Optimal Error Estimates of a Mixed Finite Element Method for Parabolic Integro-Differential Equations with Non Smooth Initial Data

by

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Deepjyoti Goswami, Amiya K. Pani and Sangita Yadav
Department of Mathematics
Industrial Mathematics Group
Indian Institute of Technology Bombay
Powai, Mumbai-400076 (India).

Abstract

In this article, a new mixed method is proposed and analyzed for parabolic integro-differential equations (PIDE) with nonsmooth initial data. Compared to mixed methods for PIDE, the present method does not bank on a reformulation using a resolvent operator. Based on energy arguments and without using parabolic type duality technique, optimal $L^2$-error estimates are derived for semidiscrete approximations, when the initial data is in $L^2$. Due to the presence of the integral term, it is, further, observed that estimate in dual of $H(\text{div})$-space plays a role in our error analysis. Moreover, the proposed analysis follows the spirit of the proof technique used for deriving optimal error estimates of finite element approximations to PIDE with smooth data and therefore, it unifies both the theories, i.e., one for smooth data and other for nonsmooth data. Finally, the proposed analysis can be easily extended to other mixed method for PIDE with rough initial data and provides an improved result.

Key Words. Parabolic integro-differential equation, extended mixed finite element method, semidiscrete Galerkin approximation, optimal error estimate, nonsmooth initial data.

1 Introduction.

In this paper, we propose and analyse an extended mixed finite element method for the following parabolic integro-differential equation:

\begin{align*}
\tag{1.1} u_t - \nabla \cdot \left( A \nabla u - \int_0^t B(t, s) \nabla u(s) \, ds \right) &= 0 \quad \text{in } \Omega \times J, \\
\tag{1.2} u(x, t) &= 0 \quad \text{on } \partial \Omega \times J, \\
\tag{1.3} u(x, 0) &= u_0(x) \quad \text{in } \Omega,
\end{align*}
with \( u_0 \in L^2(\Omega) \), where \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) is a bounded convex polygon or polyhedron, \( J = (0, T], \ T < \infty \). Here \( u_t = \frac{\partial u}{\partial t} \), \( A = [a_{ij}(x)]_{1 \leq i,j \leq d} \) and \( B(t, s) = [b_{ij}(x, t, s)]_{1 \leq i,j \leq d} \) are two \( d \times d \) matrices with smooth coefficients in \( \Omega \). Further, we assume that \( A \) is uniformly positive definite in \( \Omega \). For simplicity of exposition, we have assumed that \( A \) is independent of time. However, the proposed analysis can be easily extended to include the case, when \( A \) depends on time. Equations of the type described above arise naturally in many physical phenomena, for example, nonlocal flows in porous media (cf. Cushman and Glinn [6] and Dagan [7]) and heat transfer through materials with memory (cf. Renardy et al. [20]).

In the literature, optimal error estimates are derived for Galerkin approximations to parabolic integro-differential equations with smooth initial data (cf. [3],[13], [16]-[17]). For nonsmooth initial data, that is, when \( u_0 \in L^2(\Omega) \), Thomée and Zhang [23] have applied semigroup theoretic approach combined with a use of the inverse of the associated positive definite uniformly elliptic operator to discuss optimal error estimates in \( L^2 \)-norm for the problem (1.1)-(1.3). Pani and Peterson [19] have analysed semidiscrete Galerkin method with numerical quadrature and have derived optimal error estimates in \( L^2 \) and \( L^\infty \)-norms under the assumption that the initial data \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \). Subsequently, based on energy argument and parabolic type duality, Pani and Sinha [18] have proved optimal \( L^2 \)-error estimates for a more general parabolic integro-differential equation with nonsmooth initial data. In the context of finite volume methods applied to PIDE (1.1)-(1.3), we refer to [8]-[10].

In the past, mixed finite element methods are proposed and analyzed for the problem (1.1)-(1.3) in [11]-[13] and [22] with smooth initial data. In these articles, the problem (1.1) is often rewritten by introducing a new variable

\[
\sigma(t) = A\nabla u - \int_0^t B(t, s)\nabla u(s) \, ds,
\]

or equivalently

\[
\alpha \sigma(t) = \nabla u - \int_0^t A^{-1} B(t, s) \nabla u(s) \, ds,
\]

as

\[
u_t - \nabla \cdot \sigma(t) = 0,
\]

where \( \alpha = A^{-1} \). Further, (1.5) is again rewritten in terms of \( \nabla u \) using resolvent operator in the form

\[
\nabla u(t) = \alpha \sigma(t) + \int_0^t M(t, s) \sigma(s) \, ds,
\]

where \( M(t, s) = R(t, s)A^{-1} \) and \( R(t, s) \) is the resolvent of the matrix \( A^{-1}B(t, s) \) given by

\[
R(t, s) = A^{-1}B(t, s) + \int_s^t A^{-1}B(t, \tau)R(\tau, s) \, d\tau, \quad t > s \geq 0.
\]

With \( W = L^2(\Omega) \) and \( V = H(div, \Omega) \), the weak formulation for the mixed problem (1.5)-(1.6), which forms a basis of the mixed finite element Galerkin formulation in [11]-[13] and [22] is to seek a pair of functions \((u, \sigma) : (0, T] \rightarrow W \times V \) satisfying

\[
(\alpha \sigma, v) + \int_0^t (M(t, s) \sigma(s), v) \, ds + (\nabla \cdot v, u) = 0, \quad v \in V,
\]

\[
(u_t, w) - (\nabla \cdot \sigma, w) = 0, \quad w \in W,
\]
with \( u(0) = u_0 \). In all these articles, a suitable reformulation as has been given in (1.9)-(1.10) is employed using resolvent operator and optimal error estimates in \( L^\infty(L^2) \) for both \( u \) and \( \sigma \) are established for smooth initial data using mixed Ritz or Ritz-Volterra projections. Another related article for mixed method applied to PIDE which is valid in some special case is discussed in [15]. Subsequently, Ewing et al. [13] have derived maximum norm estimates and superconvergence results for mixed semidiscrete approximation to PIDE using mixed Ritz-Volterra projection and a tool of approximate Green’s function.

To the best of our knowledge, there is hardly any result except for a recent article [22], available concerning optimal estimates of mixed approximations to problem (1.1)-(1.3) with nonsmooth initial data. Sinha et al. [22] have analysed a semidiscrete mixed formulation based on the weak formulation (1.9)-(1.10) for the problem (1.1)-(1.3) and derived an optimal rate \( O(ht^{-1}) \) for the velocity \( \sigma(t) \) in \( L^2 \)-norm and suboptimal convergence rate \( O(ht^{-1/2}) \) for the pressure \( u(t) \) in \( L^2 \)-norm with \( t \in (0,T] \), when \( u_0 \in L^2(\Omega) \). However, optimal rate \( O(h^2t^{-1}) \) for \( u \) is only established for a class of problems when \( A = aI \) and \( B = b(t,s)I \), where \( a \) and \( b \) are independent of spatial variable \( x \). The proof technique used in [22] is based on energy arguments and parabolic type duality technique.

In this paper, an extended mixed method for (1.1) is proposed and analyzed. To motivate this new mixed method, we now introduce two variables:

\[
(1.11) \quad q = \nabla u, \quad \text{and} \quad \sigma = Aq - \int_0^t B(t,s)q(s)ds.
\]

Then the problem (1.1) takes the form

\[
\frac{du}{dt} - \nabla \cdot \sigma = 0.
\]

Now we write the weak mixed formulation of (1.1) which forms a basis of our mixed Galerkin formulation as follows:

Find \((u, q, \sigma) : J \to W \times V \times V \) satisfying

\[
(1.12) \quad (q, v) + (u, \nabla \cdot v) = 0 \quad v \in V,
\]

\[
(1.13) \quad (\sigma, z) - (Aq, z) + \int_0^t B(t,s)q(s), z)ds = 0 \quad z \in V,
\]

\[
(1.14) \quad (u_t, w) - (\nabla \cdot \sigma, w) = 0 \quad w \in W
\]

with \( u(0) = u_0 \). In the context of elliptic and parabolic problems, similar mixed methods called expanded mixed methods are analyzed in [1], [4]-[5], and [24]. The main goal of this article is to establish optimal convergence rate \( O(h^2t^{-1}) \) for \( u(t) \) in \( L^2 \)-norm and \( O(ht^{-1}) \) for \( \sigma(t) \) and \( q \) in \( L^2 \)-norm with \( t \in (0,T] \), when \( u_0 \in L^2(\Omega) \). Our analysis is again based on energy argument without using parabolic type duality technique. Essentially, the proof technique is based on a combination of an elementary energy method and a repeated use of integral operators of the form \( \hat{\phi} \) and \( \hat{\phi} \), where \( \hat{\phi}(t) := \int_0^t \phi(s)ds \). This tool is also successfully applied to linear parabolic problems with nonsmooth data in [14]. Further, due to the presence of the integral term, it is observed that estimate in the dual of \( H(div) \)-space plays a role in our error analysis. We note that compared to the formulation (1.9)-(1.10), we have, in this proposed method, introduced one more variable leading to a computation of one more extra variable. The proposed method will have an advantage, if we are not only interested to estimate the gradient but also the flux variable. Moreover, we do not resort to
an introduction of a resolvent operator to rewrite the problem in an equivalent form and this new formulation seems to be a more natural as well as direct mixed formulation for PIDE. Further, the analysis adopted here is quite elementary and can be easily extended to derive optimal $L^2$-error estimates for semidiscrete mixed finite element approximations to both $u$ and $\sigma$ discussed in Sinha et al. [22] for rough initial data. Hence, it provides an improvement over the results in [22], where optimality of $u$ is proved under the restriction that the coefficients $A, B$ are independent of the spatial variable, i.e., $A = a(t)I$ and $B = b(t)I$, and $a$ and $b$ are independent of spatial variable $x$.

Throughout this article, we denote by $C$, a generic positive constant which may very from context to context.

The paper is organized as follows. In Section 2, we discuss a priori bounds and regularity results for (1.12)-(1.14) with smooth as well as nonsmooth intial data. In Section 3, we briefly present the finite element approximation of the extended mixed formulation (1.12)-(1.14). In Section 4, we introduce and analyse extended mixed Ritz-Volterra projections. Section 5 is devoted to optimal error estimates for Galerkin approximations of $u, \sigma$ and $q$. Finally, this section is concluded with a superconvergence result.

2 A Priori Estimates and Regularity Results

In this section, we discuss some a priori bounds and regularity results to be used in the rest of this article for the problem (1.12)-(1.14), when the initial data $u_0$ is in $L^2(\Omega)$.

We use usual notations for $L^2(\Omega), H^1_0(\Omega), H^m(\Omega)$ and $H(div; \Omega)$-spaces and their norms and seminorms. To be more specific, $L^2(\Omega)$ is equipped with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. The $H(div; \Omega)$ is equipped with norm

$$\| v \|_{H(div; \Omega)} = (\|v\|^2 + \|\nabla \cdot v\|^2)^{1/2}.$$  

and the standard Sobolev space $H^m(\Omega)$ of order $m$ is equipped with its standard norm $\| \cdot \|_{H^m(\Omega)}$, which we call it simply as $\| \cdot \|_m$.

Since the matrix $A$ in (1.1) is positive definite, there exist positive constants $\alpha_0$ and $\alpha_1$ such that

$$\alpha_0\|\sigma\| \leq \|\sigma\|_A \leq \alpha_1\|\sigma\|,$$

where $\|\sigma\|_A^2 := (A\sigma, \sigma)$.

Moreover, under our assumptions on the domain and on the coefficient matrix $A$, we note that the following elliptic regularity holds:

$$\|\nabla \cdot (A\nabla \phi)\| \geq \beta\|\phi\|_2, \quad \phi \in H^1_0 \cap H^2,$$

where, $\beta$ is a positive constant independent of $\phi$.

Below, we state without proof a priori estimates and regularity results for $u, q$ and $\sigma$ satisfying (1.12)-(1.14). Note that the proof of Lemma 2.1 can be found in Theorem 4 of [19] and Lemma 2.2 of [18].

**Lemma 2.1** Let $(u, q, \sigma)$ satisfy (1.12)-(1.14) and let $0 \leq j, k \leq 2$ with $0 \leq i \leq 4$. 


(i) If \(0 \leq k + 2j - i \leq 2\), then
\[
t^i \| \frac{\partial^j u}{\partial t^j} (t) \|_k^2 \leq C \| u_0 \|_{k+2j-i}^2.
\]

(ii) Further, if \(0 \leq k + 2j - i - 1 \leq 2\), then
\[
\int_0^t s^i \| \frac{\partial^j u}{\partial s^j} (s) \|_k^2 \, ds \leq C \| u_0 \|_{k+2j-i-1}^2.
\]

For our subsequent use, we define notation \(\hat{\phi}\) as
\[
\hat{\phi}(t) = \int_0^t \phi(s) \, ds.
\]

Below, we discuss some more regularity results involving \(\sigma\).

**Lemma 2.2** Let \((u, q, \sigma)\) satisfy (1.12)-(1.14) and let \(0 \leq i, j \leq 2\).

(i) If \(0 \leq 2j - i \leq 2\), then
\[
\int_0^t s^i \| \frac{\partial^j \sigma}{\partial s^j} (s) \|_k^2 \, ds \leq C \| u_0 \|_{2j-i}^2.
\]

(ii) If \(0 \leq 2j - i - 1 \leq 2\), then
\[
t^i \| \frac{\partial^j \sigma}{\partial t^j} (t) \|_k^2 \leq C \| u_0 \|_{2j-i+1}^2.
\]

(iii) Moreover, the following estimates hold:
\[
\| \hat{u} \|_2, \| \nabla \cdot \hat{\sigma} \|, \| t \nabla \cdot \sigma \|, \| t^2 \nabla \cdot \sigma_t \| \leq C \| u_0 \|.
\]

Proof. Note that a straightforward modification of the proof of Lemma 2.2 in [22] completes the estimates (2.3) - (2.4). Moreover, from Lemma 2.2 of [18], we obtain the estimate \(\| \hat{u} \|_2\).

For the rest of the estimates in (2.5), we observe, from (1.11), that
\[
t \| \sigma \|_1^2 \leq Ct \| u \|_1^2 + Ct \int_0^t \| u(s) \|_1^2 \, ds
\]
and using (2.4), we obtain
\[
t \| \sigma \|_1^2 \leq C \| u_0 \|_1^2.
\]

Next, we use integration by parts, to rewrite (1.11) as
\[
\sigma = Aq - B(t, t)\hat{q} + \int_0^t B_s(t, s)\hat{q}(s) \, ds.
\]

Therefore, using the first set of estimates, we find that
\[
t^2 \| \nabla \cdot \sigma \|^2 \leq Ct^2 \left( \| u \|_2^2 + \| \hat{u} \|_2^2 + \int_0^t \| \hat{u}(s) \|_2^2 \, ds \right),
\]
\[
\leq C \| u_0 \|^2.
\]
Finally, we integrate (1.11) to obtain

\[ \hat{\sigma} = A\hat{q} - \int_0^t B(s, s)\hat{q}(s) \, ds + \int_0^t \int_0^s B_r(s, \tau)\hat{q}(\tau) \, d\tau \, ds. \]

Thus, using again the first set of estimates, we arrive at

\[ \| \nabla \cdot \hat{\sigma} \|^2 \leq C \| \hat{u} \|^2_2 + C \int_0^t (\| \hat{u}(s) \|^2_2 + \int_0^s \| \hat{u}(\tau) \|^2_2 \, d\tau) \, ds \]

\[ \leq C \| u_0 \|^2. \]

Finally, we differentiate (1.11) to obtain

\[ \sigma_t = Aq_t - B(t, t)q - \int_0^t B_t(t, s)q(s) \, ds \]

\[ = Aq_t - B(t, t)q - B_t(t, t)\hat{q} + \int_0^t B(t, s)q(s) \, ds \]

and using the first set of estimates, it follows that

\[ \| \nabla \cdot \sigma_t \| \leq C \left( \| u_t \|_2 + \| u \|_2 + \int_0^t \| \hat{u}(s) \|_2 \, ds \right) \]

\[ \leq C t^{-2} \| u_0 \|. \]

This completes the rest of the proof. \( \Box \)

### 3 Extended Mixed Finite Element Method

In this section, we introduce an extended mixed finite element Galerkin method which is based on the mixed formulation (1.12)-(1.14) for the problem (1.1)-(1.3).

Let \( T_h \) be a regular triangulation of \( \Omega \). Let \( V_h \times W_h \) denote a pair of finite element spaces satisfying the following conditions:

\[ (i) \quad \nabla \cdot V_h \subset W_h, \text{ and} \]

\[ (ii) \quad \text{there exists a linear operator } \Pi_h : V \rightarrow V_h \text{ such that } \nabla \cdot \Pi_h = P_h(\nabla \cdot), \]

where \( P_h : W \rightarrow W_h \) is the \( L^2 \)-projection defined by

\[ (\phi - P_h \phi, w_h) = 0, \quad \forall w_h \in W_h, \phi \in W. \]

Further, we assume that the finite element spaces satisfy the following approximation properties:

\[ (3.1) \quad \| \sigma - \Pi_h \sigma \| \leq C h \| \nabla \cdot \sigma \|, \quad \| u - P_h u \| \leq C h^r \| u \|_r, \quad r = 1, 2. \]

Examples of such finite dimensional spaces \( W_h \) and \( V_h \) satisfying the above properties can be found in [2].

We further note that \( P_h \) and \( \Pi_h \) satisfy

\[ (3.2) \quad (\nabla \cdot (\sigma - \Pi_h \sigma), w_h) = 0, \quad w_h \in W_h; \quad (u - P_h u, \nabla \cdot v_h) = 0, \quad v_h \in V_h. \]
Now, we define the corresponding semidiscrete mixed finite element approximations as a triplet \((u_h, q_h, \sigma_h)\) : \((0, T) \rightarrow W_h \times V_h \times V_h\) satisfying

\[
\begin{align*}
(3.3) \quad & (q_h, v_h) + (u_h, \nabla \cdot v_h) = 0 \quad \forall \quad v_h \in V_h, \\
(3.4) \quad & (\sigma_h, z_h) - (Aq_h, z_h) + \int_0^t (B(t, s)q_h(s), z_h)ds = 0 \quad \forall \quad z_h \in V_h, \\
(3.5) \quad & (u_{h,t}, w_h) - (\nabla \cdot \sigma_h, w_h) = 0 \quad \forall \quad w_h \in W_h
\end{align*}
\]

with \(u_h(0) = P_h u_0\). Since \(W_h\) and \(V_h\) are finite dimensional spaces, the discrete problem (3.3)-(3.5) leads to a linear system consisting of differential, integral and algebraic equations. Using Picard’s method, it is straightforward to check that the system (3.3)-(3.5) has a unique solution.

Below, we state the main result of this article.

**Theorem 3.1.** Let \((u, q, \sigma)\) and \((u_h, q_h, \sigma_h)\) satisfy (1.12)-(1.14) and (3.3)-(3.5), respectively with \(u_h(0) = P_h u_0\). Then, there exists a positive constant \(C\) independent of the discretizing parameter \(h\) such that for \(t \in (0, T)\),

\[
||u(t) - u_h(t)|| \leq C h^2 t^{-1} ||u_0||,
\]

and

\[
||q(t) - q_h(t)|| + ||\sigma(t) - \sigma_h(t)|| \leq C h t^{-1} ||u_0||.
\]

The proof of the above theorem, that is, Theorem 3.1 will be achieved through a series of Lemmas which are proved in the next two Sections. Finally, the proof will be given in Section 5.

4 Mixed Ritz-Volterra Type Projections

In this section, we shall introduce and analyse extended mixed Ritz-Volterra projections.

We now introduce extended mixed Ritz-Volterra projections as follows:

Given \((u(t), q(t), \sigma(t)) \in W \times V \times V\), for \(t \in (0, T)\), define \((\tilde{u}_h, \tilde{q}_h, \tilde{\sigma}_h) : (0, T) \rightarrow W_h \times V_h \times V_h\) satisfying

\[
\begin{align*}
(4.1) \quad & (\eta_q, v_h) + (\eta_u, \nabla \cdot v_h) = 0, \quad v_h \in V_h, \\
(4.2) \quad & (\eta_{\sigma}, z_h) - (A\eta_q, z_h) + \int_0^t (B(t, s)\eta_q(s), z_h)ds = 0, \quad z_h \in V_h, \\
(4.3) \quad & (\nabla \cdot \eta_{\sigma}, w_h) = 0, \quad w_h \in W_h
\end{align*}
\]

where \(\eta_u := (u - \tilde{u}_h), \quad \eta_q = (q - \tilde{q}_h)\) and \(\eta_{\sigma} := (\sigma - \tilde{\sigma}_h)\). Since \(W_h\) and \(V_h\) are finite dimensional spaces, the discrete problem (4.1)-(4.3), for a given triplet \(\{u, q, \sigma\}\), leads to a linear Volterra system for \(\{\tilde{u}_h, \tilde{q}_h, \tilde{\sigma}_h\}\). Note that when \(B = 0\), the system has a unique solution, see [4]. Now using theory of linear Volterra equations of second kind specially Picard’s iteration, it is straightforward to prove that the system (4.1)-(4.3), for a given triplet \(\{u, q, \sigma\}\), has a unique solution \(\{\tilde{u}_h, \tilde{q}_h, \tilde{\sigma}_h\}\).
Set
\[ e_q = q - q_h = (q - \bar{q}_h) - (q_h - \bar{q}_h) =: \eta_q - \xi_q, \]
\[ e_u = u - u_h = (u - \bar{u}_h) - (u_h - \bar{u}_h) =: \eta_u - \xi_u, \]
\[ e_\sigma = \sigma - \sigma_h = (\sigma - \bar{\sigma}_h) - (\sigma_h - \bar{\sigma}_h) =: \eta_\sigma - \xi_\sigma. \]

In this section, we shall discuss estimates of \(\eta_u, \eta_q,\) and \(\eta_\sigma\). Now, using definitions of \(P_h\) and \(\Pi_h\), we rewrite \(\eta_u, \eta_q,\) and \(\eta_\sigma\) as
\[
\begin{align*}
\eta_q &= q - q_h = (q - P_h q) - (q_h - P_h q) =: \theta_q - \rho_q, \\
\eta_u &= u - u_h = (u - P_h u) - (u_h - P_h u) =: \theta_u - \rho_u, \\
\eta_\sigma &= \sigma - \sigma_h = \sigma - \Pi_h \sigma =: \theta_\sigma.
\end{align*}
\]
Since estimates of \(\theta_u, \theta_q,\) and \(\theta_\sigma\) are known, it is enough to estimate \(\rho_u, \rho_q\). Now rewrite (4.1)-(4.3) as
\[
\begin{align*}
(\rho_q, v_h) + (\rho_u, \nabla \cdot v_h) &= (\theta_q, v_h), \quad \forall v_h \in V_h, \\
- (A \rho_q, z_h) + \int_0^t (B(t, s) \rho_q(s), z_h) ds &= (\theta_\sigma, z_h) - (A \theta_q, z_h) \\
&\quad + \int_0^t (B(t, s) \theta_q(s), z_h) ds, \quad \forall z_h \in V_h, \\
(\nabla \cdot \theta_\sigma, w_h) &= 0, \quad \forall w_h \in W_h,
\end{align*}
\]
Below, we discuss estimates of \(\eta_q\) and \(\bar{\eta}_q\).

**Lemma 4.1** Let \((\eta_u, \eta_q, \eta_\sigma)\) be such that the system (4.1)-(4.3) is satisfied. Then there exists a constant \(C\) independent of \(h\) such that for \(t \in (0, T)\),
\[
\|\eta_q(t)\| \leq Cht^{-1}\|u_0\|,
\]
and
\[
\|\bar{\eta}_q(t)\| \leq Ch\|u_0\|.
\]

Proof. Put \(z_h = \rho_q\) in (4.5) and rewrite it as
\[
\|A^{1/2} \rho_q\|^2 = -(\theta_\sigma, \rho_q) + (A \theta_q, \rho_q) - (B(t, t) \theta_q(t), \rho_q) + \int_0^t (B_s(t, s) \theta_q(s), \rho_q) ds
\]
\[\quad + (B(t, t) \rho_q, \rho_q) - \int_0^t (B_s(t, s) \rho_q(s), \rho_q(t)) ds.\]
A use of the Cauchy-Schwarz inequality yields
\[
\|A^{1/2} \rho_q\| \leq C \left( \|\theta_\sigma\| + \|\theta_q\| + \|\hat{\theta}_q\| + \int_0^t \|\hat{\theta}_q(s)\| ds + \|\hat{\rho}_q\| + \int_0^t \|\hat{\rho}_q(s)\| ds \right).
\]
Since, the matrix \(A\) is a positive definite, we arrive at
\[
\|\rho_q\| \leq C \left( \|\theta_\sigma\| + \|\theta_q\| + \|\hat{\theta}_q\| + \int_0^t \|\hat{\theta}_q(s)\| ds \right) + C \left( \|\hat{\rho}_q\| + \int_0^t \|\hat{\rho}_q(s)\| ds \right).
\]
Since estimates of $\theta_{\sigma}$, and $\theta_q$ are known, it is enough to estimate the last term on the right hand side of (4.9). For $\|\hat{\rho}_q\|$, we integrate (4.5) to obtain

$$
(A\hat{\rho}_q, z_h) = \int_0^t (B(s, s)\hat{\rho}_q(s), z_h) \, ds + \int_0^t \int_0^s (B_\tau(s, \tau)\hat{\rho}_q(\tau), z_h) \, d\tau \, ds - (\theta_{\sigma}, z_h)
$$

$$
+ (A\theta_q, z_h) - \int_0^t (B(s, s)\theta_q(s), z_h) \, ds + \int_0^t \int_0^s (B_\tau(s, \tau)\theta_q(\tau), z_h) \, d\tau \, ds.
$$

Choose $z_h = \hat{\rho}_q$, apply the Cauchy-Schwarz inequality and use positive definiteness of the matrix $A$ to arrive at

$$
\|\hat{\rho}_q\| \leq C(T, \alpha) \left( \|\theta_{\sigma}\| + \|\theta_q\| + \int_0^t (\|\theta_q(s)\| + \|\hat{\rho}_q(s)\|) \, ds \right).
$$

Note that using (1.11) and Lemma 2.2, we find that

$$
\|\hat{\theta}_q\| = \|\hat{q} - P_h\hat{q}\| \leq C h \|\hat{q}\|_1 \leq C h \|u_0\|, \quad \text{and} \quad \|\hat{\theta}_{\sigma}\| \leq C h \|u_0\|.
$$

On substituting (4.11) in (4.10), we now obtain

$$
\|\hat{\rho}_q\| \leq C h \|u_0\| + C \int_0^t \|\hat{\rho}_q(s)\| \, ds.
$$

Now apply Gronwall’s Lemma to arrive at

$$
\|\hat{\rho}_q\| \leq C h \|u_0\|.
$$

Use the projection properties (3.1) and Lemma 2.2 to obtain

$$
\|\theta_{\sigma}\|, \|\theta_q\| \leq C h^{-1} \|u_0\|.
$$

Hence, from (4.9), we find that

$$
\|\rho_q\| \leq C h^{-1} \|u_0\|,
$$

and now, a use of triangle inequality establishes the estimate (4.7).

For the second estimate (4.8), we again appeal to triangle inequality along with (4.12) to complete the rest of the proof. \(\square\)

For our future use, we need to estimate $\|\tilde{\eta}_h\|$ and $\|\tilde{\eta}_q\| (H(div; \Omega))^*$, where $(H(div; \Omega))^*$ is the topological dual of $H(div; \Omega)$.

Now, integrating the system of equations (4.1)-(4.3) with respect to time, we obtain the following system of equations

$$
(\tilde{\eta}_q, v_h) + (\tilde{\eta}_h, \nabla \cdot v_h) = 0 \quad v_h \in V_h,
$$

$$
(\tilde{\eta}_\sigma, z_h) - (A\tilde{\eta}_q, z_h) + \int_0^t (B(s, s)\tilde{\eta}_q(s), z_h) \, ds
$$

$$
- \int_0^t \int_0^s (B_\tau(s, \tau)\tilde{\eta}_q(\tau), z_h) \, d\tau \, ds = 0 \quad z_h \in V_h,
$$

$$
(\nabla \cdot \tilde{\eta}_\sigma, w_h) = 0 \quad w_h \in W_h.
$$

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Lemma 4.2 Let $(\hat{\eta}_u, \hat{\eta}_q, \hat{\eta}_\sigma)$ satisfy the system (4.15)-(4.17). Then, there is a positive constant $C$ independent of $h$ such that for $t \in (0,T]$,

\begin{align}
\|\hat{\eta}_u(t)\| &\leq Ch^2\|u_0\|, \\
\|\hat{\eta}_q(t)\|_{(H(\text{div}; \Omega))^*} &\leq Ch^2\|u_0\|,
\end{align}

where $(H(\text{div}; \Omega))^*$ is the dual of $H(\text{div}; \Omega)$.

Proof. Now we consider the following auxiliary problem:

\begin{align}
\nabla \cdot (A\nabla \zeta) &= \hat{\eta}_u \quad \text{in } \Omega \\
\zeta &= 0 \quad \text{on } \partial\Omega.
\end{align}

With

$$p = \nabla \zeta, \quad \psi = Ap$$

we find that

$$\nabla \cdot \psi = \hat{\eta}_u.$$

From the elliptic regularity result, we obtain

$$\|\zeta\|_{L^2}, \|\nabla \cdot p\|, \|\nabla \cdot \psi\| \leq C\|\hat{\eta}_u\|.$$

The following system of equations is satisfied for all $(w, v, z) \in W \times V \times V$

\begin{align}
(p, v) + (\zeta, \nabla \cdot v) &= 0 \\
(\psi, z) - (Ap, z) &= 0 \\
(\nabla \cdot \psi, w) &= (\hat{\eta}_u, w).
\end{align}

Put $w = \hat{\eta}_u$, $z = \hat{\eta}_q$ and $v = \hat{\eta}_\sigma$ in (4.23)-(4.25) and then add them to arrive at

\begin{align}
\|\hat{\eta}_u\|^2 &= (\nabla \cdot (\psi - \Pi_h \psi), \hat{\eta}_u) + (\psi - \Pi_h \psi, \hat{\eta}_q) - (A(p - \Pi_h p), \hat{\eta}_q) + (p - \Pi_h p, \hat{\eta}_\sigma) \\
&+ (\zeta - P_h \zeta, \nabla \cdot \hat{\eta}_\sigma) - \int_0^t (B(s, s)\hat{\eta}_q(s), \Pi_h p) \, ds + \int_0^t \int_\Omega (B(\tau, \tau)\hat{\eta}_q(\tau), \Pi_h p) \, d\tau \\
&\leq \|\nabla \cdot (\psi - \Pi_h \psi)\|\|\hat{\eta}_u\| + C(\|\psi - \Pi_h \psi\| + \|p - \Pi_h p\|)\|\hat{\eta}_q\| + \|p - \Pi_h p\|\|\hat{\eta}_\sigma\| \\
&+ \|\zeta - P_h \zeta\|\|\nabla \cdot \hat{\eta}_\sigma\| + C \int_0^t \|\hat{\eta}_q(s)\|_{(H(\text{div}; \Omega))^*} \, ds\|\Pi_h p\|_{H(\text{div}; \Omega)} \\
&\leq C h^2 \|\hat{\eta}_u\|_2 \|\nabla \cdot \psi\| + Ch \|\hat{\eta}_q\| \left(\|\nabla \cdot \psi\| + \|\nabla \cdot p\|\right) + Ch^2 \|\nabla \cdot \hat{\eta}_\sigma\| \|\zeta\|_2 \\
&+ C \int_0^t \|\hat{\eta}_q(s)\|_{(H(\text{div}; \Omega))^*} + \int_0^t \|\hat{\eta}_q(\tau)\|_{(H(\text{div}; \Omega))^*} \, d\tau \, ds \|\Pi_h p\|_{H(\text{div}; \Omega)}.
Using elliptic regularity (4.22) and estimates of $\|\hat{\eta}_q\|$ and $\|\hat{\eta}_q\|_{(H(\text{div};\Omega))^{\ast}}$, we now arrive at

$$
(4.26) \quad \|\hat{\eta}_u\| \leq C h^2 \|u_0\| + C \int_0^t \|\hat{\eta}_q(s)\|_{(H(\text{div};\Omega))^{\ast}} \, ds.
$$

To complete the proof of the estimate (4.26), it is enough to estimate $\|\hat{\eta}_q\|_{(H(\text{div};\Omega))^{\ast}}$. For $\|\hat{\eta}_q\|_{(H(\text{div};\Omega))^{\ast}}$, we proceed as follows. For any $v \in V$ and using (4.15), we obtain

$$
(4.27) \quad (\hat{\eta}_q, v) = (\hat{\eta}_q, v - \Pi_h v) + (\hat{\eta}_q, \Pi_h v) \\
= (\hat{\eta}_q, v - \Pi_h v) - (\hat{\eta}_u, \nabla \cdot \Pi_h v) \\
\leq \|\hat{\eta}_q\| \|v - \Pi_h v\| + \|\hat{\eta}_u\| \|\nabla \cdot \Pi_h v\| \\
\leq C(h^2 \|u_0\| + \|\hat{\eta}_u\|) \|v\|_{(H(\text{div};\Omega))}.
$$

Taking supremum over $0 \neq v \in V$ in (4.27), it follows that

$$
(4.28) \quad \|\hat{\eta}_q\|_{(H(\text{div};\Omega))^{\ast}} := \sup_{0 \neq v \in V} \frac{|(\hat{\eta}_q, v)|}{\|v\|_{(H(\text{div};\Omega))}} \leq C(h^2 \|u_0\| + \|\hat{\eta}_u\|).
$$

Substitute (4.28) in (4.26) to obtain

$$
\|\hat{\eta}_u\| \leq C h^2 \|u_0\| + C \int_0^t \|\hat{\eta}_u(s)\| \, ds.
$$

An application of Gronwall’s Lemma yields

$$
(4.29) \quad \|\hat{\eta}_u\| \leq C h^2 \|u_0\|.
$$

This, in turn, is substituted in (4.28) to complete the rest of the proof. \(\square\)

**Lemma 4.3** Let the hypotheses of Lemma 4.1 hold true. Then there is a positive constant $C$, independent of $h$, such that for $t \in (0, T]$,

$$
\|\eta_u(t)\| + \|\eta_q(t)\|_{(H(\text{div};\Omega))^{\ast}} \leq C h^2 t^{-1} \|u_0\|,
$$

and

$$
\|\eta_{u,t}(t)\| + h \|\eta_{q,t}(t)\| \leq C h^2 t^{-2} \|u_0\|.
$$

Proof. For finding estimate of $\|\eta_u\|$, we again consider the auxiliary problem (4.20)-(4.21) with replacing $\hat{\eta}_u$ on the right hand side of (4.20) by $\eta_u$.

Setting

$$
\mathbf{p} = \nabla \zeta, \quad \psi = A \mathbf{p}
$$

we obtain

$$
\nabla \cdot \psi = \eta_u.
$$

From the standard regularity results, it follows that

$$
(4.30) \quad \|\zeta\|_{L^2}, \ |\nabla \cdot \mathbf{p}|, \ |\nabla \cdot \psi| \leq C \|\eta_u\|.
$$
Now

\[
\| \eta \|^2 = (\nabla \cdot \psi, \eta) + (\psi, \eta) - (Ap, \eta) + (p, \eta) + (\zeta, \nabla \cdot \eta)
\]

\[
= (\nabla \cdot (\psi - \Pi_h \psi), \eta_u) + (\psi - \Pi_h \psi, \eta) - (A(p - \Pi_h p), \eta) + (p - \Pi_h p, \eta)
\]

\[
+ (\zeta - P_h \zeta, \nabla \cdot \eta) + (\nabla \cdot \Pi_h \psi, \eta_u) + (\Pi_h \psi, \eta) - (A\eta, \Pi_h p) + (\Pi_h p, \eta)
\]

\[
+ (P_h \zeta, \nabla \cdot \eta).
\]

Using (4.1)-(4.3) with \( v_h = \Pi_h \psi, z_h = \Pi_h p \) and \( w_h = P_h \zeta \), we obtain

\[
\| \eta \|^2 = (\nabla \cdot (\psi - \Pi_h \psi), \eta_u) + (\psi - \Pi_h \psi, \eta) - (A(p - \Pi_h p), \eta) + (p - \Pi_h p, \eta)
\]

\[
+ (\zeta - P_h \zeta, \nabla \cdot \eta) - \int_0^t (B(t, s) \eta_q(s), \Pi_h p) \, ds.
\]

Now proceed as in the proof of Lemma 4.2 for \( \| \tilde{\eta} \| \) to arrive at

\[
(4.31) \quad \| \eta \|^2 \leq \text{Ch}^2 t^{-1} \| u_0 \| \| \eta \| + | \int_0^t (B(t, s) \eta_q(s), \Pi_h p) \, ds |.
\]

We estimate the last term on the right-hand side of (4.31) as

\[
| \int_0^t (B(t, s) \eta_q(s), \Pi_h p) \, ds | = |(B(t, t) \tilde{\eta}_q(t), \Pi_h p) - \int_0^t (B_u(t, s) \tilde{\eta}_q(s), \Pi_h p) \, ds |
\]

\[
\leq C \| \tilde{\eta}_q \|_{H(div; \Omega)} \| \Pi_h p \|_{H(div; \Omega)}
\]

\[
+ C \left( \int_0^t \| \tilde{\eta}_q(s) \|_{H(div; \Omega)} \, ds \right) \| \Pi_h p \|_{H(div; \Omega)}.
\]

Substitute \( \| \tilde{\eta}_q \|_{H(div; \Omega)} \), from the Lemma 4.2, use stability property of \( \Pi_h \) and (4.30) to obtain

\[
(4.32) \quad | \int_0^t (B(t, s) \eta_q(s), \Pi_h p) \, ds | \leq \text{Ch}^2 \| u_0 \| \| \eta \|.
\]

Substitute (4.32) in (4.31) to obtain

\[
(4.33) \quad \| \eta \| \leq \text{Ch}^2 t^{-1} \| u_0 \|.
\]

From (4.1), we, as in the estimate (4.27), obtain

\[
(\eta_q, v) = (\eta_q, v - \Pi_h v) + (\eta_q, \Pi_h v)
\]

\[
= (\eta_q, v - \Pi_h v) - (\eta_u, \nabla \cdot \Pi_h v)
\]

\[
\leq \| \eta_q \| \| v - \Pi_h v \| + \| \eta_u \| \| \nabla \cdot \Pi_h v \|
\]

\[
\leq \text{Ch}^2 t^{-1} \| u_0 \| \| v \|_{H(div; \Omega)}.
\]

Taking supremum over \( 0 \neq v \in \mathbf{V} \) in (4.34), it follows that

\[
(4.35) \quad \| \eta_q \|_{H(div; \Omega)}^* := \sup_{0 \neq v \in \mathbf{V}} \frac{|(\eta_q, v)|}{\| v \|_{H(div; \Omega)}} \leq \text{Ch}^2 t^{-1} \| u_0 \|.
\]
To estimate \( \| \eta_{u,t} \| \), we now differentiate equations (4.4)-(4.6) to arrive at
\[
(4.36) \quad (\rho_{q,t}, v_h) + (\rho_{u,t}, \nabla \cdot v_h) = 0, \quad v_h \in V_h,
\]
\[
(4.37) \quad -(A \rho_{q,t}, z_h) + (B(t, t) \rho_q(t), z_h) + \int_0^t (B(t, s) \rho_q(s), z_h) ds = (\theta_{\sigma,t}, z_h)
\]
\[- (A \theta_{\sigma,t}, z_h) + (B(t, t) \theta_q(t), z_h) + \int_0^t (B(t, s) \theta_q(s), z_h) ds, \quad z_h \in V_h,
\]
\[
(4.38) \quad (\nabla \cdot \theta_{\sigma,t}, w_h) = 0, \quad w_h \in W_h.
\]
Put \( z_h = \rho_{q,t} \) in (4.37) and use integration by parts for the integral term to obtain
\[
(A \rho_{q,t}, \rho_{q,t}) = -(\theta_{\sigma,t}, \rho_{q,t}) + (A \theta_{\sigma,t}, \rho_{q,t}) - (B(t, t) \theta_q(t), \rho_{q,t})
\]
\[
= - (\theta_{\sigma,t}, \rho_{q,t}) + (A \theta_{\sigma,t}, \rho_{q,t}) - (B(t, t) \theta_q(t), \rho_{q,t}) - (B(t, t) \dot{\theta}_q(t), \rho_{q,t})
\]
\[
\quad + \int_0^t B_{ts}(t, s) \dot{\theta}_q(s), \rho_{q,t} ds + (B(t, t) \theta_q(t), \rho_{q,t}) + (B(t, t) \dot{\theta}_q(t), \rho_{q,t})
\]
\[- \int_0^t B_{ts}(t, s) \dot{\rho}_q(s), \rho_{q,t}) ds.
\]
and hence, using positive definiteness property of \( A \), we find that
\[
(4.39) \quad \| \rho_{q,t} \| \leq C \left( \| \theta_{\sigma,t} \| + \| \theta_{q,t} \| + \| \theta_q \| + \int_0^t \| \dot{\theta}_q(s) \| ds \right)
\]
\[
\quad + C \left( \| \rho_q \| + \| \dot{\rho}_q \| + \int_0^t \| \dot{\rho}_q(s) \| ds \right).
\]
Observe from Lemma 2.2 that
\[
\| \theta_{\sigma,t} \| = \| \sigma_t - \Pi_h \sigma_t \| \leq Ch \| \sigma_t \|_1 \leq Ch^{-2} \| u_0 \|,
\]
\[
\| \theta_{q,t} \| = \| q_t - \Pi_h q_t \| \leq Ch \| q_t \|_1 = Ch \| u_t \|_2 \leq Ch^{-2} \| u_0 \|,
\]
Now, use Lemma 4.1 and (4.11)-(4.14) in (4.39) to obtain
\[
\| \rho_{q,t} \| \leq Ch^{-2} \| u_0 \|,
\]
and hence,
\[
(4.40) \quad \| \eta_{q,t} \| \leq \| \rho_{q,t} \| + \| \theta_{q,t} \| \leq Ch^{-2} \| u_0 \|.
\]

To estimate \( \| \eta_{u,t} \| \), we again consider the auxiliary problem (4.20)-(4.21) with replacing \( \hat{\eta}_u \) on the right hand side of (4.20) by \( \eta_{u,t} \).

With the notations
\[
p = \nabla \zeta, \quad \psi = Ap,
\]
we obtain the following system
\[
(4.41) \quad (p, v) + (\zeta, \nabla \cdot v) = 0,
\]
\[
(4.42) \quad (\psi, z) - (Ap, z) = 0,
\]
\[
(4.43) \quad (\nabla \cdot \psi, w) = (\eta_{u,t}, w).
\]
From the standard regularity results, we note that
\[
(4.44) \quad \|\zeta\|_2, \|\nabla \cdot \mathbf{p}\|, \|\nabla \cdot \mathbf{q}\| \leq C\|u_0\|.
\]
Next, differentiate the equations (4.1)-(4.3) to obtain
\[
(4.45) \quad (\eta_{\mathbf{u}, t}, \mathbf{v}_h) + (\eta_{\mathbf{u}, t}, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,
\]
\[
(4.46) \quad (\eta_{\chi, t}, \mathbf{z}_h) - (A(\eta_{\mathbf{q}, t}, \mathbf{z}_h) + (B(t, t)\eta_{\mathbf{q}}, \mathbf{z}_h) + \int_0^t (B(t, s)\eta_{\mathbf{q}}(s), \mathbf{z}_h)ds = 0, \quad \mathbf{z}_h \in \mathbf{V}_h,
\]
\[
(4.47) \quad (\nabla \cdot \eta_{\chi, t}, w_h) = 0, \quad w_h \in W_h.
\]
As done earlier for finding estimate of \(\|u_t\|\), using (4.41)-(4.47) at the appropriate steps, we arrive at
\[
\|u_t\| \leq (\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \eta_{\mathbf{u}, t}) + (\nabla \cdot \Pi_h \mathbf{q}, \eta_{\mathbf{u}, t})
\]
\[
= (\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \eta_{\mathbf{u}, t}) + (\eta_{\mathbf{q}, t}, \mathbf{p} - \Pi_h \mathbf{p}) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \eta_{\mathbf{q}, t}) - (\Pi_h \mathbf{p}, A\eta_{\mathbf{q}, t})
\]
\[
= (\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \eta_{\mathbf{u}, t}) + (\eta_{\mathbf{q}, t}, \mathbf{p} - \Pi_h \mathbf{p}) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \eta_{\mathbf{q}, t})
\]
\[
+ (A(\mathbf{p} - \Pi_h \mathbf{p}), \Pi_h \mathbf{p}) - (B(t, t)\eta_{\mathbf{q}}), \Pi_h \mathbf{p}) - \int_0^t (B(t, s)\eta_{\mathbf{q}}(s), \Pi_h \mathbf{p})ds
\]
\[
(4.48) \quad = (\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \eta_{\mathbf{u}, t}) + (\eta_{\mathbf{q}, t}, \mathbf{p} - \Pi_h \mathbf{p}) - (A(\mathbf{p} - \Pi_h \mathbf{p}), \eta_{\mathbf{q}, t})
\]
\[
+ (A(\mathbf{p} - \Pi_h \mathbf{p}), \Pi_h \mathbf{p}) - (B(t, t)\eta_{\mathbf{q}}), \Pi_h \mathbf{p}) - \int_0^t (B(t, s)\eta_{\mathbf{q}}(s), \Pi_h \mathbf{p})ds
\]
Since all other terms except last three terms on the right hand side of (4.48) can be estimated easily using Lemma 2.2.4.2 and 4.3, the estimate (4.40) and the stability results of projections \(P_h\) and \(\Pi_h\), we need to estimate the last three terms as follows. Now using integration by parts, we obtain easily:
\[
| (\zeta - P_h \zeta, \nabla \cdot \eta_{\chi, t}) | = | (\nabla (\zeta - P_h \zeta), \eta_{\chi, t}) | \leq \| \nabla (\zeta - P_h \zeta) \| \| \eta_{\chi, t} \| \leq C\|\zeta\|_2.
\]
For the last but one term on the right hand side of (4.48), we use (4.35)to find that
\[
(\mathbf{B}(t, t)\eta_{\mathbf{q}}), \Pi_h \mathbf{p}) \leq M\|\eta_{\mathbf{q}}\|_{(H(div; \Omega))}^2 \|\Pi_h \mathbf{p}\|_{(H(div; \Omega))} \leq C\|\mathbf{u}_0\|\|\mathbf{p}\|_{(H(div; \Omega))}.
\]
Similarly for the last term on the right hand side of (4.48), we note that
\[
| \int_0^t (B(t, s)\eta_{\mathbf{q}}, \Pi_h \mathbf{p})ds | = |(B(t, t)\eta_{\mathbf{q}}(t), \Pi_h \mathbf{p}) - \int_0^t (B(t, t)\eta_{\mathbf{q}}(t), \Pi_h \mathbf{p})ds |
\]
\[
\quad \leq M(\|\eta_{\mathbf{q}}(t)\|_{(H(div; \Omega))} + \int_0^t \|\eta_{\mathbf{q}}(s)\|_{(H(div; \Omega))}ds) \|\Pi_h \mathbf{p}\|_{(H(div; \Omega))}
\]
\[
\quad \leq C\|\mathbf{u}_0\|\|\mathbf{p}\|_{(H(div; \Omega))}.
\]
All together, we obtain using the Cauchy-Schwarz inequality, Lemma 2.2, 4.2 and 4.3, the estimate (4.40) and the stability results of projections \(P_h\) and \(\Pi_h\), with elliptic regularity result (4.44) in (4.48) the following estimate:
\[
(4.49) \quad \|\eta_{\mathbf{u}, t}\| \leq C\|\mathbf{u}_0\|.
\]
Note that in the previous estimate, we have also used
\begin{equation}
(4.50) \quad \|\eta_{\sigma,t}\| \leq Cht^{-2}\|u_0\|,
\end{equation}
which can be derived from (4.46) by putting $z_h = \eta_{\sigma,t}$. This now completes the rest of the proof. \hfill \square

5 Semidiscrete Error Estimates

In this section, we discuss the proof of our main theorem, that is, Theorem 3.1.

Recall that
\[ e_u = u - u_h := \eta_u - \xi_u, \quad e_q := q - q_h := \eta_q - \xi_q \quad \text{and} \quad e_\sigma = \sigma - \sigma_h := \eta_\sigma - \xi_\sigma, \]
where, $(u, q, \sigma)$ and $(u_h, q_h, \sigma_h)$ are solutions of (1.1)-(1.3) and (1.12)-(1.14), respectively.

Note that, $(e_u, e_q, e_\sigma)$ satisfy the following equations
\begin{align}
(5.1) \quad (e_q, v_h) + (e_u, \nabla \cdot v_h) &= 0, \quad v_h \in V_h, \\
(5.2) \quad (e_\sigma, z_h) - (Ae_q, z_h) + \int_0^t (B(t, s)e_q(s), z_h)ds &= 0, \quad z_h \in V_h, \\
(5.3) \quad (e_{u,t}, w_h) - (\nabla \cdot e_\sigma, w_h) &= 0, \quad w_h \in W_h.
\end{align}

Since estimates of $\eta_u, \eta_q$ and $\eta_\sigma$ are known from Lemmas 4.1-4.3, it is enough to estimates $\xi_u, \xi_q$ and $\xi_\sigma$. Now using mixed Ritz-Volterra projections (4.1)-(4.3), we rewrite (5.1)-(5.3) as
\begin{align}
(5.4) \quad (\xi_q, v_h) + (\xi_u, \nabla \cdot v_h) &= 0, \quad v_h \in V_h, \\
(5.5) \quad (\xi_\sigma, z_h) - (A\xi_q, z_h) + \int_0^t (B(t, s)\xi_q(s), z_h)ds &= 0, \quad z_h \in V_h, \\
(5.6) \quad (\xi_{u,t}, w_h) - (\nabla \cdot \xi_\sigma, w_h) &= (\eta_{u,t}, w_h), \quad w_h \in W_h.
\end{align}

Below, we provide a proof of our main Theorem.

**Proof of theorem 3.1**

Choose $v_h = t^3\xi_\sigma$ in (5.4), $z_h = t^3\xi_q$ in (5.5) and $w_h = t^3\xi_u$ in (5.6) to obtain
\begin{equation}
(5.7) \quad \frac{1}{2} \frac{d}{dt}(t^3\|\xi_u\|^2) + t^3\|\xi_q\|^2_A = t^3(\eta_{u,t}, \xi_u) + \frac{3}{2}t^2\|\xi_u\|^2 - t^3 \int_0^t (B(t, s)\xi_q(s), \xi_q) ds
\end{equation}

Use integration by parts twice for the last term on the right hand side of (5.7) to find that
\begin{align}
(5.8) \quad \int_0^t (B(t, s)\xi_q(s), \xi_q) ds &= (B(t, t)\dot{\xi}_q, \xi_q) - \int_0^t (B_s(t, s)\dot{\xi}_q(s), \xi_q) ds \\
&= (B(t, t)\dot{\xi}_q, \xi_q) - (B(t, t)\dot{\xi}_q, \xi_q) \\
&\quad + \int_0^t (B_{ss}(t, s)\ddot{\xi}_q(s), \xi_q) ds.
\end{align}
Substitute (5.8) in (5.7) and use the Cauchy-Schwarz inequality along with the Young’s inequality and kickback argument. Then integrate the resulting inequality with respect to time and use positive definite property of $A$ to arrive at

$$
t^3\|\xi_u\|^2 + \int_0^t s^3\|\xi_q\|^2\,ds \leq C(\alpha, T) \int_0^t s^4\|\eta_{u,t}\|^2\,ds
$$

(5.9)

$$
+ C(\alpha, T) \int_0^t \left( s^2\|\xi_u\|^2 + s\|\dot{\xi}_q\|^2 + \|\ddot{\xi}_q\|^2 + \int_0^s \|\dddot{\xi}_q(\tau)\|^2\,d\tau \right)\,ds.
$$

Suppose for the last term on the right hand side of (5.9), we estimate it as:

$$
\int_0^t \left( s^2\|\xi_u\|^2 + s\|\dot{\xi}_q\|^2 + \|\ddot{\xi}_q\|^2 + \int_0^s \|\dddot{\xi}_q(\tau)\|^2\,d\tau \right)\,ds \leq C_t h^4\|u_0\|^2.
$$

(5.10)

Then using Lemma 4.3 and (5.10) in (5.9), we obtain

$$
t^3\|\xi_u\|^2 + \int_0^t s^3\|\xi_q(s)\|^2\,ds \leq C_t h^4\|u_0\|^2,
$$

and hence, for $t \in (0, T]$

$$
\|\xi_u(t)\| \leq C h^2 t^{-1}\|u_0\|.
$$

(5.11)

Now use triangle inequality along with (5.11) and Lemma 4.3 to conclude $t \in (0, T]$,

$$
\|\varepsilon_u(t)\| \leq \|\eta_{u}(t)\| + \|\xi_{u}(t)\| \leq C h^2 t^{-1}\|u_0\|.
$$

This completes the estimate of $\|\varepsilon_u(t)\|$.

To obtain the estimates of $\varepsilon_q$ and $\varepsilon_{\sigma}$, it is enough to estimate $\xi_q$ and $\xi_{\sigma}$. Rewrite (5.5) using integration by parts as in (5.8), for $z_h \in V_h$, as

$$
(A\xi_q, z_h) = (\xi_{\sigma}, z_h) + \int_0^t (B(t, s)\xi_q(s), z_h)\,ds
$$

(5.12)

$$
= (\xi_{\sigma}, z_h) + B(t, t)(\dot{\xi}_q, z_h) - B_u(t, t)(\dot{\xi}_q, z_h) + \int_0^t B_{ss}(t, s)(\ddot{\xi}_q, z_h)\,ds.
$$

Now, set $v_h = \xi_{\sigma}$ in (5.4) and $z_h = \xi_q$ in (5.12) and add the resulting equations. Then apply the Cauchy-Schwarz inequality with the Young’s inequality and kickback argument to obtain

$$
\|\xi_q\|^2 \leq C\|\xi_u\|\|\nabla \cdot \xi_{\sigma}\| + C \left( \|\dot{\xi}_q\|^2 + \|\ddot{\xi}_q\|^2 + \int_0^t \|\dddot{\xi}_q(s)\|^2\,ds \right).
$$

(5.13)

In addition to (5.10), we further assume that

$$
\|\dot{\xi}_q\|^2 \leq C h^4 t^{-1}\|u_0\|^2.
$$

(5.14)

Then using (5.10),(5.14) and (5.11) in (5.13), we find that

$$
\|\xi_q\|^2 \leq C h^2 t^{-1}\|u_0\|\|\nabla \cdot \xi_{\sigma}\| + C h^4 t^{-1}\|u_0\|^2.
$$

(5.15)

Note that, by definition

$$
\|\nabla \cdot \xi_{\sigma}\| \leq \|\nabla \cdot \sigma_h\| + \|\nabla \cdot \tilde{\sigma}_h\|.
$$
and from (4.3), it is easy to check as \( \tilde{\sigma}_h = \Pi_h \sigma \) that
\[
\| \nabla \cdot \tilde{\sigma}_h \| \leq \| \nabla \cdot \sigma \|.
\]

Analogous to the continuous case, we can easily obtain
\[
\| \nabla \cdot \sigma_h \| \leq C t^{-1} \| P_h u_0 \|.
\]

Using Lemma 2.2 and the stability property of \( P_h \), we arrive at
\[\text{(5.16)}\]
\[
\| \nabla \cdot \xi \| \leq C t^{-1} \| u_0 \|.
\]

On substituting (5.16) in (5.12), we obtain for \( t \in (0, T] \),
\[\text{(5.17)}\]
\[
\| \xi(t) \| \leq C h t^{-1} \| u_0 \|.
\]

Finally, in order to obtain an estimate for \( \xi \), we put \( z_h = \xi \) in (5.12) to arrive at
\[\text{(5.18)}\]
\[
\| \xi \| \leq C \left( \| \xi \| + \| \hat{\xi}_q \| + \int_0^t \| \hat{\xi}_q(s) \| \, ds \right).
\]

Use (5.17), (5.14) and (5.10) in (5.18) to conclude for \( t \in (0, T] \),
\[\text{(5.19)}\]
\[
\| \xi(t) \| \leq C h t^{-1} \| u_0 \|.
\]

Now using the definition of \( \Pi_h \) with (3.1) and Lemma 2.2, we obtain
\[\text{(5.20)}\]
\[
\| \eta \| = \| \theta \| = \| \sigma - \Pi_h \sigma \| \leq C h \| \nabla \cdot \sigma \| \leq C h t^{-1} \| u_0 \|.
\]

and hence, for \( t \in (0, T] \),
\[
\| e(t) \| \leq \| \eta(t) \| + \| \xi(t) \| \leq C h t^{-1} \| u_0 \|.
\]

This completes the rest of the proof. \( \square \)

To complete the proof, we need to estimate (5.10) and (5.14). In the following two Lemmas we shall achieve the proof of these estimates. Note that we need estimates of \( \hat{\xi}_q \) and \( \hat{\xi}_q \). Therefore, we now integrate (5.1)-(5.3) with respect to time and use the fact that \( u_h(0) = P_h u_0 \) to obtain
\[\text{(5.21)}\]
\[
(\dot{e}_q, v_h) + (\dot{e}_u, v_h) = 0, \quad v_h \in V_h,
\]
\[\text{(5.22)}\]
\[
(\dot{e}_\sigma, z_h) - (\int_0^t (B(s, s)\dot{e}_q(s), z_h) \, ds
- \int_0^t \int_0^s (B_r(s, \tau)\dot{e}_q(\tau), z_h) \, d\tau \, ds = 0, \quad z_h \in V_h,
\]
\[\text{(5.23)}\]
\[
(e_u, w_h) - (\nabla \cdot \dot{e}_\sigma, w_h) = 0, \quad w_h \in W_h.
\]
Using (4.15)-(4.17) in (5.21)-(5.23), we arrive at

\[(5.24) \quad (\hat{\xi}_q, v_h) + (\hat{\xi}_u, \nabla \cdot v_h) = 0,\]

\[(5.25) \quad (\hat{\xi}_u, z_h) - (A\hat{\xi}_q, z_h) + \int_0^t (B(s, s)\hat{\xi}_q(s), z_h) ds - \int_0^t \int_0^s (B_{\tau}(s, \tau)\hat{\xi}_q(\tau), z_h) d\tau ds = 0,\]

\[(5.26) \quad (\hat{\xi}_u, w_h) - (\nabla \cdot \hat{\xi}_\sigma, w_h) = (\hat{\eta}_u, w_h).\]

On integrating (5.24)-(5.26) with respect to time, we rewrite as

\[(5.27) \quad (\hat{\xi}_q, v_h) + (\hat{\xi}_u, \nabla \cdot v_h) = 0,\]

\[(5.28) \quad (\hat{\xi}_u, z_h) - (A\hat{\xi}_q, z_h) + \int_0^t (B(s, s)\hat{\xi}_q(s), z_h) ds - 2\int_0^t \int_0^s (B_{\tau}(\tau, \tau)\hat{\xi}_q(\tau), z_h) d\tau ds\]

\[+ \int_0^t \int_0^s \int_0^\tau (B_{\tau', \tau}(\tau, \tau')\hat{\xi}_q(\tau'), z_h) d\tau' d\tau ds = 0,\]

\[(5.29) \quad (\hat{\xi}_u, w_h) - (\nabla \cdot \hat{\xi}_\sigma, w_h) = (\hat{\eta}_u, w_h).\]

In the following two Lemmas, we shall discuss estimates of \(\hat{\xi}_u, \hat{\xi}_q\) and \(\hat{\xi}_u\) and thereby, complete the estimates of (5.10) and (5.14).

**Lemma 5.1** Let \((\hat{\xi}_u, \hat{\xi}_q, \hat{\xi}_\sigma)\) satisfy the set of equations (5.27)-(5.29). Then, there exists a positive constant \(C\), independent of \(h\), such that for \(t \in (0, T]\),

\[\|\hat{\xi}_u(t)\|^2 + \alpha_0 \int_0^t \|\hat{\xi}_q(s)\|^2 \leq C t^4 \|u_0\|^2,\]

and

\[\|\hat{\xi}_q(t)\|^2 + \int_0^t \|\hat{\xi}_u(s)\|^2 ds \leq C t^4 \|u_0\|^2.\]

**Proof.** Put \(v_h = \hat{\xi}_\sigma\) in (5.27), \(z_h = \hat{\xi}_q\) in (5.28) and \(w_h = \hat{\xi}_u\) in (5.29) to arrive at

\[\frac{1}{2} \frac{d}{dt}\|\hat{\xi}_u\|^2 + (A\hat{\xi}_q, \hat{\xi}_q) = \int_0^t (B(s, s)\hat{\xi}_q(s), \hat{\xi}_q) ds - 2\int_0^t \int_0^s (B_{\tau}(\tau, \tau)\hat{\xi}_q(\tau), \hat{\xi}_q) d\tau ds\]

\[+ \int_0^t \int_0^s \int_0^\tau (B_{\tau', \tau}(\tau, \tau')\hat{\xi}_q(\tau'), \hat{\xi}_q) d\tau' d\tau ds + (\hat{\eta}_u, \hat{\xi}_u).\]

Using the positive definite property of \(A\), boundedness of derivatives of \(B\), the Cauchy-Schwarz inequality with the Young’s inequality, and Lemma 2.2, we find that

\[\frac{d}{dt}\|\hat{\xi}_u\|^2 + \alpha_0\|\hat{\xi}_q\|^2 \leq \|\hat{\eta}_u\|^2 + \|\hat{\xi}_u\|^2 + C(T) \int_0^t \|\hat{\xi}_q\|^2 ds\]

\[\leq C t^4 \|u_0\|^2 + C(T) \left(\|\hat{\xi}_u\|^2 + \alpha_0 \int_0^t \|\hat{\xi}_q\|^2 ds\right).\]

Integrate it from 0 to \(t\) to obtain

\[\|\hat{\xi}_u\|^2 + \alpha_0 \int_0^t \|\hat{\xi}_q(s)\|^2 ds \leq C t^4 \|u_0\|^2 + C(T) \left(\|\hat{\xi}_u\|^2 + \alpha_0 \int_0^t \|\hat{\xi}_q(\tau)\|^2 d\tau\right) ds.\]
Use Gronwall’s Lemma to conclude the first estimate.
Next, set $v_h = \dot{\xi}_\sigma$ in (5.24), $z_h = \dot{\xi}_q$ in (5.28) and $w_h = \dot{\xi}_u$ in (5.29) and add them to obtain
\[
\|\ddot{\xi}_u\| + \frac{1}{2} \frac{d}{dt} \|\dot{\xi}_q\|^2_A = \int_0^t (B(s, s) \dot{\xi}_q(s), \ddot{\xi}_q(s))ds - 2 \int_0^t \int_0^t (B_{\tau, \tau}(\tau, \tau) \ddot{\xi}_q(\tau), \dot{\xi}_q)d\tau ds \\
+ \int_0^t \int_0^t \int_0^t (B_{\tau, \tau'}(\tau, \tau') \dot{\xi}_q(\tau'), \dot{\xi}_q)d\tau' d\tau ds + (\dot{\eta}_u, \dot{\xi}_u).
\]
Integrate and then use integration by parts for the integral terms to arrive at
\[
\frac{1}{2} \|\dot{\xi}_q\|^2_A + \int_0^t \|\dot{\xi}_u(s)\|^2 ds = \int_0^t (\dot{\eta}_u(s), \dot{\xi}_u(s)) ds + \int_0^t (B(s, s) \dot{\xi}_q(s), \dot{\xi}_q) ds \\
- \int_0^t (B(s, s) \ddot{\xi}_q(s), \dot{\xi}_q(s)) ds + 2 \int_0^t \int_0^t (B_{\tau, \tau}(\tau, \tau) \dot{\xi}_q(\tau), \dot{\xi}_q) d\tau ds \\
- 2 \int_0^t \int_0^s (B_{\tau, \tau}(\tau, \tau) \dot{\xi}_q(\tau), \dot{\xi}_q(s)) d\tau ds - \int_0^t \int_0^s \int_0^t (B_{\tau, \tau'}(\tau, \tau') \dot{\xi}_q(\tau'), \dot{\xi}_q(\tau)) d\tau' d\tau ds \\
- \int_0^t \int_0^t \int_0^s (B_{\tau, \tau'}(\tau, \tau') \dot{\xi}_q(\tau'), \dot{\xi}_q(s)) d\tau' d\tau ds.
\]
Use smoothness property of $B$, the positive definite property of $A$, the Cauchy-Schwarz inequality and the Young’s inequality with kickback arguments to obtain
\[
\|\dot{\xi}_q\|^2 + \int_0^t \|\dot{\xi}_u(s)\|^2 ds \leq C(\alpha_0, T) \int_0^t \|\dot{\xi}_q(s)\|^2 ds + C(\alpha_0) \int_0^t \|\dot{\eta}_u(s)\|^2 ds.
\]
Use Lemma 2.2, and the first estimate of the current Lemma to complete the rest of the proof. \hfill \Box

Lemma 5.2 Let $(\dot{\xi}_u, \dot{\xi}_q, \dot{\xi}_\sigma)$ satisfy the set of equations (5.24)-(5.26). Then, there exists a positive constant $C'$, independent of $h$, such that for $t \in (0, T]$,
\[
(5.30) \quad t\|\dot{\xi}_u(t)\|^2 + \int_0^t s\|\dot{\xi}_q(s)\|^2 ds \leq C' t^4 \|u_0\|^2,
\]
and
\[
(5.31) \quad t^2\|\dot{\xi}_q(t)\|^2 + \int_0^t s^2\|\dot{\xi}_u(s)\|^2 ds \leq C' t^4 \|u_0\|^2.
\]
Proof. Put $v_h = t\dot{\xi}_\sigma$ in (5.24), $z_h = t\dot{\xi}_q$ in (5.25) and $w_h = t\dot{\xi}_u$ in (5.26) and add them to arrive at
\[
\frac{1}{2} \frac{d}{dt}(t\|\ddot{\xi}_u\|) + t\|\dot{\xi}_q\|^2_A = t(\dot{\eta}_u, \dot{\xi}_u) + \frac{1}{2} \|\dot{\xi}_u\|^2 + t \int_0^t (B(s, s) \dot{\xi}_q(s), \ddot{\xi}_q)ds \\
- t \int_0^t \int_0^s (B_{\tau, \tau}(\tau, \tau) \dot{\xi}_q(\tau), \dot{\xi}_q) d\tau ds
\]
An application of integration by parts for the time integral terms of (5.32) implies
\begin{align*}
(5.33) \quad & \frac{1}{2} \frac{d}{dt}(t\|\dot{\xi}_u\|^2) + t\|\dot{\xi}_q\|^2_A = t(\eta_u, \dot{\xi}_u) + \frac{1}{2}\|\dot{\xi}_u\|^2 + t(B(t, t)\dot{\xi}_q, \dot{\xi}_q) \\
& \quad - t \int_0^t (B_s(s, s)\dot{\xi}_q(s), \dot{\xi}_q) ds - t \int_0^t (B_s(t, s)\dot{\xi}_q(s), \dot{\xi}_q) ds \\
& \quad + t \int_0^t \int_0^s (B_{r\tau}(s, \tau)\dot{\xi}_q(\tau), \dot{\xi}_q) d\tau ds.
\end{align*}

Integrate (5.33) with respect to time from 0 to t and obtain
\begin{align*}
\frac{t}{2}\|\dot{\xi}_u\|^2 + \int_0^t s\|\dot{\xi}_q(s)\|^2_A ds &= \frac{1}{2} \int_0^t \|\dot{\xi}_u(s)\|^2 + \int_0^t s(\eta_u(s), \dot{\xi}_u(s)) ds \\
& \quad + \int_0^t s(B(s, s)\dot{\xi}_q(s), \dot{\xi}_q(s)) ds \\
& \quad - \int_0^t \int_0^s s((B_{r}(\tau, \tau) + B_{r}(s, \tau))\dot{\xi}_q(\tau), \dot{\xi}_q(s)) d\tau ds \\
& \quad - \int_0^t \int_0^s \int_0^\tau s(B_{r\tau\tau'}(\tau, \tau')\dot{\xi}_q(\tau'), \dot{\xi}_q(s)) d\tau' d\tau ds.
\end{align*}

Using the positive definite property of $A$, the Cauchy Schwarz inequality with the Young’s inequality and kick back arguments, we find that
\begin{align*}
t\|\dot{\xi}_u\|^2 + \alpha_0 \int_0^t s\|\dot{\xi}_q(s)\|^2_A ds \leq \int_0^t s^2(\eta_u(s), \dot{\xi}_u(s)) ds + C(\alpha_0) \int_0^t \left(\|\dot{\xi}_u(s)\|^2 + \|\dot{\xi}_q(s)\|^2\right) ds.
\end{align*}

Use Lemma 4.3 and 5.1 to complete the first estimate.

Next, put $v_h = t^2\dot{\xi}_\sigma$ in (5.4), $z_h = t^2\dot{\xi}_q$ in (5.24) and $w_h = t^2\dot{\xi}_u$ in (5.26) and add them to arrive at
\begin{align*}
(5.34) \quad & t^2\|\dot{\xi}_u\|^2 + \frac{1}{2} \frac{d}{dt}t^2\|\dot{\xi}_q\|^2_A = t\|\dot{\xi}_q\|^2_A + t^2(\eta_u, \dot{\xi}_u) + t^2 \int_0^t (B(s, s)\dot{\xi}_q(s), \dot{\xi}_q) ds \\
& \quad - t^2 \int_0^t \int_0^s (B_{r}(s, \tau)\dot{\xi}_q(\tau), \dot{\xi}_q(s)) d\tau ds.
\end{align*}

Use integration by parts for the time integral terms (5.34) and then integrate the resulting equation with respect to time, from 0 to t, to obtain
\begin{align*}
& \frac{t^2}{2}\|\dot{\xi}_q\|^2_A + \int_0^t s^2\|\dot{\xi}_u(s)\|^2 ds = \int_0^t s\|\dot{\xi}_q(s)\|^2_A ds + \int_0^t s^2(\eta_u(s), \dot{\xi}_u(s)) ds \\
& \quad + t^2 \int_0^t (B(s, s)\dot{\xi}_q(s), \dot{\xi}_q) ds + \int_0^t s^2(B(s, s)\dot{\xi}_q(s), \dot{\xi}_q(s)) ds \\
& \quad - \int_0^t \int_0^s 2s(B_\tau(\tau, \tau)\dot{\xi}_q(\tau), \dot{\xi}_q(s)) d\tau ds - t^2 \int_0^t \int_0^s (B_{r}(s, \tau)\dot{\xi}_q(\tau), \dot{\xi}_q) d\tau ds \\
& \quad + \int_0^t \int_0^s s^2(B_{r}(\tau, \tau')\dot{\xi}_q(\tau'), \dot{\xi}_q(s)) d\tau ds + \int_0^t \int_0^s \int_0^\tau 2s(B_{r\tau\tau'}(\tau, \tau')\dot{\xi}_q(\tau'), \dot{\xi}_q(s)) d\tau' d\tau ds.
\end{align*}
Again an application of integration by parts yields

\[
\frac{t^2}{2} \| \hat{\xi}_q \|_A^2 + \int_0^t s^2 \| \xi_u(s) \|^2 \, ds = \int_0^t s \| \tilde{\xi}_q(s) \|^2_\delta \, ds + \int_0^t s^2 (\eta_u(s), \xi_u(s)) \, ds
\]

\[
+ t^2 (B(t, t) \tilde{\xi}_q, \xi_q) - t^2 \int_0^t (B_s(s, s) \tilde{\xi}_q(s), \xi_q) \, ds + \int_0^t s^2 (B(s, s) \xi_q(s), \tilde{\xi}_q(s)) \, ds
\]

\[
- \int_0^t 2s (B(s, s) \tilde{\xi}_q(s), \xi_q(s)) \, ds + \int_0^t \int_0^s 2s (B_{s\tau}(\tau, \tau) \tilde{\xi}_q(\tau), \xi_q(s)) \, d\tau \, ds
\]

\[
- t^2 \int_0^t (B_s(t, s) \tilde{\xi}_q(s), \xi_q) \, ds + t^2 \int_0^t \int_0^{t'} (B_{s\tau}(s, \tau) \tilde{\xi}_q(\tau), \xi_q) \, d\tau \, ds
\]

\[
+ \int_0^t \int_0^s \int_0^{t'} 2s (B_{s\tau\tau}(\tau, \tau') \tilde{\xi}_q(\tau'), \xi_q(s)) \, d\tau' \, d\tau \, ds.
\]

(5.35)

While bounds for all other terms except for the 9th term on the right hand side of (5.35) are easy to obtain, we estimate below the ninth term. A use of the Cauchy-Schwarz inequality with (5.36), we arrive at

\[
\leq C(T) \int_0^t \int_0^s \| \tilde{\xi}_q(\tau) \| \| \xi_q(s) \| \, d\tau \, ds
\]

(5.36)

\[
\leq C(T) \int_0^t s^{1/2} \left( \int_0^s \| \hat{\xi}_q(\tau) \|^2_\delta \, d\tau \right)^{1/2} \| \xi_q(s) \| \, ds
\]

\[
\leq C(T) \left( \int_0^t s \| \hat{\xi}_q(s) \|^2 \, ds + \int_0^t \| \hat{\xi}_q(s) \|^2 \, ds \right).
\]

Using the positive definite property of \( A \), the Cauchy-Schwarz inequality and the Young’s inequality with (5.36), we arrive at

\[
t^2 \| \xi_q \|^2 + \int_0^t s^2 \| \xi_u(s) \|^2 \, ds \leq C(\alpha_0, T) \int_0^t (s \| \tilde{\xi}_q(s) \|^2 + s^2 \| \eta_u(s) \|^2 + \| \tilde{\xi}_q(s) \|^2) \, ds
\]

(5.37)

\[
+ C(\alpha_0, T) \| \hat{\xi}_q \|^2.
\]

A use of Lemmas 4.3-5.2 in (5.37) now completes the rest of the proof. \( \square \)

**Remark 5.1 (Superconvergence of \( q_h - \hat{q}_h \))** As a consequence of our main Theorem 3.1, we obtain below, superconvergence result for \( \xi_q \).

Now differentiate (5.4) and put \( v_h = t^4 \xi_{\sigma} \). Then, choose \( z_h = t^4 \xi_{q,t} \) in (5.12) and \( w_h = t^4 \xi_{u,t} \) and add all the three resulting equations to obtain

\[
\frac{1}{2} \frac{d}{dt} (t^4 \| \xi_q \|_A^2) + t^4 \| \xi_{q,t} \|^2 = 2t^3 \| \xi_q \|_A^2 + t^4 (\eta_{u,t}, \xi_{u,t}) + t^4 B(t, t) (\hat{\xi}_q, \xi_{q,t})
\]

\[
- t^4 B_s(t, s)|_{s=t} (\hat{\xi}_q, \xi_{q,t}) + t^4 \int_0^t B_{s\tau}(t, s) (\tilde{\xi}_q(s), \xi_{q,t}) \, ds.
\]
Integrate and use integration by parts to find that
\[
\frac{t^4}{2} \| \xi_q \|^2 + \int_0^t s^4 \| \xi_{u,s} \|^2 \, ds = 2 \int_0^t s^3 \| \xi_q \|^2 \, ds + \int_0^t s^4 (\eta_{u,s}, \xi_{u,s}) \, ds + t^4 B(t, t) (\hat{\xi}_q, \xi_q)
\]
\[
- \int_0^t (4s^3 B(s, s) + s^4 B_s(s, s)) (\hat{\xi}_q, \xi_q) \, ds + \int_0^t s^4 B(s, s) (\xi_q, \xi_q) \, ds
\]
\[
- t^4 B_s(t, s)_{s=0} (\hat{\xi}_q, \xi_q) - \int_0^t (4s^3 B_s(s, \tau)_{\tau=s} + s^4 \frac{d}{ds} (B_s(s, \tau)_{\tau=s})) (\hat{\xi}_q, \xi_q) \, ds
\]
\[
- \int_0^t s^4 B_\tau(s, \tau)_{\tau=s} (\hat{\xi}_q(t), \xi_q) \, ds - \int_0^t s^4 B_\tau(t, s) (\hat{\xi}_q(s), \xi_q) \, ds
\]
\[
- \int_0^t s^4 \int_0^s \frac{d}{ds} (B_\tau(s, \tau)) (\hat{\xi}_q(t), \xi_q) \, d\tau ds.
\]
Now, use the Cauchy-Schwarz inequality, the Young’s inequality with kickback arguments and the positive definite property of \( A \) to obtain
\[
t^4 \| \xi_q \|^2 + \int_0^t s^4 \| \xi_{u,s} \|^2 \, ds \leq C \int_0^t (s^3 + s^4) \| \xi_q \|^2 \, ds + C \int_0^t s^4 \| \eta_{u,s} \|^2 \, ds + C t^4 \| \hat{\xi}_q \|^2
\]
\[
+ C \int_0^t (s^3 + s^4) (\| \hat{\xi}_q \|^2 + \| \hat{\xi}_q \|^2) \, ds + C t^4 \| \hat{\xi}_q \|^2 + C t^4 \int_0^t \| \hat{\xi}_q(s) \|^2 \, ds.
\]
Thus, we conclude using Lemmas 5.1-5.2 that
\[
t^4 \| \xi_q \|^2 + \int_0^t s^4 \| \xi_{u,s} \|^2 \, ds \leq C t^4 h^4 \| u_0 \|^2.
\]
and for \( t \in (0, T) \),
\[
(5.38) \quad \|q_h(t) - \bar{q}_h(t)\| \leq C h^2 t^{-3/2} \| u_0 \|,
\]
where \( \bar{q}_h \) is obtained through mixed Ritz-Volterra projections \((4.1)-(4.3)\). Note that we have derived a superconvergence result for \( \| \xi_q(t) \| \) for \( t \in (0, T) \).

**Remark 5.2** The proposed analysis in this article is quite elementary and can be easily extended to derive optimal \( L^2 \)-error estimates for semidiscrete mixed finite element approximations to both \( u \) and \( \sigma \) discussed in Sinha et al. \cite{22}, when the initial data \( u_0 \in L^2 \). Thus, it provides an improvement over the results in \cite{22}, where optimality of \( u \) is only proved under the restriction that the coefficients \( A, B \) are independent of the spatial variable, i.e., \( A = a(t) I \) and \( B = b(t) I \), and \( a \) and \( b \) are independent of spatial variable \( x \).

Since proof technique is quite similar and only with appropriate modifications results can be proved for the mixed method in \cite{22}, we shall not pursue it further.

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