R}}_{\Omega\times\Delta}$, the germ δ has representative $\tilde{\delta} \in \mathcal{O}^{\mathcal{R}}_{\Omega}\setminus\{0\}$, and the coefficients of the polynomials Q_j, G, H_i, F_i have representatives in $\mathcal{O}^{\mathcal{R}}_{\Omega}$.

Let $\vec{z}$ denote $(z_1,\ldots,z_k)$, and let $\tilde{Q}_j(\vec{z},t)$, $\tilde{G}(\vec{z},t)$, $\tilde{H}_i(\vec{z},t)$ and $\tilde{F}_i(\vec{z},z_{k+1},\ldots,z_n)$ be the polynomials (in t or $(z_{k+1},\ldots,z_n)$) whose coefficients are the representatives on Ω of the coefficients of Q_j, G, H_i, F_i respectively. In particular this means that $\tilde{\delta}$ is the discriminant of $\tilde{G}$.

Then equations (5.2)–(5.5) imply

$$\tilde{\delta}(\vec{z})z_j - \tilde{Q}_j(\vec{z},\tilde{w}(z)) = \sum_{i=1}^{r} \tilde{a}_{ij}(z)\tilde{f}_i(z) \text{ in } \Omega\times\Delta,\ j=k+1,\ldots,n, \tag{5.6}$$

$$\tilde{G}(\vec{z},\tilde{w}(z)) = \sum_{i=1}^{r} \tilde{b}_i(z)\tilde{f}_i(z) \text{ in } \Omega\times\Delta, \tag{5.7}$$

$$\tilde{\delta}(\vec{z})^m\tilde{f}_i(z) = \tilde{F}_i(\vec{z},\tilde{\delta}(\vec{z})z_{k+1},\ldots,\tilde{\delta}(\vec{z})z_n) \text{ in } \Omega\times\mathbb{C}^{n-k},\ i=1,\ldots,r, \tag{5.8}$$

and

$$\tilde{F}_i\big(\vec{z},\tilde{Q}_{k+1}(\vec{z},t),\ldots,\tilde{Q}_n(\vec{z},t)\big) = \tilde{G}(\vec{z},t)\tilde{H}_i(\vec{z},t) \text{ in } \Omega\times\mathbb{C},\ i=1,\ldots,r. \tag{5.9}$$

Now we set $Z = Z^*\cup\{\tilde{\delta}=0\}$. Then the set $X' = X\cap\pi^{-1}(\Omega\setminus Z)$ defined in the statement of the theorem can be written

$$X' = \{z\in\mathbb{C}^n : \vec{z}\in\Omega\setminus Z,\ \tilde{f}_i(z)=0, i=1,\ldots,r\}.$$

I claim that $X' = \pi^*(\Lambda)$ where

$$\Lambda = \{(z,t)\in\mathbb{C}^{n+1} : \vec{z}\in\Omega\setminus Z,\ \tilde{G}(\vec{z},t)=0, \tilde{\delta}(\vec{z})z_j = \tilde{Q}_j(\vec{z},t),\ j=k+1,\ldots,n\}$$

and $\pi^* : \mathbb{C}^{n+1} \to \mathbb{C}^n, (z,t) \mapsto z$. Certainly Λ is non-empty, since for any $\vec{z} \in \Omega \setminus Z$ there is $z \in \pi^{-1}(\{\vec{z}\}) \cap X$, and then $t = \tilde{w}(z)$ satisfies $\tilde{G}(\vec{z},t) = 0$. Then equation (5.6) implies that the other equations defining Λ are satisfied at $(\vec{z}, z_{k+1}, \ldots, z_n, t)$. And at any such point the function

$$\Phi : (\vec{z}, z_{k+1}, \ldots, z_n, t) \mapsto (\tilde{\delta}(\vec{z})z_{k+1} - \tilde{Q}_{k+1}(\vec{z},t), \ldots, \tilde{\delta}(\vec{z})z_n - \tilde{Q}_n(\vec{z},t), \tilde{G}(\vec{z},t))$$

satisfies the hypotheses of the implicit function theorem (note in particular that $\vec{z}$ is not in the zero set of the discriminant of G; so where $\tilde{G}(\vec{z},t) = 0$, $\partial\tilde{G}/\partial t(\vec{z},t) \neq 0$), and hence Λ is a locally topographic sub-$\mathcal{R}$-manifold of $\Omega \times \mathbb{C}^{n-k+1}$. Moreover if U is a small neighbourhood of any $(z,t) \in \Lambda$ then $\pi^* \restriction \Lambda \cap U$ is locally biholomorphic (and in $\mathcal{R}$), so the set $\pi^*(\Lambda)$ does indeed contain the germ of a sub-$\mathcal{R}$-manifold of dimension k at each point.

But now finally, if $(z,t) \in \Lambda$ it follows from equations (5.8) and (5.9) and the condition $(\tilde{\delta}(\vec{z}) \neq 0)$ that $\tilde{f}_i(z) = 0$, each i; and hence $z \in X'$. Conversely, if $z \in X'$ then we have seen that equations (5.6) and (5.7) imply that the point $(z, \tilde{w}(z)) \in \Lambda$. Hence $X' = \pi^*(\Lambda)$, and is therefore of the desired form. $\square$

Note that there exist a cover of $\Omega \setminus Z$ by finitely many $\mathbb{R}_{an}$-definable open sets $U_s \subseteq \Omega$, and implicit representations on U_s over $\mathcal{O}^{\mathcal{R}}_{\Omega\times\Delta\times\Gamma}$,

$$\langle n-k+1, \Phi, (\lambda_{s,l,k+1}, \ldots, \lambda_{s,l,n}, \lambda_{s,l,n+1})\rangle, \tag{5.10}$$

where $\Gamma \subseteq \mathbb{C}$ is $\mathbb{R}_{an}$-definable and sufficiently large to contain $\mathrm{cl}(\tilde{w}(\Omega))$, $\lambda_{s,l} = (\lambda_{s,l,k+1}, \ldots, \lambda_{s,l,n+1}) : U_s \to \Delta \times \mathbb{C}$, representing the defining functions of the locally topographic sub-$\mathcal{R}$-manifold

$$\Lambda = (\mathrm{Id}_k, \lambda_{s,1})(U_s) \cup \ldots (\mathrm{Id}_k, \lambda_{s,p})(U_s)$$

on U_s ($l = 1, \ldots, p$, where p is the multiplicity of the covering of $\Omega \setminus Z$ by X'). The domain U_s may be chosen small enough that these implicit representations extend to $\mathrm{cl}(U_s)$ (since $\mathrm{cl}(\Lambda \cap \pi^{-1}(U_s))$ is a subset of $\{H(\vec{z}, z_{k+1}, \ldots, z_n, t) = 0\}$, a closed subset of a neighbourhood of $\mathrm{cl}(\Omega \times \Delta \times \Gamma)$, and the representations extend to points on the boundary of this set). The first $n-k$ coordinates $\lambda^*_{s,l} = (\lambda_{s,l,k+1}, \ldots, \lambda_{s,l,n})$ are, in the same manner, defining functions of the locally topographic sub-$\mathcal{R}$-manifold X'.

Corollary 5.13 *Under the assumptions of the Descriptive Lemma, the image under the projection onto $\mathbb{C}^k$ of any representative of the germ $A = V(I)$ contains a neighbourhood of the origin.*

Proof. Let X be a representative of A on $\Omega \times \Delta$ as found in the Descriptive Lemma. Such an Ω can be taken to be arbitrarily small, and contains the origin. The projection $\pi : X \mapsto \Omega$ is proper, by conclusion (1) of the Lemma, and hence a closed map. That is, the projection of any subset of X which is closed in $\pi^{-1}(\Omega)$ is closed in Ω. But X is closed in $\Omega \times \Delta$ and $\pi(X) \cap \pi^{-1}(\Omega \setminus Z) = \Omega \setminus Z$. Hence

$$\pi(X) \supseteq \mathrm{cl}\big(\pi(X) \cap \pi^{-1}(\Omega \setminus Z)\big) \cap \Omega = \mathrm{cl}(\Omega \setminus Z) \cap \Omega = \Omega. \qquad \square$$

The Rückert Descriptive Lemma and corollary 5.13 give a sufficiently good description of the $\mathcal{R}$-analytic germs to prove the Nullstellensatz for $\mathcal{O}_n^{\mathcal{R}}$. To show that an irreducible $\mathcal{R}$-analytic germ is of constant analytic dimension we need a finer understanding of the behaviour of the representative X on the exceptional set $\pi^{-1}(Z)$.

Definition. Let $N \cong \mathbb{C}^n$ be a vector space. A polynomial mapping of form $P(\eta_1, \ldots, \eta_p, v) \to \mathbb{C}^s$ on the space N^{p+1} is called a *collector* if it is symmetric with respect to $\eta = (\eta_1, \ldots, \eta_p)$ and

$$P^{-1}(0) = \{(\eta, v) : v = \eta_1 \vee \ldots \vee v = \eta_p\}.$$

Recall that a closed nowhere dense subset Z of a complex manifold M is called *thin* if for every open set $\Omega \subseteq M$ any holomorphic function on $\Omega \setminus Z$ which is locally bounded near Z has holomorphic extension to Ω. Let M be a connected manifold and N be a vector space. A *quasi-cover* in $M \times N$ is a pair (Z, Λ) where Z is a thin subset of M and Λ is a closed locally topographic submanifold of $(M \setminus Z) \times N$ such that the natural projection $\pi : \mathrm{cl}(\Lambda) \to M$ is proper (where cl denotes closure in $M \times N$). Then every fibre of the projection $\pi : \Lambda \to M$ contains the same finite number of points, p, say; then we say that the quasi-cover is *p-sheeted*.

Proposition 5.14 (Łojasiewicz, "The first lemma on quasi-covers") *For every vector space N and $p \in \mathbb{N}$ there exists a collector $P : N^{p+1} \to \mathbb{C}^s$ (for some s; we may take $s \geq p \cdot p!$). Let (Z, Λ) be a p-sheeted quasi-cover on $M \times N$. Then there exists a unique holomorphic mapping $F = F_P : M \times N \to \mathbb{C}^s$ such that*

$$F(u,v) = P(\eta_1, \dots, \eta_p, v) \text{ where } \{\eta_1, \dots, \eta_p\} = \pi^{-1}(u) \cap \Lambda,$$

for each $(u, v) \in (M \setminus Z) \times N$. Then $\mathrm{cl}(\Lambda) = F^{-1}(0)$.

Indeed, we may write the collector $P(\eta, v) = \sum_{|\nu| \leq d} a_\nu(\eta) v^\nu$, with $a_\nu : N^p \to \mathbb{C}^s$ symmetric polynomial mappings. Then the mappings $c_\nu^0 : M \setminus Z \to \mathbb{C}^s$, well-defined by $c_\nu^0(u) = a_\nu(\eta_1, \dots, \eta_p)$ where $\{\eta_1, \dots, \eta_p\} = \pi^{-1}(u) \cap \Lambda$, are holomorphic and locally bounded near Z. So they have holomorphic extensions $c_\nu : M \to \mathbb{C}^s$, and the mapping

$$F(u,v) = \sum_{|\nu| \leq d} c_\nu(u) v^\nu$$

has the required property.

Thus *if* the quasi-cover (Z, Λ) of $\Omega \times \mathbb{C}^n$ satisfies: Ω and Z are both $\mathbb{R}_{an}$-definable, and Λ is a sub-$\mathcal{R}$-manifold such that at every point of Λ the graph witnessing that Λ is locally topographic at that point may be chosen from among finitely many $\mathcal{R}$-holomorphic mappings, and the implicit representations of these $\mathcal{R}$-holomorphic mappings defining Λ extend to $\mathrm{cl}(\Lambda) \subseteq \mathrm{cl}(\Omega \times \mathbb{C}^n)$; *then* theorem 4.14 (analytic continuation) implies that the function $F(u, v)$ of proposition 5.14 lies in $(\mathcal{O}_\Omega^{\mathcal{R}})^s$.

In the situation of the Descriptive Lemma the pair (Z, X') is a quasi-cover, and by the observation following its proof these additional conditions are also satisfied on sufficiently small $\Omega \times \Delta$. Hence the set $\tilde{X} = \mathrm{cl}(X \setminus \pi^{-1}(Z)) \cap (\Omega \times \Delta) = F^{-1}(0)$ is $\mathcal{R}$-analytic in $\Omega \times \Delta$; and since Z is $\mathcal{R}$-analytic, so is the set $X_Z = X \cap \pi^{-1}(Z)$. We have $X = \tilde{X} \cup X_Z$, so, taking germs at 0, $A = \tilde{X}_0 \cup (X_Z)_0$. But A is irreducible and $A \not\subseteq (X_Z)_0$ (by corollary 5.13); hence $A = \tilde{X}_0$. So, provided Ω is sufficiently small, there is equality between the representatives of A and $\tilde{X}_0$; that is, $X \subseteq \mathrm{cl}(X \setminus \pi^{-1}(Z))$. We have proved, therefore, the first part of the following statement.

Corollary 5.15 *In the Descriptive Lemma the set $\Omega \times \Delta$ may be chosen to be small enough that the sub-$\mathcal{R}$-manifold X' is dense in the representative X of A. In this case, moreover, X' is connected.*

Proof. It remains to show that X' is connected. If we suppose for a contradiction that we can write $X' = X'_1 \cup X'_2$, a disjoint union with X'_i non-empty and open in X', then (Z, X'_i) is a quasi-cover in $\Omega \times \mathbb{C}^{n-k}$. (Because in particular X' is closed in $(\Omega \setminus Z) \times \mathbb{C}^n$, and the projection $\pi \restriction X'$ is proper since it has uniformly bounded fibres, so $\pi(X')$ is closed in $\Omega \setminus Z$; but X' is a locally topographic sub-$\mathcal{R}$-manifold so its image under the projection is also open; and $\Omega \setminus Z$ is connected.)

Further, since X' is a locally topographic sub-$\mathcal{R}$-manifold and the $\mathcal{R}$-holomorphic mappings whose graphs cover X' have implicit representations extending continuously to $\mathrm{cl}(X')$, a subcollection of these mappings have graphs covering X'_i. Thus we can conclude from proposition 5.14 that $\mathrm{cl}(X'_i) = F_i^{-1}(0)$ for some $F_i \in (\mathcal{O}^{\mathcal{R}}_{\Omega})^s$.

But $X = \mathrm{cl}(X') = \mathrm{cl}(X'_1) \cup \mathrm{cl}(X'_2)$, and hence we can write the germ $A = X_0 = (F_1^{-1}(0))_0 \cup (F_2^{-1}(0))_0$. By the same reasoning as in corollary 5.13, as the X'_i contain points arbitrarily close to the origin in $\mathbb{C}^n$, neither of the germs of $\mathcal{R}$-analytic sets $(\mathrm{cl}(X'_i))_0 = (F_i^{-1}(0))_0$ is empty or contained in the other. Thus we have a proper decomposition of $A = V(I)$ by $\mathcal{R}$-analytic germs, contradicting (in view of lemma 5.4) our hypothesis that I is a prime ideal of $\mathcal{O}^{\mathcal{R}}_n$. $\square$

We can carry this analysis further with a lemma towards the theorem that the analytic set X^* of the singular points of an $\mathcal{R}$-analytic set X is itself $\mathcal{R}$-analytic.

Definition. Let $N \cong C^n$ be a vector space. A polynomial mapping of form $Q(\eta_1, \ldots, \eta_p, v) \to \mathbb{C}^s$ on the space N^{p+1} is called a 2-*collector* if it is symmetric with respect to $\eta = (\eta_1, \ldots, \eta_p)$ and

$$Q^{-1}(0) = \{(\eta, v) : \bigvee_{1\leq i\neq j\leq p} v = \eta_i = \eta_j\}.$$

Proposition 5.16 (Łojasiewicz, "The second lemma on quasi-covers") *For every vector space N and $p \in \mathbb{N}$ there exists a 2-collector $Q : N^{p+1} \to \mathbb{C}^{s'}$ (for some s'; we may take $s' \geq \frac{1}{2}p(p+1)p!$). Let (Z, Λ) be a p-sheeted quasi-cover on $M \times N$. Then there exists a unique holomorphic mapping $G : M \times N \to \mathbb{C}^s$ such that*

$$G(u, v) = Q(\eta_1, \ldots, \eta_p, v) \text{ where } \{\eta_1, \ldots, \eta_p\} = \pi^{-1}(u) \cap \Lambda,$$

for each $(u, v) \in (M \setminus Z) \times N$. The set Σ of points of $\mathrm{cl}(\Lambda) \subseteq M \times N$ *at which* $\mathrm{cl}(\Lambda)$ *is not a topographic manifold is analytic in $M \times N$, satisfying*

$$\Sigma = \mathrm{cl}(\Lambda) \cap G^{-1}(0).$$

If we return to the situation of the Descriptive Lemma and let $\Omega \times \Delta$ be a neighbourhood as found in the proof of corollary 5.15, then the same reasoning as for the First Lemma tells us that this $G \in (\mathcal{O}^{\mathcal{R}}_{\Omega})^{s'}$.

Corollary 5.17 *In the situation of corollary 5.15, the subset Σ of X of points at which X is* not *a topographic sub-$\mathcal{R}$-manifold is $\mathcal{R}$-analytic in $\Omega \times \Delta$.* □

5.3 The Nullstellensatz for $\mathcal{O}^{\mathcal{R}}$

Theorem 5.18 (Nullstellensatz) *Let $a \in \mathbb{C}^n$ and let I be an ideal of the ring $\mathcal{O}^{\mathcal{R}}_a$. Then $\mathcal{I}^{\mathcal{R}}(V(I)) = \mathrm{rad}(I)$.*

Proof. First, suppose I is prime, so $I = \mathrm{rad}(I)$. Evidently $I \subseteq \mathcal{I}^{\mathcal{R}}(V(I))$, so it is enough to show the opposite inclusion. We may assume that $a = 0$ and that I is k-regular for some $k \geq 0$. Then the corollary 5.13 to the Descriptive Lemma implies

$$\mathcal{O}^{\mathcal{R}}_k \cap \mathcal{I}^{\mathcal{R}}(V(I)) = 0. \tag{5.11}$$

For if $\tilde{g}$ is independent of $z_{k+1}, \ldots, z_n$ and vanishes on a representative of $V(I)$, then $\tilde{g}$ vanishes on Ω and hence is identically zero.

Now let $f \in \mathcal{O}_n^{\mathcal{R}} \setminus I$. Then $\hat{f} \in \hat{\mathcal{O}}_n^{\mathcal{R}} \setminus 0$, and since $\hat{\mathcal{O}}_n^{\mathcal{R}}$ is integral over $\hat{\mathcal{O}}_k^{\mathcal{R}}$, $\hat{f}\hat{g} \in \hat{\mathcal{O}}_k^{\mathcal{R}} \setminus 0$ for some $g \in \mathcal{O}_n^{\mathcal{R}}$. Thus

$$fg \in h + I \subseteq h + \mathcal{I}^{\mathcal{R}}(V(I))$$

for some $h \in \mathcal{O}_k^{\mathcal{R}} \setminus I$. But then (5.11) implies $h \notin \mathcal{I}^{\mathcal{R}}(V(I))$. Thus $fg \notin \mathcal{I}^{\mathcal{R}}(V(I))$ and so $f \notin \mathcal{I}^{\mathcal{R}}(V(I))$.

Consider now an arbitrary proper ideal I. Since $\mathcal{O}_a^{\mathcal{R}}$ is Noetherian, I has a primary decomposition $I = I_1 \cap \cdots \cap I_r$ (where each I_j is such that $\operatorname{rad}(I_j)$ is prime; there is a unique irredundant decomposition of I). Then as the ideals $\operatorname{rad} I_i$ are prime, we have $\mathcal{I}^{\mathcal{R}}(V(I)) = \bigcap_{i=1}^r \mathcal{I}^{\mathcal{R}}(V(I_i)) = \bigcap_{i=1}^r \mathcal{I}^{\mathcal{R}}(V(\operatorname{rad} I_i)) = \bigcap_{i=1}^r \operatorname{rad} I_i = \operatorname{rad} I$. □

Corollary 5.19 *1. The (non-empty) irreducible $\mathcal{R}$-analytic germs at a are exactly those germs of form $V(I)$ with I a prime ideal of $\mathcal{O}_a^{\mathcal{R}}$.*

2. If $k \geq 0$, then any k-regular $\mathcal{R}$-analytic germ at a is k-dimensional. Moreover its dimension as an analytic germ is also k.

3. If an $\mathcal{R}$-analytic germ at a is k-dimensional then it is k-regular with respect to some choice of coordinates (indeed, to any choice from a dense subset of $L_0(\mathbb{C}^n, \mathbb{C}^n)$).

4. Every $\mathcal{R}$-analytic germ has dimension equal to its dimension as an analytic germ.

Proof. Part (1) is immediate from theorem 5.18 and lemma 5.4.

For part (2), first suppose that A is irreducible, and so by part (1) there is some prime k-regular ideal I such that $A = V(I)$. Then, as in the proof of the descriptive lemma, any representative of the germ A is contained in a Weierstrass set $W(\tilde{p}_{k+1}, \ldots, \tilde{p}_n)$, so $\dim A \leq k$; but the descriptive lemma and corollary 5.13 imply that each representative of A contains a k-dimensional sub-$\mathcal{R}$-manifold. Hence $\dim A = k$.

Moreover, the dimension of the Weierstrass set as an analytic set is also less than or equal to k; so since the dimension of A as an analytic germ is no less than its dimension as an $\mathcal{R}$-analytic germ, the two dimensions agree.

In general, if $A = V(I)$ is an $\mathcal{R}$-analytic germ for I a k-regular ideal of $\mathcal{O}_a^{\mathcal{R}}$, we can write the primary decomposition of I as $I = I_1 \cap \cdots \cap I_r$. Each primary ideal I_j is k-normal by equivalent (2) of lemma 5.6, so by proposition 5.9 there is a change of the coordinates $z_1, \ldots, z_k$ under which they are each k_j-regular, for some $k_j \leq k$, respectively. If each $k_j < k$ then I would be $(k-1)$-normal, and so include a distinguished member of $\mathcal{O}_{k-1}^{\mathcal{R}}[z_k]$, contradicting k-regularity. Hence $k = \max(k_j) = \dim V(I)$.

Now part (3) follows from part (2) and proposition 5.9.

Finally, if A is an $\mathcal{R}$-analytic germ let $A = A_1 \cup \cdots \cup A_r$ be its decomposition into irreducible germs. Each of these is k_j-regular with respect to some coordinates, and of dimension k_j respectively (by (2) and (3)); then the dimension of A as an analytic germ equals $\max(k_j)$, by part (2). □

Lemma 5.20 *Let I be a prime ideal in $\mathcal{O}_n^{\mathcal{R}}$, and let $\langle I\mathcal{O}_n \rangle$ denote the ideal generated by I in $\mathcal{O}_n$. Then $I = \langle I\mathcal{O}_n \rangle \cap \mathcal{O}_n^{\mathcal{R}}$.*

Proof. I is prime in $\mathcal{O}_n^{\mathcal{R}}$, so by the Nullstellensatz,

$$\begin{aligned} I = \mathcal{I}^{\mathcal{R}}(V(I)) &= \{f \in \mathcal{O}_n^{\mathcal{R}} : f \restriction V(I) = 0\} \\ &= \mathcal{I}(V(I)) \cap \mathcal{O}_n^{\mathcal{R}}. \end{aligned}$$

But $\langle I\mathcal{O}_n \rangle \subseteq \mathcal{I}(V(I))$, so

$$I \subseteq \langle I\mathcal{O}_n \rangle \cap \mathcal{O}_n^{\mathcal{R}} \subseteq \mathcal{I}(V(I)) \cap \mathcal{O}_n^{\mathcal{R}} = I,$$

as required. □

Proof of theorem 5.1. Let A be an irreducible germ at $a \in \mathbb{C}^n$. We may assume that $a = 0$, and $A = V(I)$ where I is a k-regular prime ideal of $\mathcal{O}_n^{\mathcal{R}}$. Write $J = \langle I\mathcal{O}_n \rangle$, the ideal generated by I in $\mathcal{O}_n$.

Let $J = J_1 \cap \cdots \cap J_r$ be the primary decomposition of J, and suppose that $\mathrm{rad}(J_1), \ldots, \mathrm{rad}(J_{r'})$ are the minimal elements of $\{\mathrm{rad}(J_1), \ldots, \mathrm{rad}(J_r)\}$. Then $A = V(J_1) \cup \cdots \cup V(J_{r'})$, and this is an irredundant expression of A as a union of irreducible analytic germs.

Now let X be a representative of A as found in the Rückert lemma for I, on a sufficiently small set $\Omega \times \Delta$ that, by corollary 5.15, writing $X' = X \cap \pi^{-1}(\Omega \setminus Z)$ it is the case that $X = \mathrm{cl}(X') \cap (\Omega \times \Delta)$; and also that the generators of the J_i have representatives on $\Omega \times \Delta$ too. Then X is uniformly k-dimensional as an analytic set, since $X = \mathrm{cl}(X')$ and the analytic dimension of the sub-$\mathcal{R}$-manifold X' is equal to k. Hence each analytically irreducible germ $V(J_i)$, $i \leq r'$, is of dimension k, as their representatives on $\Omega \times \Delta$ are.

Thus by part (3) of corollary 5.19 and its analytic analogue (Łojasiewicz, [8], III.4.2), there is a dense subset of $L_0(\mathbb{C}^n, \mathbb{C}^n)$ such that each J_i ($i \leq r'$), as well as I, is k-regular with respect to any choice of coordinates in the subset. We may therefore assume without loss of generality that each J_i is indeed k-regular in $\mathcal{O}_n$, whence $\mathrm{rad}\, J_i$ is a k-regular prime ideal of $\mathcal{O}_n$.

So by Rückert's descriptive lemma in the analytic case there are, on sufficiently small $\Omega \times \Delta$ (on which the conclusions of corollary 5.15 for I also hold), representatives $X_i \subseteq X$ of $V(\mathrm{rad}\, J_i)$, and analytic nowhere dense subsets Z_i of Ω such that $(Z_i, X_i \cap \pi^{-1}(Z_i))$ is a quasi-cover in $\Omega \times \mathbb{C}^{n-k}$. Now let $Z' = Z \cup Z_1 \cup \cdots \cup Z_{r'}$, so this Z' is again an analytic nowhere dense subset of Ω, and therefore thin. Let $X'' = X \cap \pi^{-1}(Z')$ and for each i let $X_i'' = X_i \cap \pi^{-1}(Z')$. Then X'' is dense in X and connected, since X' is. (The proof of corollary 5.15 follows through with $Z := Z'$.)

Each X_i'' so defined is a closed subset of X'', since it is a subset and is analytic in $(\Omega \setminus Z_i)$; but X_i'' is also a locally topographic manifold and so for every point a of X_i'' there is a neighbourhood U of a such that $X_i'' \cap U$ is the graph of a holomorphic map; since $X_i'' \subseteq X''$ and X'' is also such a graph, we must have $X_i'' \cap U = X'' \cap U$ so X_i'' is also open in X''. Thus $X'' = X_i''$, so $X = \mathrm{cl}(X'') \cap (\Omega \times \Delta) \subseteq X_i$. Hence $X = X_i$ for each i; so taking germs $A = V(\mathrm{rad}\, J_i) = V(J_i)$, each i.

Thus there is exactly one germ in the irredundant analytic decomposition of A, and A is irreducible as an analytic germ. $\square$

5.4 Geometry for $\mathcal{R}$-analytic sets

We pass now from considering germs to sets. In this section we establish that if the class of countable unions of irreducible $\mathcal{R}$-analytic sets satisfies axiom (CP) then it satisfies all the quasi-Zariski axioms, and also show that the singular subset X^* of an $\mathcal{R}$-analytic set X is $\mathcal{R}$-analytic.

It will be necessary to consider, "locally", the projections of $\mathcal{R}$-analytic sets, in particular for axioms (AF) and (FC). The principal tool is the rank theorem, which we have already used once in proving lemma 5.10.

Definition. Recall that if $f : M \mapsto \mathbb{C}^n$ is a holomorphic mapping between manifolds, then the *rank of f at $a \in M$* is well defined by

$$\operatorname{rank}_a(f) = \operatorname{rank}(d_a(f)) = \operatorname{rank}\big(\frac{\partial(f \circ \phi^{-1})}{\partial z}(\phi(a))\big)$$

(where ϕ is a chart at a) and the *rank of f* by $\operatorname{rank}(f) = \max\{\operatorname{rank}_a(f) : a \in M\}$. This definition makes sense in particular if M is an $\mathcal{R}$-manifold and f is an $\mathcal{R}$-holomorphic function on M.

If $a \in M$ then $\operatorname{rank}_a(f) = \min\{\operatorname{rank}_z(f) : z \in U\}$ for some open neighbourhood U of a. It follows that $\operatorname{rank}_a(f)$ attains its maximum on some open subset M' of M. If $M' = M$ then f is of *constant rank*.

Proposition 5.21 (Rank theorem, Łojasiewicz C.4.1) *Let $f : M \mapsto N$ be a holomorphic mapping of constant rank r between manifolds, and let $a \in M$. Then, for some sufficiently small neighbourhoods U of a and V of $f(a)$, there exist charts $\phi : U \to \mathbb{C}^m$, $\psi : V \to C^n$, such that*

$$\psi \circ (f \restriction U) \circ \phi^{-1} = \lambda \restriction \phi(U)$$

where $\lambda : \mathbb{C}^m \to \mathbb{C}^n$ is a linear mapping of rank r. Moreover $f(U)$ is an r-dimensional submanifold of N and the fibres of $f \restriction U$ are $(m-r)$-dimensional submanifolds of M.

Indeed we may suppose $N = \mathbb{C}^n$ and then if $p : \ker d_a(f) \to N$ and $q : N \to \operatorname{im} d_a(f)$ are projections then we may take $\phi = (p, q \circ f) \restriction U$ and $\psi = \mathrm{Id}_{\mathbb{C}^n}$. Hence if f is $\mathcal{R}$-holomorphic on the $\mathcal{R}$-analytic manifold M, then the mappings ϕ and ψ are in $\mathcal{O}^{\mathcal{R}}_U$ and $\mathcal{O}^{\mathcal{R}}_V$ respectively.

Lemma 5.22 *Let $X \subseteq \mathbb{C}^{m+n}$ be locally $\mathcal{R}$-analytic and consider the natural projection $\pi : \mathbb{C}^{m+n} \to \mathbb{C}^m$. If $l \in \mathbb{N}$ and the dimension of the fibre* $\dim(\pi^{-1}(z) \cap X)$ *is bounded above by l for all $z \in \pi(X)$, then $\pi(X)$ contains a sub-$\mathcal{R}$-manifold of dimension r for some* $r \geq \dim X - l$.

Proof. Let $\Gamma \subseteq \mathbb{C}^{m+n}$ be a sub-$\mathcal{R}$-manifold contained in X and consider $a \in \Gamma$ such that $\mathrm{rank}_a(\pi \restriction \Gamma)$ is maximal. Then on some neighbourhood of a in Γ this mapping has constant rank; so there is an open subset U of Γ containing a such that $\pi(U)$ is a sub-$\mathcal{R}$-manifold and the fibres of $\pi \restriction U$ are sub-$\mathcal{R}$-manifolds of dimension $\dim(U) - \dim(\pi(U)) \leq l$.

Thus $\dim(\pi(U)) \geq \dim(\Gamma) - l$. If we take Γ to be of maximal dimension it follows that $\dim \pi(U) \geq \dim X - l$. □

Corollary 5.23 *Let $W = W(P_{k+1}, \ldots, P_{k+n}) \subseteq \Omega \times \mathbb{C}^n$ be a Weierstrass set in $\mathbb{C}^{k+n}$. Then* $\dim W \leq k$.

Proof. Let $\pi : \mathbb{C}^{k+n} \to \mathbb{C}^k$. For any $z = (z_1, \ldots, z_k) \in \pi(W)$, the fibre of W above z contains only finitely many points and hence is zero-dimensional. Any locally $\mathcal{R}$-analytic subset of $\pi(W)$ is contained in $\mathbb{C}^k$ and is therefore of dimension no greater than k. By the lemma, therefore, $\dim W \leq k + 0 = k$. □

Hitherto in this section we have naturally made no use of the results of section 5.2. Now we return to considering the consequences of theorem 5.1.

Definition. Let $X \subseteq \mathbb{C}^n$ be locally analytic. Recall that the points of X at which it is a submanifold (of dimension k) are called *regular (regular of dimension k)*. Those points at which X is not regular are called *singular*. The sets X^0, $X^{(k)}$, X^* are respectively the subsets consisting of the regular points, the regular points of dimension k, and the singular points of X. If $g : X \to \mathbb{C}^m$ is holomorphic then we define $\operatorname{rank} g = \operatorname{rank}(g \restriction X^0)$.

More technically, we define a *k-complete* sequence of linear coordinate systems $\phi_1, \ldots, \phi_r$ for a vector space $M \cong \mathbb{C}^n$ as follows: if $(e_1, \ldots, e_n)$ is the canonical basis for $\mathbb{C}^n$, and $\phi \in L_0(M, \mathbb{C}^n)$ is a coordinate system, let

$$\lambda(\phi) = \phi^{-1}(e_{k+1}) \wedge \ldots \wedge \phi^{-1}(e_n)$$

in the exterior product space $\bigwedge^{n-k} M$. Then the sequence $\phi_1, \ldots, \phi_r$ is k-complete if $\lambda(\phi_1), \ldots, \lambda(\phi_r)$ generate $\bigwedge^{n-k} M$.

Proposition 5.24 (Łojasiewicz, IV.2.3) *For every finite-dimensional vector space $\mathbb{C}^n$ and every $k \leq n$, a k-complete sequence of coordinate systems exists of some length r; then the k-complete sequences form an open dense subset of $(L_0(\mathbb{C}^n, \mathbb{C}^n))^r$. If a set $X \subseteq \mathbb{C}^n$ is a k-dimensional submanifold at a point $a \in X$, then for any k-complete sequence, the set X is a topographic submanifold at a in one of the systems of the sequence.*

We have already observed that the implicit function theorem for $\mathcal{O}^{\mathcal{R}}$ implies that wherever a locally $\mathcal{R}$-analytic set is a topographic submanifold, it is a topographic sub-$\mathcal{R}$-manifold.

Theorem 5.25 *Let $X \subseteq \mathbb{C}^n$ be locally $\mathcal{R}$-analytic. Then X^* is locally $\mathcal{R}$-analytic, and if X is $\mathcal{R}$-analytic then so is X^*. Moreover* $\dim X^* < \dim X$ *if X is non-empty.*

Proof. It is enough to show that in some neighbourhood U of each $a \in X$, the set $X^* \cap U$ is $\mathcal{R}$-analytic in U of dimension $< k$.

First suppose the germ X_a is irreducible of dimension k. Then there is a k-complete sequence $\phi_1, \ldots, \phi_r$ of coordinate systems such that X_a is k-regular with respect to each ϕ_j (by proposition 5.24 and (3) of corollary 5.19). So we may apply, for each $1 \leq j \leq r$, the descriptive lemma and corollary 5.17 of the second lemma on quasi-covers to the germ $\phi_j(X_a)$. We get a neighbourhood $U_j = \phi_j^{-1}(\Omega_j \times \Delta_j)$ and a representative W_j of X_a on U_j such that the set Σ_j

of points at which W_j is not a k-dimensional topographic sub-$\mathcal{R}$-manifold with respect to the coordinate system ϕ_j is $\mathcal{R}$-analytic in U_j.

On any sufficiently small neighbourhood $U \subseteq \bigcap_{j=1}^{r} U_j$, each of the representatives $W_j \cap U$ coincides with $X \cap U$. Proposition 5.24 then implies that

$$X^* \cap U = \bigcap_{j=1}^{r} (\Sigma_j \cap U)$$

and so $X^* \cap U$ is $\mathcal{R}$-analytic. Since $X^* \cap U \subsetneq X \cap U$ and U has been chosen small enough that $X \cap U$ is irreducible of dimension k, we must have in this case that $\dim(X^* \cap U) < k$. Indeed, the rank theorem tells us that any Σ_j is of dimension less than k, since the projection of $\phi_j(\Sigma_j)$ onto Ω_j is nowhere dense in Ω_j (and hence contains no k-dimensional submanifold), while the fibres of this projection are discrete.

Now for a general X_a, we may decompose X_a irredundantly as a union of irreducible germs $X_a = (X_1)_a \cup \ldots (X_s)_a$, finding U small enough that X_i^* is $\mathcal{R}$-analytic on U for each i. We may ensure also that the representatives X_i of $(X_i)_a$ are irreducible in U and contain a connected dense sub-$\mathcal{R}$-manifold X_i^0 (by corollary 5.15 of the first lemma on quasi-covers, after a suitable change of coordinates).

Then I claim that

$$X^* \cap U = \bigcup_{i=1}^{s} (X_i^*) \cup \bigcup_{1 \le i < j \le s} (X_i \cap X_j). \tag{5.12}$$

Clearly if $b \in X \cap U$ does not lie in the set described by the right hand side of equation (5.12), then $X_b = (X_i)_b$ for some unique i and so X_b is the germ of an $\mathcal{R}$-manifold. So the inclusion "$\subseteq$" holds.

For the converse, note first that if $j \neq i$ then the intersection $X_i \cap X_j$ is nowhere dense in X_j; since otherwise the locally $\mathcal{R}$-analytic set $X_i \cap X_j^0$ has nonempty interior in the connected $\mathcal{R}$-manifold X_j^0 and so $X_j^0 \subseteq X^i$; thus $X_j \subseteq X_i$, contradicting the irredundancy of the decomposition $(X_1, \ldots, X_s)$.

So if $b \in X_i \cap X_j$, in any neighbourhood V of b the set X contains points in $X_i \backslash X_j$ and $X_j \backslash X_i$. If $X \cap V$ were a sub-$\mathcal{R}$-manifold, and assuming without loss that X_i is of maximal dimension with $b \in X_i$, then $X_i \cap V$ would be a locally

$\mathcal{R}$-analytic subset of $X \cap V$ with nonempty interior, and so $X_i \cap V = X \cap V$; a contradiction. Moreover if $b \in X_i^* \setminus \bigcup_{j \neq i} X_j$, then X is not a sub-$\mathcal{R}$-manifold in any neighbourhood of b. Thus the inclusion "$\supseteq$" holds also, and the claim is proved.

Thus $X^* \cap U$ is an $\mathcal{R}$-analytic subset of U. What is more, since for each $i \neq j$ it is the case that

$$X_i \cap X_j \subseteq ((X_i \cap X_j) \cap X_j^0) \cup X_j^*,$$

and the intersection $(X_i \cap X_j)$ is nowhere dense in the sub-$\mathcal{R}$-manifold X_j^0, it follows that $\dim(X_i \cap X_j) < \dim X_j$. Thus we see from the first part of the proof and equation (5.12) that $\dim X^* \cap U < k$. □

Instead of giving the whole proof of theorem 5.25 it would of course have been enough to demonstrate the first case, with X_a irreducible, and then to have appealed to the corresponding theorem for analytic sets in the light of theorem 5.1. Similarly the following propositions would follow for the $\mathcal{R}$-analytic case by routine arguments from theorem 5.25 and lemmas we have already proved.

Proposition 5.26 (Structure theorem) *Let $X \subseteq \mathbb{C}^n$ be analytic. The analytic components of X are the sets of form $\mathrm{cl}(Y)$ where Y is a connected component of X^0.*

Hence (by theorem 5.1), if X is $\mathcal{R}$-analytic, these are its irreducible components. They form a locally finite family. A connected component of X^0 is a connected component of $X^{(k)}$ for some k.

Proposition 5.27 *Let $W \subseteq X$ be analytic sets and suppose X is irreducible. Then*

$$W \subsetneq X \leftrightarrow (W \textit{ is nowhere dense in } X) \leftrightarrow \dim W < \dim X.$$

Definition. Let $\mathcal{C}_n(\mathcal{R})$ denote the collection of all countable unions of irreducible $\mathcal{R}$-analytic subsets of $\mathbb{C}^n$, and let $\mathcal{C}(\mathcal{R}) = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$.

If $X \in \mathcal{C}$, define $\dim X$ to be the greatest k such that X contains a sub-$\mathcal{R}$-manifold of dimension k as a subset; if X is $\mathcal{R}$-analytic then this is its dimension as a $\mathcal{R}$-analytic set (as defined at the start of section 5.2).

Theorem 5.28 *The triple $\langle \mathbb{C}, \mathcal{C}(\mathcal{R}), \dim \rangle$ satisfies all the axioms to be a QZ-structure except possibly (CP) and (AF). The irreducible $\mathcal{R}$-analytic sets are irreducible in the sense of the QZ geometry. If (CP) is satisfied then so is (AF).*

Proof. Let $X \subseteq \mathbb{C}^n$ be a $\mathcal{R}$-analytic irreducible set, and let $\{W_i : i \in \omega\}$ be a countable collection of $\mathcal{R}$-analytic subsets of X with $\bigcup_{i\in\omega} W_i = X$. Then $X = W_i$ for some i; for if any W_i is a proper subset of X then W_i is nowhere dense in X, by proposition 5.27. But X is a Baire space (being a locally compact Hausdorff space); so a countable union of nowhere dense subsets of X is nowhere dense in X. If $X = \bigcup_{j\in\omega} X_j$ with each $X_j \in \mathcal{C}(\mathcal{R})$, then each $X_j = \bigcup_{i\in\omega} W_{ji}$ for some countable collection of $\mathcal{R}$-analytic subsets of X $\{W_{ji} : j, i \in \omega\}$; so $X = X_j$ for some j. Thus X is irreducible in $\mathcal{C}(\mathcal{R})$.

Now the first six axioms are straightforward to check.

1 (L) *Language.* Evidently the singleton sets $\{a\}$ are $\mathcal{R}$-analytic; indeed if $a \in \mathbb{C}$ then $z - a \in \mathcal{R}$. $\mathbb{C}$ itself is $\mathcal{R}$-analytic; as is the graph of $(z_i - z_j)$ in $\mathbb{C}^n$. If $X, Y \in \mathcal{C}(\mathcal{R})$ then we may write $X = \bigcup_{i\in\omega} X_i$, $Y = \bigcup_{j\in\omega} Y_j$ with the X_i and Y_j $\mathcal{R}$-analytic; and the intersection, union, and cartesian product of X with Y are respectively countable unions of the intersections, unions, and products of the X_i and Y_j; which are in turn $\mathcal{R}$-analytic.

2 (IC) *Irreducible components.* We have shown that any irreducible $\mathcal{R}$-analytic set is irreducible in $\mathcal{C}(\mathcal{R})$; after this identification, axiom (IC) is immediate from the definition of $\mathcal{C}(\mathcal{R})$.

3 (CU) *Countable union.* This is also immediate from the definition.

4 (DP) *Dimension of a point.* A singleton set $\{a\}$ contains the zero-dimensional manifold $\{a\} \cong \mathbb{C}^0$ as a subset; this is clearly maximal.

5 (IM) *Irreducible model.* The domain $\mathbb{C}$ is irreducible of dimension 1.

6 (DU) *Dimension of unions.* Let $X = \bigcup_{i\in\omega} X_i \in \mathcal{C}_n(\mathcal{R})$; then clearly $\dim X \geq \max\dim(X_i)$. Suppose $M \subseteq X$ is an $\mathcal{R}$-analytic manifold of dimension $k > \max\dim(X_i)$. Then M is $\mathcal{R}$-analytic in some neighbourhood U of M in $\mathbb{C}^n$; we can find an irreducible $\mathcal{R}$-analytic component M_1 of M with dimension k. Some $X_i \cap M_1$ has nonempty interior in the Baire space M_1. But then $M_1 \subseteq (X_i \cap U)$, a contradiction. Hence $\dim X = \max\dim(X_i)$.

7 (DI) *Dimension of irreducible sets.* Let $W \in \mathcal{C}(\mathcal{R})$ be a proper subset of the irreducible set X; and let $W = \bigcup_{i\in\omega} W_i$ be an expression of W as a union of $\mathcal{R}$-analytic sets. Then proposition 5.27 implies that $\dim W_i < \dim X$; so by axiom (DU) $\dim W < \dim X$.

We skip axiom 8! Axioms (FC) and (AF) require more work.

Lemma 5.29 *Let $X \subseteq \mathbb{C}^n$ be an irreducible $\mathcal{R}$-analytic set of dimension k, and let $g : V \to \mathbb{C}^m$ be an $\mathcal{R}$-holomorphic map on some open $V \supseteq X$. Let $r \in \mathbb{N}$ and define*

$$C(X, g, r) = \{z \in X^0 : \mathrm{rank}_z(g \restriction X^0) \leq r\} \cup X^*.$$

Then this set $C(X, g, r)$ is $\mathcal{R}$-analytic.

Proof. Note that $g\restriction X_0$ is a mapping of manifolds so the definition of $C(X, g, r)$ makes sense; write $C = C(X, g, r)$. Clearly (since the rank is a lower-semicontinuous function) C is a closed set, so it is enough to show that for any $a \in X$ there is a neighbourhood U of a in which C is $\mathcal{R}$-analytic.

Fix such an a and assume, without loss, that the $\mathcal{R}$-analytic germ X_a is k-regular. So the Rückert descriptive lemma applies to X_a and (by corollary 5.15) there exist a neighbourhood $U = \Omega \times \Delta$ of a, and a nowhere dense $\mathcal{R}$-analytic subset Z of Ω, such that $X' = X \cap \pi^{-1}(\Omega \setminus Z)$ is a dense sub-$\mathcal{R}$-manifold of $X^0 \cap U$. Indeed there is an $\mathcal{R}$-holomorphic atlas for X' whose charts are of form

$$(\mathrm{Id}_k, \lambda^*_{s,l})^{-1} = (z_1, \ldots, z_k, \lambda_{s,l,k+1}, \ldots, \lambda_{s,l,n})^{-1} : \mathrm{graph}(\lambda^*_{s,l}) \to U_s$$

where the functions $\lambda_{s,l}$ and sets U_s are those of the implicit representations of Λ given by formula (5.10) on page 88.

If Σ is the set of points at which X is not a topographic submanifold, then $X' \subseteq (X \setminus \Sigma) \subseteq X^0 \subseteq \mathrm{cl}(X')$. Every point of $X \setminus \Sigma$ is in the closure of the graph of some $\lambda^*_{s,l}$; such a function $\lambda^*_{s,l}$ has holomorphic extension to a neighbourhood of this point.

Now if $b \in X'$ then $X'_b = X^0_b$, so $\mathrm{rank}_b(g \restriction X^0) = \mathrm{rank}_b(g \restriction X')$. Recall that the function $\tilde{w} : U \to \mathbb{C}$ is such that $(z, \tilde{w}(z)) \in \Lambda$ for each $z \in U$. Pick any s, l for which $(b, \tilde{w}(b))$ is in the graph of $\lambda_{s,l}(U, s)$. Then, considering the chart corresponding to s, l in the definition of $\mathrm{rank}_z(g \restriction X')$, we see that

$$\begin{aligned} \mathrm{rank}_b(g \restriction X') &= \mathrm{rank}\Big(\frac{\partial(g \circ (\mathrm{Id}_k, \lambda^*_{s,l}))}{\partial \vec{z}}(\vec{b})\Big) \\ &= \mathrm{rank}\left(\Big(\frac{\partial g}{\partial z}(b)\Big)\Big(\frac{\partial \lambda^*_{s,l}}{\partial \vec{z}}(\vec{b})\Big)\right). \end{aligned} \tag{5.13}$$

But we know from the implicit representation (5.10) that $\lambda^*_{s,l}$ is (the first $n-k$ coordinates of) the function found by applying the implicit function theorem to Φ at the point $(\vec{b}, \lambda_{s,l}(\vec{b})) = (b, \tilde{w}(b))$, and hence

$$\begin{aligned} \frac{\partial \lambda^*_{s,l}}{\partial \vec{z}}(\vec{b}) &= \pi^* \left(\frac{\partial \Phi}{\partial z_{k+1}, \ldots, z_n, t}(b, \tilde{w}(b))\right)^{-1} \left(\frac{\partial \Phi}{\partial \vec{z}}(b, \tilde{w}(b))\right) \\ &= \frac{1}{D(b)} \cdot \pi^* \,\mathrm{adj}\left(\frac{\partial \Phi}{\partial z_{k+1}, \ldots, z_n, t}(b, \tilde{w}(b))\right) \left(\frac{\partial \Phi}{\partial \vec{z}}(b, \tilde{w}(b))\right) \end{aligned} \tag{5.14}$$

where $\pi^* \in L_0(\mathbb{C}^{n-k+1}, \mathbb{C}^{n-k})$ is the projection onto the first $n-k$ coordinates, adj denotes the adjutant matrix, and

$$D(b) = \det(\partial \Phi / \partial z_{k+1}, \ldots, z_n, t)(b, \tilde{w}(b)).$$

Multiplying by $D(b)$ and substituting this expression into equation (5.13) we see that $\mathrm{rank}_b(g \restriction X^0)$ is (independently of our choice of s, l) equal to the rank of a matrix whose coefficients are $\mathcal{R}$-holomorphic functions on U, evaluated at b; this is true for any $b \in X'$.

Now the rank of this matrix is strictly less than r if and only if the determinants of all its r-minors take value zero. These determinants are $\mathcal{R}$-holomorphic functions on U, which we may denote $h_1, \ldots, h_q$, say. Thus $C \cap$

$X' = Z(h_1, \ldots, h_q) \cap X'$. Let C_0 denote the $\mathcal{R}$-analytic subset of $Z(h_1, \ldots, h_q) \cap X$ containing those of its components which have nonempty intersection with X'.

We may exploit equation (5.14) further. Recall that by proposition 4.4 there is $v \in \mathbb{C}^n$ such that $D(z)$ is regular with respect to v everywhere in an open neighbourhood of $\mathrm{cl}(U)$, of order no greater than d, say. Then we may write a partition of the topographic manifold $(X \cap U) \setminus \Sigma = X' \cup \cdots \cup X^{(d+1)}$, where $X^{(j+1)}$ is the subset of $(X \cap U) \setminus \Sigma$ containing the points at which $D(z)$ is regular with respect to v of order j.

Applying L'Hôpital's rule to equation (5.14), by differentiating both D and the coefficients of the matrix j times with respect to v, we get an expression for $\partial\lambda^*_{s,l}/\partial\vec{z}(\vec{b})$ valid on $X^{(j+1)}$. Again we may substitute into (5.13) and find the $\mathcal{R}$-holomorphic determinants of the r-minors, $h^j_1, \ldots, h^j_{q_j}$, say. So $C \cap X^{(j+1)} = Z(h^j_1, \ldots, h^j_{q_j}) \cap X^{(j+1)}$. Let C_j denote the subset of

$$Z(h^j_1, \ldots, h^j_{q_j}) \cap \{b \in X : D(b) = \cdots = \frac{\partial^{j-1} D}{\partial v^{j-1}}(b) = 0\}$$

containing those of its components which have nonempty intersection with $X^{(j)}$. Then $C_j \subseteq C \setminus (X' \cup \cdots \cup X^{(j)})$ but $C_j \supseteq C \cap X^{(j+1)}$.

Thus we see that

$$C \cap (X \setminus \Sigma) \subseteq C_0 \cup \cdots \cup C_d \subseteq \mathrm{cl}(C \setminus \Sigma) \subseteq C.$$

If we extend the natural coordinates to a k-complete sequence of coordinate systems with respect to each of which X_a is k-regular (as without loss of generality we assume that we can), then we may repeat this argument in each coordinate system—perhaps reducing the neighbourhood U—to cover all of $(C \cap X^0) \cap U$ with finitely many $\mathcal{R}$-analytic subsets of $C \cap U$. But $C = (C \cap X^0) \cup X^*$, so we deduce that $C \cap U$ is indeed $\mathcal{R}$-analytic in U, as required. □

Theorem 5.30 (Cartan-Remmert, Łojasiewicz V.3.3) *Let $X \subseteq \mathbb{C}^n$ be $\mathcal{R}$-analytic, and let $g : X \to \mathbb{C}^m$ be an $\mathcal{R}$-holomorphic map. For $a \in X$ let $F_a(g)$ denote the germ at a of the fibre $g^{-1}(g(a))$. Then for any $r \in \mathbb{N}$ the set $P(X, g, r) = \{a \in X : \dim F_a(g) \geq r\}$ is $\mathcal{R}$-analytic.*

Proof. We proceed by induction on $\dim X$, and may assume that X is irreducible (since if $X = \bigcup X_i$, then $F_a(g) = \bigcup F_a(g \restriction X_i))$. The case $\dim X = 0$ is trivial, so let X be irreducible of dimension $k > 0$.

Let $d(g) = \min\{\dim F_a(g) : a \in X\}$ be the minimal dimension of a fibre of g. If $r \leq d(g)$, then $\{a \in X : \dim F_a(g) \geq r\} = X$, and we are done. Otherwise, consider the set $C = C(X, g, \mathrm{rank}(g)-1)$, which we proved $\mathcal{R}$-analytic in lemma 5.29. Certainly $\dim C < k$, as C is an analytic subset of X omitting those a for which $\mathrm{rank}_a(g)$ is maximal, while X is irreducible. So it is enough, inductively, to prove that $P(X, g, r) = P(C, g \restriction C, r)$. The inclusion "$\supseteq$" is clear.

On applying the rank theorem to $g \restriction (X \cap U)$ for arbitrarily small neighbourhoods U of b), we have that $\dim X = \dim F_b(g) + \mathrm{rank}\, f$ for any $b \in (X \setminus C)$, since g is of constant rank on the submanifold $X \setminus C$; and hence $\dim F_b(g) = d(g)$ for every such b. We conclude that $P(X, g, r) \subseteq C$.

So if $a \in P(X, g, r)$, consider the fibre $F = g^{-1}(g(a))$. Its subset $F \setminus C$ is a submanifold of constant dimension $d(g)$, and so, taking germs, $\dim(F \setminus C)_a \leq d(g) < \dim(F_a(g))$.

But $F = (F \setminus C) \cup (F \cap C)$, and $F \cap C = (f \restriction C)^{-1}((f \restriction C)(a))$. So $F_a(g) = F_a(g \restriction C) \cup (F \setminus C)_a$, and thus $\dim F_a(g \restriction C) = \dim F_a(g) \geq r$. Thus $a \in P(C, g \restriction C, r)$, as required; and the induction is complete. □

We can now complete our verification of the axioms.

9 (FC) *Fibre condition.* It is sufficient, by axiom (DU), to suppose that $X \subseteq \mathcal{C}_{n+m}$ is irreducible. Recall that $\mathcal{P}^{n.m}(X, k) = \{a \in M^n : \dim(X \cap \pi_{n,m}^{-1}(a)) > k\}$, where $\pi_{n,m}$ is the natural projection from $\mathbb{C}^{n+m}$ to $\mathbb{C}^n$. By theorem 5.30 the set $P(X, \pi_{n,m}, k) = \{a \in X : \dim(\pi_{n,m}^{-1}(\pi_{n,m}(a))) \geq r\}$ is $\mathcal{R}$-analytic and hence in $\mathcal{C}(\mathcal{R})$. So

$$\mathcal{P}^{n.m}(X, k) = \mathcal{P}^{n.m}(P(X, \pi_{n,m}), k) = \pi_{n,m}(P(X, \pi_{n,m}), k),$$

as required.

10 (AF) *Additive formula.* Let X be irreducible. Then by the rank theorem applied to π, the set $\pi(X)$ contains submanifolds of dimension $\mathrm{rank}(\pi \restriction X)$, but not of any greater dimension. Moreover, as shown in the proof of theorem 5.30, $\text{rank}(\pi \restriction X) = \min\{\dim(\pi^{-1}(a)) : a \in \pi(X)\}$. Thus

$$\dim X = \dim \pi(X) + \min\{\dim(\pi^{-1}(a)) : a \in \pi(X)\}.$$

We have not succeeded in proving (AF) unconditionally, however, since the set $\pi(X)$ has dimension only in the $\mathcal{R}$-analytic sense and in the absence of (CP) it may be that the closure (in the sense of $\mathcal{C}(\mathcal{R})$) of $\pi(X)$ has greater dimension than $\pi(X)$ itself.

Assuming (CP), though, there is $X' \in \mathcal{C}(\mathcal{R})$ such that $Y := \pi(X) \cup X' \in \mathcal{C}(\mathcal{R})$ and $\dim X' < \dim Y$. Writing X' as a countable union of $\mathcal{R}$-analytic sets $X' = \bigcup_{i \in \Omega} Y_i$, we have that, in the $\mathcal{R}$-analytic sense, $\dim Y = \max(\{\dim Y_i : i \in \omega\} \cup \{\dim \pi(X)\})$. For $\pi(X)$ is a countable union of submanifolds $\pi(X) = \bigcup_{j \in \omega} M_j$, say, and any irreducible component of Y is a Baire space; so if $Y = \bigcup_i Y_i \cup \bigcup_j M_j$, then one of the Y_i or M_j has nonempty interior in Y, and we know it cannot be one of the Y_i.

This completes the proof of theorem 5.28. □

Chapter 6

The projection axiom and the exponential function

In chapters 3, 4 and 5 we have investigated the local behaviour of $\mathcal{R}$-analytic sets for any $\mathcal{R}$ with the properties (R1)–(R6) given on page 24. In particular, if $\mathcal{S}$ is *any* set of entire holomorphic functions on $\mathbb{C}$ which is closed under partial derivatives and $\mathfrak{C}$ is the expansion of the complex field by the functions of $\mathcal{S}$ and a constant symbol for each complex number, then the set of tuples of terms of the language of $\mathfrak{C}$ has these properties.

At the cost of much labour in chapter 4 we have established a reasonable geometry for a class of sets that are all locally defined by existentially definable functions (in the sense given by the construction of $\mathcal{O}^{\mathcal{R}}$) in such a $\mathfrak{C}$. For the local results of chapter 5 we have used only the formal properties of $\mathcal{O}^{\mathcal{R}}$, such as Weierstrass division, and not its detailed construction. This chapter discusses how we might gain the reward of our labour in a strategy for verifying the "global" property (CP) for the countable unions of $\mathcal{R}$-analytic sets in certain particular cases, and gives a partial result for the motivating example where $\mathfrak{C}$ is the complex exponential field $\mathbb{C}_{\exp}$.

One can hope to exploit the existential definition of the functions of $\mathcal{O}^{\mathcal{R}}$. In $\mathcal{C}(\mathcal{R})$, (CP) is a statement about the projection by one dimension of the sets of simultaneous solutions of functions from $\mathcal{O}^{\mathcal{R}}$. Our intention is to find equivalent statements, exchanging the consideration only of the varieties of functions in $\mathcal{R}$ for the consideration of *arbitrary* projections of these varieties. Indeed, it will be enough to consider the varieties of functions from a subclass generating $\mathcal{R}$ under substitution. For $\mathfrak{C} = \mathbb{C}_{\exp}$, the exponential polynomials, and also the exponential polynomials of degree at most 1, are generating subclasses.

Evidently no such strategy can work for $\mathcal{O}^{\mathcal{R}}$ in general, since the Fatou-Bieberbach example (found in Bochner and Martin, [3], or Nishino, [13]) of an entire function which is a homeomorphism between $\mathbb{C}^2$ and a proper open subset of $\mathbb{C}^2$ cannot live in any QZ structure.

6.1 Equivalents of (CP) for $\mathcal{R}$-analytic sets

If $\mathcal{R}$ is any collection of entire functions satisfying (R1)–(R6), recall that the $\mathcal{R}$-analytic subsets of $\mathbb{C}^n$ are those sets X for which, at every point $a \in \mathbb{C}^n$, there is a neighbourhood U of a and a collection of functions $f_1, \ldots, f_r \in \mathcal{O}^{\mathcal{R}}_U$, such that $X \cap U = \{z \in U : f_1(z) = \cdots = f_r(z) = 0\}$. We have defined $\mathcal{C}(\mathcal{R})$ to contain all the countable unions of $\mathcal{R}$-analytic subsets of $\mathbb{C}^n$, for each n. Countable union and projection commute, so (CP) for the structure $\langle \mathbb{C}, \mathcal{C}(\mathcal{R}), \dim \rangle$ is equivalent to the following statement:

(CP for $\mathcal{R}$) If X is an $\mathcal{R}$-analytic subset of $\mathbb{C}^{n+1}$ and $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is the projection $\pi_{n,1}$, then there are countable collections $\{X'_i : i \in \omega\}$, $\{Y_j : j \in \omega\}$ of $\mathcal{R}$-analytic subsets of $\mathbb{C}^n$, such that $\bigcup_{j\in\omega} Y_j = \pi(X) \cup \bigcup_{i\in\omega} X'_i$ and $\max\{\dim X'_i : i \in \omega\} < \max\{\dim Y_j : j \in \omega\}$.

Recall also that if $f \in \mathcal{O}^{\mathcal{R}}_U$ then there is a partition of U into $S_1 \cup \cdots \cup S_t$ and on each S_s an implicit representation $\langle k_s, F_s(x, z_s), \tilde{f}_s \rangle$ of f with continuous extension to $\mathrm{cl}(S_s) \cap U$. So if $X \subseteq \mathbb{C}^{n+1}$ is any $\mathcal{R}$-analytic set, say

$$(x, y) \in X \leftrightarrow \bigvee_{i\in\omega} \big((x, y) \in U_i \wedge f_{i1}(x, y) = \cdots = f_{ir_i}(x, y) = 0\big),$$

we have, after partitioning U_i into $\bigcup_{s\le t_i} S_{is}$ compatibly with the representations of each f_{ij}, and assuming—for convenience and without loss of generality—that for each i there is a fixed k_i such that for each $j \leq r_i$ and $s \leq t_i$, f_{ij} is represented on S_{is} by functions of exactly $n+1+k_i$ variables, and writing $K_i = r_i \cdot k_i$:

$$(x,y) \in X \leftrightarrow \bigvee_{i\in\omega} \bigvee_{s\leq t_i} \exists z_1,\ldots,z_{K_i} \Big(x,y,z_1,\ldots,z_{K_i}) \in V_{is} \wedge \\ \wedge\ z_1 = z_{k_i+1} = z_{(r_i-1)\cdot k_i+1} = 0 \ \wedge\ F_{is}(x,y,z_1,\ldots,z_{K_i}) = 0\Big)$$

where $F_{is} : \mathbb{C}^{n+1+K_i} \to \mathbb{C}^{K_i} \in \mathcal{R}$, and $V_{is} \subseteq \mathbb{C}^{n+1+K_i}$ is chosen such that $\pi_{n+1,K_i} V_{is} = S_{is}$, and each fibre of V_{is} in this projection is open and such that $(\tilde{f_{i1}}(x,y),\ldots,\tilde{f_{ir_i}}(x,y))$ is the unique point of $Z(F_{is}) \cap (\pi^{-1}_{n+1,K_i}(x,y) \cap V_{is})$. Hence, with a permutation σ of the z variables to bring the value $\bar{0} \in \mathbb{C}^{r_i}$ to the front, and writing $V^* = \sigma(V), F^* = F \circ \sigma^{-1}$, we have

$$x \in \pi(X) \leftrightarrow \bigvee_{i\in\omega} \bigvee_{s\leq t_i} \exists y, z_{r_i+1},\ldots,z_{K_i} \Big((x,y,\bar{0},z_{r_i+1},\ldots,z_{K_i}) \in V^*_{is} \\ \wedge\ F^*_{is}(x,y,\bar{0},z_{r_i+1},\ldots,z_{k(i,j)}) = 0\Big). \quad (6.1)$$

From this representation we can see at once that the following two conditions are equivalent:

(ASY1) Given any $F \in \mathcal{R}$ with $F : \mathbb{C}^{n+m} \to \mathbb{C}^r$, if $\pi_{n,m}(Z(F))$ has non-empty interior in $\mathbb{C}^n$ then there is $Y \in \mathcal{C}_n(\mathcal{R})$ with $\dim(Y) < n$ such that for all $a \in \mathbb{C}^n$, if for every $\varepsilon > 0$ the set

$$\{(x,z) \in \mathbb{C}^{n+m} : \|x-a\| < \varepsilon \wedge \dim_{(x,z)}(\pi^{-1}_{n,m}(x) \cap Z(F)) = 0\}$$

has unbounded connected components, then $a \in Y$.

(ASY2) If $X \subseteq \mathbb{C}^{n+m}$ is $\mathcal{R}$-analytic, and $\pi(X)$ has non-empty interior in $\mathbb{C}^n$, then there is $Y \in \mathcal{C}_n(\mathcal{R})$ with $\dim(Y) < n$ such that for all $a \in \mathbb{C}^n$, if for every $\varepsilon > 0$ the set

$$\{(x,z) \in \mathbb{C}^{n+m} : \|x-a\| < \varepsilon \wedge \dim_{(x,z)}(\pi^{-1}_{n,m}(x) \cap X) = 0\}$$

has unbounded connected components, then $a \in Y$. Call such an a an *asymptotic point* of $\pi \restriction X$.

Indeed if (ASY1) holds and X is $\mathcal{R}$-analytic, and a is an asymptotic point of $\pi(X)$, then a is an asymptotic point of $\pi^*(Z(F^*_{is}))$ in the representation of X given by (6.1) for some i, s and $\pi^* = \pi_{(n+m+r_i),(K_i-r_i)}$. The countable union of the corresponding Y_{ij} is then a witness for (ASY2) for X. The converse implication is immediate.

If $f : \mathbb{C}^n \to \mathbb{C}^m$ then by an *asymptotic point of* f I mean an asymptotic point of $\pi_{n,m} \restriction \mathrm{Graph}(f)$.

Lemma 6.1 *If the condition (ASY1) holds, and* $f \in \mathcal{O}^{\mathcal{R}}_U$ *with continuous extension to* $\mathrm{cl}(U) \subseteq \mathbb{C}^n$*, then those* $z \in \mathbb{C}^n$ *which lie in the domain of* no *holomorphic extension of* f *are contained in some set* $Y \in \mathcal{C}_n(\mathcal{R})$ *with* $\dim Y < n$.

Proof. If $\langle k, F, \tilde{f}\rangle$ is an implicit representation of f on some open $V \subseteq U$ (in the finite partition defining f), then by (ASY2) there is $Y \in \mathcal{C}_n(\mathcal{R})$, $\dim(Y) < n$, containing all the asymptotic points of $\pi_{n,k} \restriction X$, where X is the irreducible component of $Z(F)$ containing the graph of $\tilde{f}$. Moreover if we let $X' = \{(x, y, z) \in \mathbb{C}^{n+k+1} : (x, y) \in X \wedge (z \cdot \det((\partial F/\partial y)(x, y)) = 1)\}$ then $\pi_{n,k+1}(X')$ has complement contained in some $Y' \in \mathcal{C}_n(\mathcal{R})$, and $\dim(Y') < n$ also. Clearly $Y \subseteq Y'$; and if γ is any simple path in $\mathbb{C}^n \setminus Y'$ with $\gamma(0) \in U$ and $\gamma(1) = b \in \mathbb{C}^n \setminus Y'$ then we may extend $\tilde{f}$ holomorphically to a domain $\Omega(b)$ containing γ since at every point of $X \cap \pi^{-1}(\gamma)$ the conditions of the implicit function theorem hold. So the union of all such $\Omega(b)$ covers $\mathbb{C}^k \setminus Y'$. □

Theorem 6.2 *If the condition (ASY1) holds then the condition (CP for* $\mathcal{R}$*) holds.*

Proof. Let $X \subseteq \mathbb{C}^n$ be an irreducible $\mathcal{R}$-analytic set and let $\pi = \pi_{n-1,1}$. If every fibre of $\pi \restriction X$ is of dimension 1 then $\pi(X)$ is $\mathcal{R}$-analytic.

Otherwise, if $\dim X = k$ then there is some projection $\pi' : \mathbb{C}^{n-1} \to M \cong \mathbb{C}^k$ such that $\pi' \circ \pi \restriction X$ has some zero-dimensional fibres, and k is minimal with this property. Then $\pi' \circ \pi(X)$ has non-empty interior in M, by the rank theorem 5.21. Assume without loss of generality that $M = \mathbb{C}^k$ and $\pi' = \pi_{k,n-k-1}$. If $z \in \mathbb{C}^n$, write $\vec{z} = \pi' \circ \pi(z)$.

Since X is irreducible, the subset $X_0 = \{z \in X : \dim((\pi' \circ \pi)^{-1}(z) \cap X) = 0\}$ is dense in X and a connected open subset. It is analytically constructible by axiom (FC) for $\mathcal{C}(\mathcal{R})$. At every point a of X_0, the germ X_a has a finite decomposition into k-regular irreducible germs $A_1, \dots, A_{t(a)}$; the ideal $\mathcal{I}^{\mathcal{R}}(A_i)$ is prime for each $i \leq t(a)$.

So for each A_i there is an open set $\Omega \subseteq \mathbb{C}^k$ containing $\vec{a}$ on which we can verify the conclusions of the Rückert descriptive lemma, theorem 5.10. In particular there are finitely many open sets U_s covering $\Omega \setminus Z$ and on each U_s implicit representations $\langle n-k+1, \Phi, (\lambda_{s,l,k+1}, \dots, \lambda_{s,l,n}, \lambda_{s,l,n+1})\rangle$ such that $X \cap \pi^{-1}_{k,n-k}(\Omega \setminus Z)$ is covered by the graphs of $(\lambda_{s,l,k+1}, \dots, \lambda_{s,l,n})$, as s, l vary. Moreover, there is a primitive element $\hat{w}$ of the ring extension $\hat{\mathcal{O}_n^{\mathcal{R}}} = \mathcal{O}_n^{\mathcal{R}}/\mathcal{I}^{\mathcal{R}}(A_i)$ over $\hat{\mathcal{O}_k^{\mathcal{R}}} = \mathcal{O}_k^{\mathcal{R}} + \mathcal{I}^{\mathcal{R}}(A_i)$; $\tilde{w}$ is a representative of the germ w on Ω, and we have for each $\vec{z} \in \Omega \setminus Z$,

$$\lambda_{s,l,n+1}(\vec{z}) = \tilde{w}(\vec{z}, \lambda_{s,l,k+1}(\vec{z}), \dots, \lambda_{s,l,n}(\vec{z})).$$

If $\vec{b}$ is an asymptotic point of $\pi_{k,n-k} \restriction X$ then for some choice of $a \in X_0$ and selection of s, l, the corresponding branch of the local covering map $\lambda = (\lambda_{k+1}, \dots, \lambda_n)$ has holomorphic extension to a domain $U \subseteq \pi_{k,n-k}(X_0)$ with $\vec{b} \subseteq \mathrm{cl}(U)$, but the graph of λ is unbounded arbitrarily close to $\vec{b}$. Suppose however that $(\lambda_{k+1}, \dots, \lambda_{n-1})$ has continuous extension to $\vec{b}$ taking values $(b_{k+1}, \dots, b_{n-1})$, say. Then $b = (\vec{b}, b_{k+1}, \dots, b_{n-1}) \in \pi(X)$.

Now, $\hat{\mathcal{O}_n^{\mathcal{R}}} = \hat{\mathcal{O}_k^{\mathcal{R}}}[\hat{z}_{k+1}, \dots, \hat{z}_n]$, and $\hat{\mathcal{O}_k^{\mathcal{R}}} \cong \mathcal{O}_k^{\mathcal{R}}$, so by the last part of the primitive element theorem (proposition 5.12), given any infinite subsets $R_{k+1}, \dots, R_n$ of $\mathcal{O}_k^{\mathcal{R}}$ there is a primitive element of form $\hat{w}_0$ where $w_0 \in R_{k+1}z_{k+1} + \cdots + R_n z_n$.

We have established, in claim 5.11, that $\hat{z}_n$ has a monic minimal polynomial over $\hat{\mathcal{O}_k^{\mathcal{R}}}$, p say of degree d, such that $p(z_n) \in \mathcal{I}^{\mathcal{R}}(X_a)$. So the representative $\tilde{p}[X]$ on Ω satisfies $\tilde{p}(\vec{z}, \lambda_n(\vec{z})) = 0$ where this composition is defined; by lemma 6.1 we may assume without loss of generality that the domain of holomorphy for each of the coefficients of $\tilde{p}$ is $U \supseteq U_s$. That is,

$$\lambda_n^d + \tilde{p}_{d-1}\lambda_n^{d-1} + \cdots + \tilde{p}_0 = 0$$

identically on U.

Suppose $|\lambda_n(\vec{z})| = \rho$ for some large $\rho \in \mathbb{R}$. Then one of the $\tilde{p}_i$ satisfies $|p_i(\vec{z})| > \rho/d$; so $|\frac{\lambda_n}{\tilde{p}_i^2}(\vec{z})| < \frac{d^2}{\rho}$. Hence we can fix some $0 < i < d$ such that the inequality

$$|\frac{\lambda_n}{\tilde{p}_i^2}(\vec{z})| < \frac{d^2}{|\lambda n(\vec{z})|}$$

holds on an open set $U' \subseteq U$, satisfying $\vec{b} \in \mathrm{cl}(U')$ and on which $|\lambda_n| \restriction U'$ grows without bound near $\vec{b}$.

Let R_1 be an infinite subset of the circle of radius $1/d$ in $\mathbb{C}$, and let $R_2 = \cdots = R_{n-1} = R_1$. If we choose a constant $c_0 \in \mathbb{C}$ which is not in the range of $\tilde{p}_i \restriction U_s$ then we may assume without loss of generality that $U' \cap \tilde{p}_i^{-1}(c_0) = \emptyset$ also. So we may let $R_n = \{c/(p_i - c_0)^2 \in \mathcal{O}_k^{\mathcal{R}} : c \in R_1\}$ and then $R_n \subseteq \mathcal{O}_k^{\mathcal{R}}$ and each element of R_n has a representative holomorphic on $U' \subseteq \Omega \setminus \tilde{p}_i^{-1}(c_0)$. Moreover each element c of R_n has continuous extension to $\vec{b}$ taking value 0 there, and its values on U' near $\vec{b}$ approach 0 sufficiently rapidly that $(c \cdot \lambda_n)$ has continuous extension to $\vec{b}$ with value 0 also.

Then there is $w_0 \in R_1 z_1 + \cdots + R_n z_n$ for which $\hat{w}_0$ is a primitive element of the ring extension $\hat{\mathcal{O}_n^{\mathcal{R}}}$ over $\hat{\mathcal{O}_k^{\mathcal{R}}}$ with a representative $\tilde{w}_0$ on $U' \times \mathbb{C}^{n-k}$. By our choice of R_n the function

$$\mu(\vec{z}) = \tilde{w}_0(\vec{z}, \lambda_{k+1}(\vec{z}), \ldots, \lambda_n(\vec{z})) \in \mathcal{O}_{U'}^{\mathcal{R}}$$

has continuous extension to $\vec{b}$ with value $\mu(\vec{b}) = \sum_{j=k+1}^{n-1} c_j b_j$ for some $c_j \in R_1$.

We may carry through the same reasoning as in the proof of the Rückert lemma on page 88, but substituting this particular primitive element $\tilde{w}_0$ as our choice of $\tilde{w}$ throughout. We conclude that there are polynomials in t $\tilde{Q}_{k+1}, \ldots, \tilde{Q}_n, \tilde{G} \in \mathcal{O}_{U'}^{\mathcal{R}}[t]$ and function $\tilde{\delta} \in \mathcal{O}_{U'}^{\mathcal{R}}$, such that if we define the function

$$\Phi_0 : (\vec{z}, z_{k+1}, \ldots, z_n, t) \mapsto (\tilde{\delta}(\vec{z})z_{k+1} - \tilde{Q}_{k+1}(\vec{z}, t), \ldots, \tilde{\delta}(\vec{z})z_n - \tilde{Q}_n(\vec{z}, t), \tilde{G}(\vec{z}, t))$$

then $\langle n-k+1, \Phi_0, (\lambda_{k+1}, \ldots, \lambda_{n-1}, \lambda_n, \mu)\rangle$ is an implicit representation of each of the λ_j and of μ on U' over $\mathcal{O}_{U' \times \Delta}^{\mathcal{R}}$ (for an appropriate polydisk Δ).

So in particular, since $\langle 1, \tilde{G}, \mu\rangle$ is an implicit representation of μ over $\mathcal{O}_{U'}^{\mathcal{R}}[t]$, there is some $\langle r, H, \tilde{h}\rangle$ representing $\tilde{G}$ over $\mathcal{R}$ on some open subset V of $U' \times \mathbb{C}$

with $(\vec{b}, \mu(\vec{b})) \subseteq \mathrm{cl}(V \cap \mathrm{Graph}(\mu))$—that is, $H \in \mathcal{R}$, $\tilde{h}_1 = \tilde{G}$, and this notation departs slightly from our usual practice due to an unfortunate clash of tildes.

Then, expanding out this representation in the manner of (6.1), we have for any $(\vec{a}, \mu(\vec{a})) \in V$ that

$$\mu(\vec{a}) = a_{n+1} \to \tilde{G}(\vec{a}, a_{n+1}) = 0, \text{ so}$$
$$\mu(\vec{a}) = a_{n+1} \to \exists u_2, \ldots, u_r\, H(\vec{a}, a_{n+1}, 0, u_2, \ldots, u_r) = (\underbrace{0, \ldots, 0}_{r\ times})$$

and hence $\mu(\vec{a}) = a_{n+1}$ implies that $(\vec{a}, a_{n+1})$ is an asymptotic point of

$$\pi^* \restriction \big(Z(H(\vec{z}, z_{n+1}, v_1, u_2, \ldots, u_r)) \cap Z(v_1 \cdot v_2 - 1)\big) = \pi^* \restriction M_1, \text{ say,}$$

where $\pi^* : (\vec{z}, z_{n+1}, v_1, v_2, u_2, \ldots, u_r) \mapsto (\vec{z}, z_{n+1})$. Indeed, for any non-zero v_1 sufficiently close to 0 there is some a'_{n+1} near a_{n+1} and some $(u_2, \ldots, u_r)$ such that H satisfies the conditions of the implicit function theorem at the point $(\vec{a}, a'_{n+1}, v_1, u_2, \ldots, u_r)$ (which is then an isolated point of the fibre of the projection of this set onto its first $k+1$ variables); and v_1 determines uniquely its multiplicative inverse v_2 which takes an unbounded set of values as a'_{n+1} approaches a_{n+1}.

But it follows now also that $(\vec{b}, \mu(\vec{b}))$ is an asymptotic point of this same projection $\pi^* \restriction M_1$. By condition (ASY1) there is some $W \in \mathcal{C}_{k+1}(\mathcal{R})$ covering the set of asymptotic points of $\pi^* \restriction M_1$ and of dimension no greater than k. In particular, since the graph of $\mu \restriction U'$ is an irreducible $\mathcal{R}$-analytic set, there is some irreducible $\mathcal{R}$-analytic W_0 with $\dim W_0 = k$ and $\mathrm{Graph}(\mu) \subseteq W_0$. Then $(\vec{b}, \mu(\vec{b})) \in W_0$ and this is an isolated point of the fibre $(\{\vec{b}\} \times \mathbb{C}) \cap W_0$ (since the function μ extends continuously to this point).

Having found this W_0 to serve as an $\mathcal{R}$-analytic representative of the graph of μ, we can return to the implicit representation by Φ_0 of the coordinate functions $\lambda_{k+1}, \ldots, \lambda_{n-1}$. These have by our hypothesis continuous extensions to $b_{k+1}, \ldots, b_{n-1}$ respectively, and from Φ_0 we can read off that $\lambda_j(\vec{z}) = \tilde{Q}(\vec{z}, \mu(\vec{z}))/\tilde{\delta}(\vec{z})$ on U' for each $k+1 \le j \le n-1$.

For each such j there is thus some open $V_j \subseteq U' \times \mathbb{C}^2$ with $(\vec{b}, \mu(\vec{b}), \lambda_j(\vec{b})) \in \mathrm{cl}(V_j \cap \mathrm{Graph}(\mu, \lambda))$ and an implicit representation $\langle r_j, F_j, \tilde{f}_j$ of $\tilde{Q}/\tilde{\delta}$ on V_j over

$\mathcal{R}$.

Then for any $(\vec{a}, \mu(\vec{a}), \lambda(\vec{a})) \in V_j$ we have

$$\lambda(\vec{a}) = a_j \to a_j = \frac{\tilde{Q}(\vec{a}, \mu(\vec{a}))}{\delta(\vec{a})}, \text{ so}$$
$$\lambda(\vec{a}) = a_j \to \exists v_1\, (\vec{a}, v_1) \in W_0 \wedge$$
$$\exists u_2, \ldots, u_{r_j}\, (F_j(\vec{a}, v_1, a_j, 0, u_2, \ldots, u_{r_j}) = 0)$$

and the familiar argument shows then that $(\vec{a}, a_j)$ is an asymptotic point of

$$\pi_j^* \restriction \big(Z(F_j(\vec{z}, v_1, z_j, v_2, u_2, \ldots, u_{r_j})) \cap (v_2 \cdot v_3 - 1) \cap ((\vec{z}, v_1) \in W_0)\big)$$
$$= \pi_j^* \restriction M_j, \text{ say.}$$

And now $(\vec{b}, b_j)$ is also an asymptotic point of this projection. By condition (ASY2) there is a set $W_j \in \mathcal{C}_{k+1}(\mathcal{R})$ containing all the asymptotic points of $\pi_j^* \restriction M_j$ and of dimension no greater than k. In particular some irreducible $\mathcal{R}$-analytic W_j' of dimension k covers the graph of λ_j, and hence in particular the point $(\vec{b}, b_j)$, which is isolated in the fibre $\{\vec{b}\} \times \mathbb{C} \cap W_j'$.

Hence the $\mathcal{R}$-analytic set

$$X' = \{(z_1, \ldots, z_k, z_{k+1}, \ldots, z_{n-1}) : \bigwedge_{j=k+1}^{n-1} (\vec{z}, z_j) \in W_j'\} \subseteq \mathbb{C}^{n-1}$$

covers $\mathrm{Graph}((\lambda_{k+1}, \ldots, \lambda_{n-1}) \restriction U') \subseteq \pi(X)$ and also the point of interest $b = (\vec{b}, b_{k+1}, \ldots, b_{n-1})$. As the point a at which we took the Rückert description varies over X_0, all points in $\mathrm{cl}(\pi(X))$ lie on the continuous extension of some $(\lambda_{s,l,k+1}, \ldots, \lambda_{s,l,n-1})$ arising in the description; but indeed the graph of this λ covers an open subset of X. So since the topology on $\mathbb{C}$ has countable basis there is a countable selection of points $\{a_i : i \in \omega\}$ for which in carrying out the above argument at every point a_i, and for every branch (indexed by s, l) of the Rückert covering at that point, we get a countable collection $\{X_i' : i \in \omega\}$ of $\mathcal{R}$-analytic sets whose union covers $\mathrm{cl}(\pi(X))$.

Finally, every point $b \in \mathrm{cl}(\pi(X)) \setminus \pi(X)$ is, by our choice of π', such that $\pi' \circ \pi(b)$ is an asymptotic point of $\pi_{k,n-k}(X)$. So there is by condition (ASY2) a set $Y \in \mathcal{C}_k(\mathcal{R})$, with $\dim(Y) < k$, such that every such b lies in $\pi_{k,n-k}^{-1}(Y)$.

So such a b lies in some X_i' from our collection; and we have also shown that b is isolated in the fibre of $\pi_{k,n-k} \restriction X_i'$. Hence $b \in X_i' \cap \pi_{k,n-k}^{-1}(Y) = Y_i'$, say, and $\dim(X_i' \cap \pi_{k,n-k}^{-1}(Y)) < k$.

That is, $\pi(X) \cup (\bigcup_{i\in\omega} Y_i') = (\bigcup_{i\in\omega} X_i') \in \mathcal{C}(\mathcal{R})$; and $\dim(Y_i') < k = \dim(X_{i'}')$ for each $i, i' \in \omega$. This is condition (CP for $\mathcal{R}$), as required. □

Inspection of the proof of theorem 6.2 shows that we in fact only use the condition (ASY1) in the case $r = m$, where $F : \mathbb{C}^{n+m} \to \mathbb{C}^m$. Considered as a system of equations in the variables $y_1, \ldots, y_m$ with parameters $x_1, \ldots, x_n$, then, $F(x, y) = 0$ in this case *exactly determined.* Moreover we can arrange that $\partial F/\partial y$ is nonsingular almost everywhere on any component of $Z(F)$. Hence the following condition is also sufficient to prove (CP for $\mathcal{R}$):

(ASY3) Given any $F \in \mathcal{R}$ with $F : \mathbb{C}^{n+m} \to \mathbb{C}^n$, if $\pi_{n,m}(Z(F))$ has non-empty interior in $\mathbb{C}^n$ then there is $Y \in \mathcal{C}_n(\mathcal{R})$ with $\dim(Y) < n$ such that for all $a \in \mathbb{C}^n$, if for every $\varepsilon > 0$ the set

$$\{z \in \mathbb{C}^m : \exists x\, (\|x - a\| < \varepsilon \wedge (x, z) \in \mathrm{Ift}(F))\}$$

has unbounded connected components, then $a \in Y$.

Further, it may not be necessary when verifying (ASY3) to consider all the exactly determined $F \in \mathcal{R}$. If $\mathcal{S} \subseteq \mathcal{R}$ satisfies

1. $\mathcal{S}$ is an additive subgroup of the functions in $\mathcal{R}$;
2. every variable $z_j \in \mathcal{S}$;
3. $\mathbb{C} \subseteq \mathcal{S}$; and
4. $\mathcal{S}$ generates $\mathcal{R}$ under substitution;

then every set of form $\pi(Z(F))$ with $F \in \mathcal{R}$ is of form $\pi'(Z(F'))$ with $F' \in \mathcal{S}^{<\omega}$.

In particular, in the case of $\mathbb{C}_{\exp}$, observe that if $F = F(\tau)$ is a tuple of terms of the language $L_{\exp}$, and the subterm τ is of form $\tau = \sigma_1 \cdot \sigma_2$, then

$$F(\tau) = 0 \leftrightarrow \big(F(z_3) = 0 \wedge z_3 = e^{z_1+z_2} \wedge e^{z_1} = \sigma_1 \wedge e^{z_2} = \sigma_2\big) \\ \vee (F(0) = 0 \wedge \sigma_1 = 0) \vee (F(0) = 0 \wedge \sigma_2 = 0)$$

Thus we can eliminate multiplication in favour of addition and exponentiation. This observation gives us two candidates for subgroups of the terms of $L_{\exp}$ to concentrate on:

1. the exponential polynomials $\mathcal{EP}$,

$$\mathcal{EP} = \{f(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) : f \in \mathbb{C}[z_1, \ldots, z_n, y_1, \ldots, y_n]\};$$

2. the exponential polynomials of degree at most 1, $\mathcal{EP}_1$, (equivalently, of degree at most 1 in the unexponentiated variables)

$$\mathcal{EP}_1 = \{c + \sum_{i=1}^{n} a_i z_i + \sum_{j=1}^{n} b_j e^{z_j} : (a, b, c) \in \mathbb{C}^{2n+1}\}.$$

In the next section we shall verify (ASY3) for an arbitrary exponential polynomial of one variable, in this sense. More general notions of exponential polynomial are available: in particular, it would be natural to consider also functions of form $f(z_1, e^{\lambda_1 z_1}, \ldots, e^{\lambda_k z_1})$ where $f \in \mathbb{C}[z_1, y_1, \ldots, y_k]$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. For such functions the distinguished role played in the proof of theorem 6.3 by the imaginary axis would be taken by other rays through the origin.

6.2 Exponential polynomials of one variable

Definition. By a *parametrized exponential polynomial* of one variable I mean a function of form

$$f(z, e^z) = \sum_{p=-m}^{m} \sum_{q=0}^{n} x^*_{pq} z^q e^{pz},$$

where for each $(p,q) \leq (n,m)$ either $x^*_{pq} = x_{pq}$, a coordinate variable of the parameter space $K \cong \mathbb{C}^{(2m+1)(n+1)}$, or $x^*_{pq} = 0$ identically; and $f(z,e^z)$ is obtained by substituting $w = e^z$ in the polynomial $f(z,w) = \sum_{j,k} x^*_{pq} w^p z^q$.

If $f(z,e^z)$ is a parametrized exponential polynomial in z then the *carrier of* f is the set $\Delta(f) : \{(p,q) : x^*_{pq} = x_{pq}\} \subseteq \mathbb{Z} \times \mathbb{N}$ of monomials featuring in $f(z,w)$.

The *Newton polygon for* f is the upper part of the convex hull of $\Delta(f)$. Namely it has vertices given by the shortest sequence $\{(p_l,q_l) : l = 0,\ldots,d\}$ such that:

1. for each $l \leq d$, $(p_l,q_l) \in \Delta(f)$;
2. $p_0 < p_1 < \cdots < p_d$;
3. every $(p,q) \in \Delta(f)$ satisfies $p_0 \leq p \leq p_d$; and
4. whenever $0 \leq l < d$ and $(p,q) \in \Delta(f)$,
$$\frac{q - q_l}{p - p_l} \leq \frac{q_{l+1} - q_l}{p_{l+1} - p_l}$$

Write, for $0 < l \leq d$, $\mu_l = (q_l - q_{l-1})/(p_l - p_{l-1})$. Then $\mu_1 > \mu_2 > \ldots \mu_d$.

Theorem 6.3 *If $f(z,e^z) = \sum_{p=-m}^{m} \sum_{q=0}^{n} x^*_{pq} z^q e^{pz}$ is any parametrized exponential polynomial, and $a = (a_{pq})_{-m \leq q \leq m, 0 \leq p \leq n} \in K$ is an asymptotic point of the graph of $f(z,e^z)$, then there is some vertex (p_l,q_l) of the Newton polygon for f for which $a_{p_l q_l} = 0$.*

Proof. We can summarize the ideas of the proof briefly. If $|z|$ is large compared with the sizes $|x_{p_l q_l}|$ of the coefficients of $f(z,w)$ at the vertices of the Newton polygon, then the zeros of $f(z,w)$ considered as a polynomial in w are, up to a factor of the fractional power of z, z^{μ_l}, close to the roots of polynomials g_l taken from the edges of the Newton polygon. In particular, $\arg(w)$ at such a zero takes one of certain values fixed by $\arg(z)$ and the coefficients $x_{p_l q_l}$. Moreover if $|z|$ is large and z, e^z satisfy a polynomial relation, then z is near to the imaginary axis of $\mathbb{C}$. Hence the argument of z is fixed near $\pm\frac{\pi}{2}$. But in any unbounded set of z near to the imaginary axis, $\arg(e^z)$ takes all values; so on no such unbounded set can $f(z, e^z) = 0$ uniformly.

Now for the detail of the proof. Let $Y = \{x \in \mathbb{C}^{(2m+1)(n+1)} : x_{p_l q_l} = 0 \text{ for some } l \leq d\}$ be the set of those x for which the coefficient at some vertex of the Newton polygon for f vanishes, and suppose that $a \notin Y$. If $d = 0$ then $f(z, w) = w^m f_0(z)$ for some power m of w and polynomial $f_0(z) = x_{mq} z^q + \cdots + x_{m0}$. So $f(z, e^z) = 0 \leftrightarrow f_0(z) = 0$. And if $a_{mq} = a_{p_0 q_0} \neq 0$ and $|x - a| < |a_{mq}|/2$, all the roots α of $f_0(z)$ satisfy

$$-x_{mq}\alpha^q = x_{m,q-1}\alpha^{q-1} + \cdots + x_{m0},$$

$$|x_{mq}||\alpha^q| \leq q(\|a\| + |a_m q|/2) \max(|\alpha^{q-1}|, 1)$$

and so

$$|\alpha| \leq \max\Big(\frac{4q\|a\|}{|a_{mq}|}, 1\Big).$$

That is, all roots of $f(z, e^z)$ for x near a are bounded, so in this trivial case a is certainly not an asymptotic point of $f(z, e^z)$.

Otherwise, if $d > 0$, we may assume without loss of generality that $p_0 = 0$ (because $f(z, e^z) = 0$ if and only if $e^{-p_0 z} f(z, e^z) = 0$). For each $l = 1, \ldots, d$, consider the function $f_l(z, u)$ obtained by substituting $u = w \cdot z^{\mu_l}$ for w in $f(z, w)$,

$$f_l(z, u) = \sum_{(p,q) \in \Delta(f)} x_{pq} z^{q - \mu_l \cdot p} u^p.$$

We have that $(q_l - \mu_l p_l) = (q_{l-1} - \mu_l p_{l-1}) = \nu$, say, and that for all $p, q \in \Delta(f)$, $(q - \mu_l p) \leq \nu$. So this ν is the highest (fractional) power of z occurring in $f_l(z, u)$ and we may write

$$\begin{aligned} f_l(z, u) &= \sum_{\substack{(p,q) \in \Delta(f) \\ q - \mu_l p = \nu}} x_{pq} z^\nu u^p + \sum_{\substack{(p,q) \in \Delta(f) \\ q - \mu_l p < \nu}} x_{pq} z^{q - \mu_l p} u^p \\ &= z^\nu g_l(u) + \sum_{\substack{(p,q) \in \Delta(f) \\ q - \mu_l p < \nu}} x_{pq} z^{q - \mu_l p} u^p. \end{aligned}$$

Then $g_l(u)$ is a polynomial in u of degree p_l with leading coefficient $x_{p_l q_l}$ and least non-zero term $x_{p_{l-1}q_{l-1}}u^{p_{l-1}}$. Hence $g_l(u)$ has exactly $p_l - p_{l-1}$ zeros not at the origin, provided that $x \notin Y$.

Now, $a \notin Y$ so we may label the zeros of g_l evaluated at a not lying at the origin by $\alpha_{p_l-1} + 1, \ldots, \alpha_{p_l}$, and the origin by α_0, say. Suppose $\varepsilon > 0$ is chosen small enough that if $\alpha_j, \alpha_{j'}$ are any two distinct zeros of g_l evaluated at a then they are a distance greater than 4ε apart. Then there is $\delta > 0$ such that if $|x - a| < \delta$ then $x \notin Y$ and every zero of g_l evaluated at x is within distance ε of a zero of g_l evaluated at a.

So if α is any zero, either one of the α_j or the origin, of $g_l(u)$ evaluated at a then for any u on the circle $\{u : |u - \alpha_j| = 2\varepsilon\} = \gamma(\alpha_j, 2\varepsilon)$, evaluating g_l at any $x \in (\|x - a\| < \delta)$, we have that $g_l(u) \neq 0$. It follows that some $R_{l,\alpha} > 0$ is sufficiently large that if $|z| > R_{l,\alpha}$ then, for any $\|x - a\| < \delta$ and any choice of the fractional power z^{μ_l} (which then determines $z^\nu = z^{q_l - \mu_l p_l}$), for all $u \in \gamma(\alpha, \varepsilon)$ we have

$$|z^\nu g_l(u)| > \Big| \sum_{\substack{(p,q)\in\Delta(f)\\ q-\mu_l p<\nu}} x_{pq} z^{q-\mu_l p} u^p \Big|.$$

So by Rouché's theorem if $\|x - a\| < \delta$ and $|z| > R_{l,\alpha}$ then $f_l(z, u)$ has exactly as many zeros, counted with multiplicity, on the disk $(|u - \alpha| < 2\varepsilon)$ as $g_l(u)$ has.

Moreover if $\rho > 0$ is chosen large enough that every zero of g_l (evaluated at any x with $\|x - a\| < \delta$) lies inside $|u| = \rho$, then there is $R_{l,\infty}$ sufficiently large that if $|z| > R_{l,\infty}$ we may apply Rouché's theorem on the circle $|u| = \rho$, to show that $f_l(z, u)$ has exactly as many zeros β with $|\beta| < \rho$ as g_l; namely p_l. It follows in particular that if $|z| > \max(R_{l,\infty}, R_{l,0})$ then $f_l(z, u)$ has $p_l - p_{l-1}$ zeros β satisfying $\varepsilon < |\beta| < \rho$. For sufficiently small ε we may take $\rho = 1/\varepsilon$.

Now let $R_l > \max\{R_{l,j} : p_{l-1} < j \leq p_l\} \cup \{R_{l,0}, R_{l,\infty}\}$. Then for all z with $|z| > R_l$, and all x with $\|x - a\| < \delta$, if $f_l(z, u) = 0$ and $2\varepsilon < u < \rho$, then there is some $p_{l-1} < j \leq p_l$ for which $|u - \alpha_j| < 2\varepsilon$; and if we count with multiplicity then this is a bijective correspondence between the zeros β of $f_l(z, u)$ satisfying $\varepsilon < |\beta| < \rho$ and $(\alpha_{p_l-1} + 1, \ldots, \alpha_{p_l})$.

For sufficiently small ε, therefore, there are δ sufficiently small and ρ and $R > \max(R_1, \ldots, R_d)$ sufficiently large that for all x with $\|x - a\| < \delta$ and all z with $|z| > R$, for each $l = 1, \ldots, d$ and each selection z^{μ_l} of the fractional root, if $2\varepsilon < |w \cdot z^{\mu_l}| < \rho$ and $f(z, w) = 0$ then $|w \cdot z^{\mu_l} - \alpha_j| < 2\varepsilon$, for some α_j a root of g_l (evaluated at a).

But also we may pick R large enough that for each $l = 1, \ldots, d$ and $p_{l-1} < j \leq p_l$,

$$|\alpha_j z^{\mu_{l+1}-\mu_l}| < \varepsilon \text{ if } l < d, \text{ and } |\alpha_j z^{\mu_l - \mu_{l-1}}| > \rho \text{ if } l > 0.$$

So we have (since indeed there are p_d zeros of g_d and g_1 has no zeros at the origin) a *partition* of the zeros of $f(z, w)$: if $|z| > R$, $\|x - a\| < \delta$ and $f(z, w) = 0$ then for some unique $l \leq d$ we have

$$|wz^{\mu_l} - \alpha_j| < 2\varepsilon \text{ for some } \alpha_j \text{ a non-zero root of } g_l \text{ evaluated at } a.$$

We shall consider the argument function as a mapping $\arg : \mathbb{C} \setminus \{0\} \to \langle \mathbb{R}/2\pi\mathbb{Z}, + \rangle$ onto the circle. Where $|wz^{\mu_l} - \alpha_j| < 2\varepsilon$, we have $|w - \alpha_j z^{-\mu_l}| < 2\varepsilon |z^{-\mu_l}|$, and hence

$$|\arg(w) - \arg(\alpha_j z^{-\mu_l})| \leq \arcsin\Big|\frac{2\varepsilon z^{-\mu_l}}{\alpha_j z^{-\mu_l}}\Big| = \arcsin\Big|\frac{2\varepsilon}{\alpha_k}\Big|. \tag{6.2}$$

So up to the factors $z^{-\mu_l}$ and this ε, each zero of $f(z, w)$ has argument determined entirely by the roots of the polynomials g_l read off the edges of the Newton polygon, provided $|z| > R$, $\|x - a\| < \delta$.

This is the first step of the proof. Now to show that large solutions of $f(z, e^z)$ for x near a must lie near the imaginary axis, write

$$f(z, e^z) = x_{p_d q_d} z^{q_d} e^{p_d z} + \sum_{\substack{(p_d, q) \in \Delta(f) \\ q < q_d}} x_{p_d, q} z^q e^{p_d z} + \sum_{\substack{(p,q) \in \Delta(f) \\ p < p_d}} x_{p,q} z^q e^{pz}.$$

Now, $a \notin Y$ so there is some δ_0 and $A > 0$ such that if $\|x - a\| < \delta_0$ then $|x_{p_d q_d}| > A$ and also $|x_{p_0 q_0}| > A$. Moreover there is B such that if $\|x - a\| < \delta_0$ and $(p, q) \in \delta(f)$ then $|x_{pq}| < B$. Then if $\|x - a\| < \delta_0$, $f(z, e^z) = 0$ implies

$$\left| x_{p_d q_d} z^{p_d} e^{q_d z} \right| = \Big| \sum_{\substack{(p_d, q) \in \Delta(f) \\ q < q_d}} x_{p_d, q} z^q e^{p_d z} + \sum_{\substack{(p,q) \in \Delta(f) \\ p < p_d}} x_{p,q} z^q e^{pz} \Big|,$$

whence (recalling that $\Delta(f)$ is bounded by $p \leq p_d, q \leq n$)

$$A|z|^{q_d} e^{p_d \cdot \mathrm{re}(z)} \leq Bq_d|z|^{q_d-1} e^{p_d \cdot \mathrm{re}(z)} + Bp_d(n+1)|z|^n e^{(p_d-1)\cdot \mathrm{re}(z)}$$

and so $A|z|e^{\mathrm{re}(z)} \leq Bp_d(n+1)|z|^{n-q_d} + Bq_d e^{\mathrm{re}(z)}$. So if $|z|$ is sufficiently large, say $|z| > 2Bp_d/A$, and $\|x-a\| \leq \delta_0$, $f(z, e^z) = 0$ implies that

$$e^{\mathrm{re}(z)} \leq \frac{B(n+1)p_d}{Aq_d} \leq |z|\frac{n+1}{q_d}.$$

That is, for some $C_d \in \mathbb{R}$, $\mathrm{re}(z) \leq C_d + (n - q_d)\log|z|$.

A similar calculation shows that as $x_{p_0q_0}$ is also bounded away from zero, there is some $C_0 \in R$ such that sufficiently large solutions z of $f(z, e^z) = 0$ also satisfy $\mathrm{re}(z) \geq C_0 + (n - q_0)\log|z|$.

So, given $\theta > 0$ we can find R such that if $\|x - a\| < \delta_0$ and $|z| > R$, if $f(z, e^z) = 0$ then $|\arg(z) - \frac{\pi}{2}| < \theta$ or $|\arg(z) + \frac{\pi}{2}| < \theta$.

But we may pick θ sufficiently small that

$$\mathcal{S} = \bigcup\big\{(\alpha_j\omega - 2\theta, \alpha_j\omega + 2\theta) : \ \alpha_j \text{ a non-zero root of some } g_l \\ 1 \leq l \leq d, \text{ evaluated at } a,\ \omega \text{ a } \mu_l\text{th root of } \pm\sqrt{-1}\big\} \quad (6.3)$$

does not cover $\mathbb{R}/2\pi\mathbb{Z}$.

And we may pick ε small enough that no zeros of any g_l evaluated at a have modulus less than 2ε unless they lie at the origin, and for each non-zero root α_j of any g_l,

$$\arcsin\left|\frac{2\varepsilon}{\alpha_j}\right| \leq 4\frac{\varepsilon}{|\alpha_j|} \leq \theta.$$

But now if $\|x - a\| < \delta$ for some sufficiently small δ, and $|z|$ is sufficiently large, it follows from (6.2) and (6.3) that

$$z \in \{z \in \mathbb{C} : \arg(e^z) \in \mathcal{S} \text{ and } \big||\arg z| - \frac{\pi}{2}\big| < \theta\},$$

and we are done. This set has no unbounded components. □

6.3 Further questions

To complete the proof that $\mathbb{C}_{\exp}$ is quasi-minimal—by verifying the quasi-Zariski axioms for the class $\mathcal{CR}$ with $\mathcal{R}$ the tuples of terms from the language $L_{\exp}$—it would be enough to extrapolate the methods of section 6.2 to an arbitrary number of variables.

Conjectural completion of the argument. In contrast with the real exponential field it is easier to consider the class $\mathcal{EP}_1$ of exponential polynomials of degree at most 1, eliminating any multiplication between the variables entirely in favour of addition and exponentiation. We should expect a notion of a Newton polytope to be important, generalising the Newton polygons of the previous section; and eliminating multiplication reduces the dimension of the polytope by half. The Newton polytope features, for example, in the analysis by Kazarnovskii, [6], of the zeros of exponential sums (exponential polynomials $f(\bar{z}, e^{z_1}, \ldots, e^{z_n})$ where $f(\bar{x}, \bar{y})$ is of degree zero in the variables $\bar{x}$, or is of degree zero in the variables $\bar{y}$). He finds a theorem on the asymptotes of sets of this type for $n = 1, 2$.

Suppose we are given an exactly determined system of parametrized functions in $\mathcal{EP}_1$,

$$f(z) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{pmatrix} \begin{pmatrix} e^{z_1} \\ \vdots \\ e^{z_n} \end{pmatrix}$$

with c_i, a_{ij}, b_{ij} variables of the parameter space $K \subseteq \mathbb{C}^{2n^2+n}$. I shall sketch a possible approach to determining where $f(z)$ can have asymptotic points (a, b, c).

If $x = (a, b, c) \in K$ is an algebraically generic point of K and such that the matrices $A = (a_{ij}), B = (b_{ij})$ are both invertible, then similar methods to those of theorem 6.3 show that (a, b, c) is not an asymptotic point of f. Indeed in this case if z is a large solution of f at x' near (a, b, c), we can find ε such that if $\|z - z'\| = \varepsilon$, then $\|Be^z - Be^{z'}\| \gg \varepsilon$. The corresponding Newton polytope in this case is the n-simplex with vertices $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ and the origin $(0, \dots, 0)$, lying in $\mathbb{R}^n \subseteq \mathbb{C}^n$.

We may assume that the matrix A is invertible, for otherwise, perhaps after a linear change of variables, the variable z_n appears in f only in exponentiated form e^{z_n}. We can then substitute $z'_n = e^{z_n}$; this increases the rank of A and decreases that of B.

In general the matrix B is not invertible. So some linear relationship holds between the z_i (and the constants c_j), and it is convenient to eliminate as many variables as possible. After a relabelling, we have f in the form

$$f(z) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 \cdot z} \\ \\ \vdots \\ e^{\lambda_m \cdot z} \end{pmatrix}$$

where $m \geq n$, $\operatorname{rank} B = n$, and each $\lambda_j(a, b, c) \in \mathbb{C}^n$.

Then $\Lambda(x) = \{\lambda_j(x) : j \leq m\} \cup \{(0, \dots, 0)\}$ is the *spectrum of f at x*. The convex hull $P(x)$ of $\Lambda \subseteq \mathbb{R}^{2n} \cong \mathbb{C}^n$ is the *Newton polytope of f at x*.

Now at least if A is invertible, the equation $f(z) = 0$ gives us an expression of the variables z in terms of $\{e^{\lambda \cdot z} : \lambda \in \Lambda\}$. If the real dimension of the polytope $P(x)$ is greater than n, then the projection $\pi_\mu(P)$ of P onto the complex line corresponding to some affine combination of the z_i, $\mu : z \mapsto \mu_1(z_1) + \dots \mu_n(z_n) + \mu_0$ say, has nonempty interior and contains the origin. Then at a solution of $f(z) = 0$ where $\mu(z)$ is large, there is at least one of the points of (Λ) for which the product $\pi_\mu(\lambda) \cdot \mu(z) \in \mathbb{C}$ has large real part. Then $|e^{\lambda z}| \gg |\mu(z)|$, so no such large solution should be expected to exist.

Thus, we may conjecture, asymptotic points of f have Newton polytope of real dimension at most n. This is not an $\mathcal{R}$-analytic condition on x! However, by a ($\mathbb{C}$-linear) change of variables we may ensure that the polytope lies in the distinguished subspace $\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \subseteq \mathbb{C}^n$. In this case if no two vertices of the polytope coalesce at x (that is, if $\lambda_j(x) \neq \lambda_{j'}(x)$ for any $j < j' \leq m$) we can find some approximation of the coordinates of the points in Λ by rational numbers which is sufficiently fine to distinguish all the vertices of P for any

x' near x. Expressing all these rational numbers with a common denominator approximates f by polynomials in $e^{z_1}, \ldots, e^{z_n}$, now with some n-by-n submatrix of B invertible. The approximation should be such that for sufficiently large z, f and its approximation have solutions close together by Rouché's theorem. That some large z' is a solution to the approximation should now imply that an n-by-n submatrix corresponding to n vertices of the leading face of the polytope at z (that is, the face on which points λ give largest values of $|e^{\lambda z}|$) is nonsingular. Then the same argument as for the case that A, B are both invertible would show that x is not an asymptotic point of our original f.

This line of reasoning, if it could be completed, would show that f has asymptotic points only at those x for which either two points of the spectrum coalesce, or the coefficients of x corresponding to some vertex of the Newton polytope take value zero, or if some matrix of linear combinations of the x is singular (and the corresponding condition is false at generic $x \in K$). All these are algebraic and hence $\mathcal{R}$-analytic conditions. This would be sufficient to imply (ASY3) and hence axiom (CP), and prove the original conjecture that $\mathbb{C}_{\text{exp}}$ is quasi-minimal.

Relations with analytic Zariski structures. Peatfield and Zilber, [15] have introduced the notion of an analytic Zariski structure, to describe in general the behaviour of "analytic" sets in a reasonable geometry expanding a compact Zariski geometry. One might profitably ask whether the classes of $\mathcal{R}$-analytic sets provide a source of examples of analytic Zariski geometries over $\mathbb{C}$; the major work left to do for this would be continuing the programme of chapter 5 to include the proper mapping theorem. Another direction would be to explore whether there is anything fruitful in the method of quasi-Zariski structures which can contribute to the study of the analytic Zariski axioms.

Liouville functions. The Liouville functions, entire holomorphic functions whose power series at the origin has rapidly diminishing coefficients, were introduced by Wilkie in [23]. They satisfy a Schanuel condition. Koiran in [7] has proved that the theory of the complex field expanded by a Liouville function H is exactly the limit theory of generic polynomials, and so finds an axiomatization for the theory of $\mathbb{C}_H$. In particular, he shows that if $\phi(x,z,y)$ is a polynomial over $\mathbb{Q}$ in the variables $(x_1,\ldots,x_m,z_1,\ldots,z_n,y_1,\ldots,y_n)$, $\xi(x)$ expresses "$Z_x(\phi) := \{(z,y) : \phi(x,z,y) = 0\}$ is irreducible in K^{2n} and not contained in any subspace of form $x_i = x_j$ or $x_i = c \in K$", and $\psi(x)$ expresses "for some projection $\pi : K^{2n} \to K^n$ either fixing any member of the canonical basis or annihilating it, $\pi(Z_x(\phi))$ is dense in K^n", in the language of fields K, then

$$\forall \bar{x}\, \exists z_1,\ldots,z_n\, ((\xi(\bar{x}) \wedge \psi(\bar{x})) \to \phi(\bar{x},z_1,\ldots,z_n,H(z_1),\ldots,H(z_n))) \quad (6.4)$$

is a theorem of $\mathbb{C}_H$.

This is a statement of the form of (ASY1). So a natural candidate for an analytic model of the QZ axioms is the expansion of the complex field by such a Liouville function H. To expand this into an $\mathcal{R}$-analytic geometry we would need to add all the derivatives of H. The derivative of a Liouville function is again a Liouville function, but we would need to prove an analogue of (6.4) mentioning arbitrarily many derivatives of H as well as H itself. It is not yet known whether $\mathbb{C}_H$ is a model of the analytic Zariski axioms either (see Peatfield, [14]).

Model theory over quasi-Zariski structures. It is not clear whether the category of quasi-Zariski structures has any model theory. Clearly not every elementary extension of the first order structure induced by a quasi-Zariski structure is quasi-Zariski. There is scope to investigate the appropriate notion of embedding for this class. In particular, a natural question is whether there is a reasonably good notion of the rank of a tuple inside a general quasi-Zariski structure. For an analytic model of the QZ axioms a natural notion of analytic rank would be interesting in its own right.

Indeed, our methods do not address Zilber's original question, whether $C_{\exp}$ is a homogeneous structure. A refinement of the $\mathcal{R}$-analytic theory with a more parsimonious use of constants would perhaps provide an approach to this problem.

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